

Los-Vauglit Test assures us that Th (ACFo) is complete. This uses: the theory has no finite models; and the theory is 2⁴⁰ categoing L t Jerzy Loś, Robert Vaught (1954) C-wordes Logic Algebre (Cauchy) complete & No -> (model) complete « Continuous y compact » No convergent yes categorical » Convected yes closed F Let L be a language and let X be the collection of all L-structures. For any set of sentences Σ over L, let $K_{\Sigma} = sd$ of of L-structures satisfying all the sentences in Σ Then X is a top. space with K_{Σ} as its basic closed set. This space is (topologically) compact. $\Im K_{0}$: θ sentence over L \Im are basic closed set. Eq. K= $\mathbb{Q}[\delta z] = \{a + b + \overline{z} : a, b \in \mathbb{Q}\}$ has two field automorphisms, $\iota(a + b + \overline{z}) = a - b + \overline{z}$. T $(a + b + \overline{z}) = a - b + \overline{z}$.

C held uncontably many actimorphisms but only two of them are continuous. Where do we get this?
$\mathbb{C} \subset \mathbb{C}[x] \subset \mathbb{C}(x) = K \subset K$
The ring C[x] has automorphisms f(x) ~ f(x+a)
$K = \mathbb{C}(x) = \begin{cases} \frac{f(x)}{q(x)} & : & f(x), g(x) \in \mathbb{C}[x] \end{cases}$
is a field extension of C and it's not alg. closed.
K[t] has irreducible polys eg. t-x e K[t]
\overline{K} is an alg. closed field of char. O $[\overline{K}] = 2^{R_0} = [\mathbb{C}]$
But there is only one alg. closed field of char. O for each uncomtable cardinality (the theory of ACF, is uncountably categorical) so $K \cong \mathbb{C}$.
Elevi PI Pi in CI lit I al l'ano
K has lots of automorphisms i.e. I has 1015 of automorphisms.
R has only one automorphism, the identify (a) = a.
R has only one automorphism, the identify $I(a) = a$. Axiang for R?
R has only one automorphism, the identify $I(a) = a$. Axians for R? Field axions
k has lots of automorphisms i.e. (has loss of automorphisms. R has only one automorphism, the identify $I(a) = a$. Axians for R? Field axions Introduce a new binary relation symbol '<' (a <b <math="" a="" for="" is="" shorthand="">R(a,b)) and axians <math>(\forall a)(\forall b)[(a<b) (a="b)]</math" (b<a))="" (b<a)]="" \neg[(a<b)="" \vee="" \wedge=""></b)></math>

(VaXVb)(Ve) ((a <b)-> (a+c < b+c) ~ (c>o)-> (ac < bc)])</b)->
R is the migne ordered field which is (Cauchy)-complete and having Q as a dense subfield.
But we cannot state "Cauchy complete" in first order theory of fields.
How much of the theory of R can be captured in first order logic ?
Ordered field axions
• $(\forall a)(a \neq 0 \rightarrow a^2 > 0)$
• $(\forall a)(a>0 \rightarrow (\exists b)(b=a))$
 Every polynomial f(x) ∈ ℝ[x] of odd degree has a root. Eq. for degree \$
$(\forall a)(\forall b)(\forall c)(\exists x)(x^3+qx^2+bx+c=0)$
The first order theory of R is complete.
However the theory is not K-categorical for any cardinality K. (No models for K finite; more than one for each infinite K.)
Eq. for $K = R_0 : \overline{Q} \cap R$
For K= 2 ⁴⁰ : IR; hyperreals *R
Any model of RCF is a real closed field. Every real closed field is <u>elementarily</u> equivalent to R (i.e. has the same first order theory). R and C are elementarily equivalent.
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Emil Artin (1927) proved the Hilbert 17th problem using mathematical logic.
Hilbert's 17th Problem such that \$70 (i.e. f(xi,, xn) 70 for all xi,, x ER).
Let $f(x_1, \dots, x_n) \in \mathbb{R}[x_1, \dots, x_n]$, Is it necessary then $f = s_1^2 + \dots + s_n^2$ for some
rational functions $s_i(x_1,, x_n) \in \mathbb{R}(x_1,, x_n)$? (Phiston: $k \leq 2^n$)
Motekin's example: $n=2$. $f(x,y) = 1-3x^2y^2 + x^2y^4 + x^4y^2$? This is not expressible as a sum of
Squares of polys out
$f(x,y) = \left(\frac{x y (x+y-2)}{x^2+y^2}\right)^{-} + \left[\frac{x y (x+y-2)}{x^2+y^2}\right]^{-} + \left[\frac{x - y^2}{x^2+y^2}\right]^{-} + \left[\frac{x - y^2}{x^2+y^2}\right]^{-}$
Note: $1 + x^{\frac{4}{y^2}} + \frac{x^{\frac{2}{y^4}}}{3} \ge (1 + x^{\frac{4}{y^2}} + x^{\frac{2}{y^4}})^{\frac{1}{3}} = x^{\frac{4}{y^2}}$ by the aritmetic-geometric mean inequality
So f(x,y) ≥ 0 Sor all xy.
If $f = s_i^2 + \cdots + s_k^2$ for some $s_i(x, y) \in \mathbb{R}[x, y]$ then deg $s_i \leq 3$, so $s_i(x, y)$ may have terms
1, x, y, x, xy, y, xy, xy, xy, xy
$S_i(x,y) = a_i + b_i x + c_i y + d_i \pi y + e_i x^2 + f_i y^2$
In R, the positive elements are squares. $S_i^2 = 2d; \pi q + \cdots$
(Not true in w) Consequence: Aut R = 1. If \$\$ E Aut R i.e. \$: R->R is bijective and \$(a+b) = \$\$(G) + \$\$(b) for, all
then $\phi(a) = a$ for all $e \in \mathbb{R}$. Why? $\phi(a^2) = \phi(a)^2$ so $\phi(a) > 0$ iff $a > 0$. $\phi(ab) = \phi(a) \phi(b)$ a, be \mathbb{R}

$S_0 \phi(a) < \phi(b) \iff \alpha < b$.	$\phi(o) = 0$
$ \overleftarrow{\phi(b)} - \phi(a) > 0 $	$\phi(z) = \phi(H) = \phi(1) + \phi(z) = 1 + 1 = 2$
	$\varphi(u) = u$
<>> 6-q>0	$\varphi(\alpha) = \alpha$ for all $\alpha \in \mathbb{R}$.
\Leftrightarrow $a < b$.	p(a) = q is q is q is q if q is q
(ompare: DIJZ] is also an ordered of	field but it has nontrival automorphism of (a+bir)- a-bir
tor all a, be w. Hilbort's 17th mally is true for n=1:	every fix) = Rix) with fix a for all x satisfies
$f(x) = q(x)^2 + h(x)^2$ for some $q(x)$, $h(x)$	ERTR]. Why? Factor
$f(x) = \lambda \widetilde{\Pi} (x - r_i)^2 \cdot \widetilde{\Pi} ((x - s_i)^2 + t_i^2)$) where $\lambda \geq 0$, $\lambda = a^2$
1=r	
$(a^{2}+b^{2})(c^{2}+d^{2}) = (ac-bd)^{2} + (ad+bc)^{2}$	
Proof of Hilbert's 17th Roblem (Artin; S	erre)
Let $f = f(x_1, \dots, x_n) \in \mathbb{R}[x_1, \dots, x_n]$. Suppose	e f is not a sum of squares of rational tunitions; we must
Show F(a,, an) < 0 For some Q,,	$a_{1} \in \mathbb{R}$
T= S suns of squares of rational funct	tions in f?
$= \{ s_1^2 + \dots + s_k^2 : s_i \in F \}$ Note: T	+T GT, TTGT, a ET for all a EF.

T defines a preorder or F , namely for $g,h \in F$, we say $g \leq h$ iff $h \cdot g \in T$.
"<" is transitive but it's a partial order in general.
It's an order $\mathcal{F} \mathcal{F} \mathcal{T} \mathcal{U}(-\mathcal{T}) = \mathcal{F}$ and $\mathcal{T} \mathcal{O}(-\mathcal{T}) = \{0\}$.
(total order) -T= \u03e3-g=T\u03e3
We are assuming fET.
Among all preorders containing T but not containing & choose a maximal preorder & using Zorn's lemma. Lotally ordered
let ?Pa: are A } be a collection of preorders on F with Pa 21, f& Pa
(i.e. for every aper, either Pasts or Pasta)
({Par} is a chain) Then P= V Par is an upper bound for the chain i.e. Pa SP
for all de A. Then P is a preorder (D. DCP PDCP 2°CP) and P2T f&P
By Zorn's Lanna there exists a maximal preorder P as above.
(i) Show $-1 \notin P$. If $-1 \in P$ then $f = \left(\frac{1+f}{2}\right)^2 + (-1)\left(\frac{1-f}{2}\right)^2 \in P$, a contradiction.
(ii) Show -f e P. Suppose -f e P and consider $\vec{P} = P - Pf = \{a - bf : a, b \in P\}$ which is a preorder.
$\tilde{P} + \tilde{P} = \tilde{S}(q_1 - b_1 f) + (q_2 - b_2 f) = (q_1 + q_2) - (b_1 + b_2)f : q_1, b_2 \in P_2^2 \leq \tilde{P}$
$\tilde{P}\tilde{P}:$ $(a_{r}-b_{r}f)(a_{r}-b_{r}f)$ \tilde{P} \tilde{P}
$= (a_1 a_2 + f_1 b_1 b_2) - (a_1 b_2 + a_2 b_1) f \in \beta \qquad P \supset P \qquad -f \notin P \\ f \in \beta \qquad f \in \beta$
T By maximality of P, fei EP
l = 0 $l = 0$

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