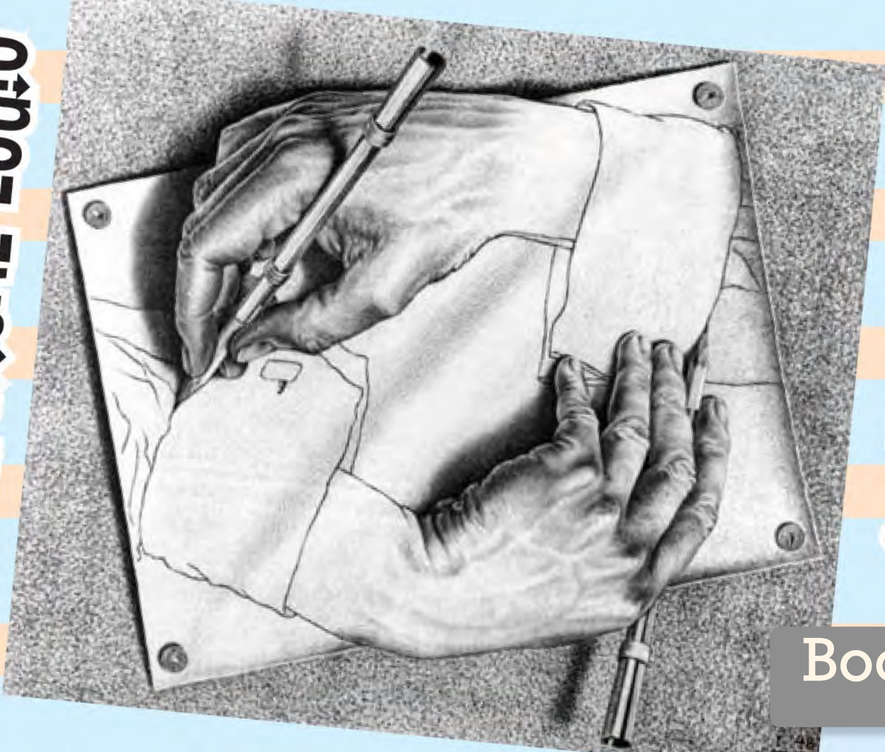
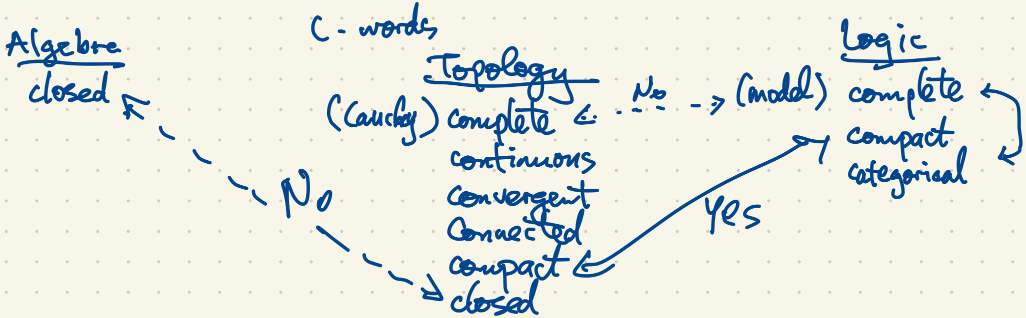
MATHEMATICAL LOGIC & SET THEORY



Book 2

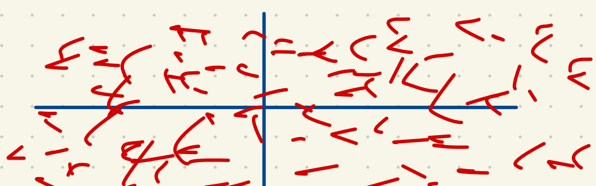
Łoś-Vaught Test assures us that $Th(ACF_0)$ is complete. This uses: the theory has no finite models; and the theory is $2^{\aleph_0}$-categorical.

Ł ł Jerzy Łoś, Robert Vaught (1954)



Let L be a language and let X be the collection of all L-structures.
For any set of sentences Σ over L, let K_Σ = set of of L-structures satisfying all the sentences in Σ (i.e. the set of models of Σ).
Then X is a top. space with K_Σ as its basic closed set.
This space is (topologically) compact. $\{K_\theta : \theta \text{ sentence over } L\}$ are sub-basic closed sets.

Eg. $K = \mathbb{Q}[\sqrt{2}] = \{a+b\sqrt{2} : a, b \in \mathbb{Q}\}$ has two field automorphisms, $\iota(a+b\sqrt{2}) = a+b\sqrt{2}$, $\tau(a+b\sqrt{2}) = a-b\sqrt{2}$.



$\mathbb{C}$ has uncountably many automorphisms but only two of them are continuous.
Where do we get this?

$$\mathbb{C} \subset \mathbb{C}[x] \subset \mathbb{C}(x) = K \subset \overline{K}$$

The polynomial ring $\mathbb{C}[x]$ has automorphisms $f(x) \longmapsto f(x+a)$

$K = \mathbb{C}(x) = \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in \mathbb{C}[x] \right\}$

is a field extension of $\mathbb{C}$ and it's not alg. closed.

$K[t]$ has irreducible polys eg. $t^2 - x \in K[t]$

$\overline{K}$ is an alg. closed field of char. 0, $|\overline{K}| = 2^{\aleph_0} = |\mathbb{C}|$

But there is only one alg. closed field of char. 0 for each uncountable cardinality (the theory of ACF_0 is uncountably categorical) so $\overline{K} \cong \mathbb{C}$.

$\overline{K}$ has lots of automorphisms i.e. $\mathbb{C}$ has lots of automorphisms.

$\mathbb{R}$ has only one automorphism, the identity $\iota(a) = a$.

Axioms for $\mathbb{R}$?

Field axioms
+ Order axioms
and axioms

Introduce a new binary relation symbol '$<$' ($a<b$ is a shorthand for $R(a,b)$)

$(\forall a)(\forall b)[((a<b) \vee (a=b) \vee (b<a)) \wedge \neg[(a<b) \wedge (b<a)] \wedge \neg[(a<b) \wedge (a=b)] \wedge \neg[(b<a) \wedge (a=b)]]$

$(\forall a)(\forall b)(\forall c)(((a<b) \wedge (b<c)) \rightarrow (a<c))$

$(\forall a)(\forall b)(\forall c)\ \big((a<b) \to [(a+c < b+c) \wedge\ (c>0) \to (ac<bc)]\big)$

$\mathbb{R}$ is the unique ordered field which is (Cauchy)-complete and having $\mathbb{Q}$ as a dense subfield.

But we cannot state "Cauchy complete" in first order theory of fields.

How much of the theory of $\mathbb{R}$ can be captured in first order logic ?

Ordered field axioms

- $(\forall a)(a \neq 0 \to a^2 > 0)$
- $(\forall a)(a>0 \to (\exists b)(b^2 = a))$
- Every polynomial $f(x) \in \mathbb{R}[x]$ of odd degree has a root. Eg. for degree 3
  $(\forall a)(\forall b)(\forall c)(\exists x)\ (x^3 + ax^2 + bx + c = 0)$.

RCF

The first order theory of $\mathbb{R}$ is complete.

However the theory is not κ-categorical for any cardinality κ. (No models for κ finite; more than one for each infinite κ.)

Eg. for $\kappa = \aleph_0$: $\bar{\mathbb{Q}} \cap \mathbb{R}$

For $\kappa = 2^{\aleph_0}$: $\mathbb{R}$; hyperreals ${}^*\mathbb{R}$

Any model of RCF is a real closed field.

Every real closed field is <u>elementarily equivalent</u> to $\mathbb{R}$ (i.e. has the same first order theory).

$\bar{\mathbb{Q}}$ and $\mathbb{C}$ are elementarily equivalent.

Emil Artin (1927) proved the Hilbert 17^{th} problem using mathematical logic.

Hilbert's 17th Problem

Let $f(x_1,\dots,x_n)\in\mathbb{R}[x_1,\dots,x_n]$, such that $f\geq 0$ (i.e. $f(x_1,\dots,x_n)\geq 0$ for all $x_1,\dots,x_n\in\mathbb{R}$). Is it necessary that $f=s_1^2+\dots+s_k^2$ for some rational functions $s_i(x_1,\dots,x_n)\in\mathbb{R}(x_1,\dots,x_n)$? (Pfister: $k\leq 2^n$)

Motzkin's example: $n=2$. $f(x,y)=1-3x^2y^2+x^2y^4+x^4y^2\geq 0$ This is not expressible as a sum of squares of poly's but

$$f(x,y)=\left[\frac{x^2y(x^2+y^2-2)}{x^2+y^2}\right]^2+\left[\frac{xy^2(x^2+y^2-2)}{x^2+y^2}\right]^2+\left[\frac{xy(x^2+y^2-2)}{x^2+y^2}\right]^2+\left[\frac{x^2-y^2}{x^2+y^2}\right]^2.$$

Note: $\frac{1+x^4y^2+x^2y^4}{3}\geq(1\cdot x^4y^2\cdot x^2y^4)^{\frac{1}{3}}=x^2y^2$ by the arithmetic-geometric mean inequality

so $f(x,y)\geq 0$ for all x,y.

If $f=s_1^2+\dots+s_k^2$ for some $s_i(x,y)\in\mathbb{R}[x,y]$ then $\deg s_i\leq 3$, so $s_i(x,y)$ may have terms

$1,x,y,x^2,xy,y^2,\cancel{x^3},\cancel{x^2y},\cancel{xy^2},\cancel{y^3}$

$s_i(x,y)=a_i+b_ix+c_iy+d_ixy+e_ix^2+f_iy^2$

$s_i^2=\underline{2d_i}xy+\cdots$

In $\mathbb{R}$, the positive elements are squares.
(Not true in $\mathbb{Q}$)

Consequence: $|\mathrm{Aut}\,\mathbb{R}|=1$. If $\phi\in\mathrm{Aut}\,\mathbb{R}$ i.e. $\phi:\mathbb{R}\to\mathbb{R}$ is bijective and $\phi(a+b)=\phi(a)+\phi(b)$, $\phi(ab)=\phi(a)\phi(b)$ for all $a,b\in\mathbb{R}$
then $\phi(a)=a$ for all $a\in\mathbb{R}$. Why? $\phi(a^2)=\phi(a)^2$ so $\phi(a)>0$ iff $a>0$.

So $\phi(a) < \phi(b) \iff a < b$.

$\iff \phi(b) - \phi(a) > 0$

$\iff \phi(b-a) > 0$

$\iff b - a > 0$

$\iff a < b$.

$\phi(0) = 0$

$\phi(1) = 1$

$\phi(2) = \phi(1+1) = \phi(1) + \phi(1) = 1 + 1 = 2$

$\vdots$

$\phi(n) = n$

$\phi(a) = a$ for all $a \in \mathbb{Q}$

$\phi(a) = a$ for all $a \in \mathbb{R}$.

Compare: $\mathbb{Q}[\sqrt{2}]$ is also an ordered field but it has a nontrivial automorphism $\phi(a+b\sqrt{2}) = a - b\sqrt{2}$ for all $a, b \in \mathbb{Q}$.

Hilbert's 17th problem is true for $n=1$: every $f(x) \in \mathbb{R}[x]$ with $f(x) \geq 0$ for all x satisfies $f(x) = g(x)^2 + h(x)^2$ for some $g(x), h(x) \in \mathbb{R}[x]$. Why? Factor

$$f(x) = \lambda \prod_{i=1}^{m} (x - r_i)^2 \cdot \prod_{j=1}^{n} \left((x - s_i)^2 + t_i^2 \right) \quad \text{where } \lambda \geq 0, \ \lambda = a^2$$

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$

Proof of Hilbert's 17th Problem (Artin; Serre)

Let $f = f(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n]$. Suppose f is not a sum of squares of rational functions; we must show $f(a_1, \ldots, a_n) < 0$ for some $a_1, \ldots, a_n \in \mathbb{R}$.

$F = \mathbb{R}(x_1, \ldots, x_n)$ = field of rational functions in $x_1, \ldots, x_n$ with real coefficients.

$T = \{$ sums of squares of rational functions in $F \}$.

$= \{ s_1^2 + \cdots + s_k^2 : s_i \in F \}$. Note: $T + T \subseteq T$, $TT \subseteq T$, $a^2 \in T$ for all $a \in F$.

T defines a preorder on F, namely for $g, h \in F$, we say $g \leq h$ iff $h - g \in T$.
"$\leq$" is transitive but it's a partial order in general.
It's an order (total order) iff $T \cup (-T) = F$ and $T \cap (-T) = \{0\}$.

$$-T = \{-g : g \in T\}$$

We are assuming $f \notin T$.
Among all preorders containing T but not containing f, choose a maximal preorder P using Zorn's lemma.
Let $\{P_\alpha : \alpha \in A\}$ be a totally ordered collection of preorders on F with $P_\alpha \supseteq T$, $f \notin P_\alpha$.
(i.e. for every $\alpha, \beta \in A$, either $P_\alpha \subseteq P_\beta$ or $P_\beta \subseteq P_\alpha$)
($\{P_\alpha\}$ is a chain) Then $P = \bigcup_{\alpha \in A} P_\alpha$ is an upper bound for the chain i.e. $P_\alpha \subseteq P$ for all $\alpha \in A$. Then P is a preorder ($P + P \subseteq P$, $PP \subseteq P$, $a^2 \in P$) and $P \supseteq T$, $f \notin P$.
By Zorn's lemma there exists a maximal preorder P as above.

(i) Show $-1 \notin P$. If $-1 \in P$ then $f = \left(\frac{1+f}{2}\right)^2 + (-1)\left(\frac{1-f}{2}\right)^2 \in P$, a contradiction.

(ii) Show $-f \in P$. Suppose $-f \notin P$ and consider $\tilde{P} = P - Pf = \{a - bf : a, b \in P\}$ which is a preorder.

$$\tilde{P} + \tilde{P} = \{(a_1 - b_1 f) + (a_2 - b_2 f) = \underbrace{(a_1 + a_2)}_{\Uparrow\, P} - \underbrace{(b_1 + b_2)}_{\Uparrow\, P} f : a_i, b_i \in P\} \subseteq \tilde{P}$$

$\tilde{P}\tilde{P}$:

$$(a_1 - b_1 f)(a_2 - b_2 f)$$

$$= \underbrace{(\underbrace{a_1 a_2}_{} + \underbrace{f^2}_{} \cdot \underbrace{b_1 b_2}_{})}_{\Uparrow\, P} - \underbrace{(a_1 b_2 + a_2 b_1)}_{\Uparrow\, P} f \in \tilde{P}$$

$\tilde{P} \supset P$ $\quad$ $-f \notin P$
$-f \in \tilde{P}$

By maximality of P, $f \in \tilde{P}$.

$f = a - bf$, some $a, b \in P$. $\quad (1+b)f = a \Rightarrow f = \frac{a}{1+b} = \underbrace{(1+b)a \cdot \frac{1}{(1+b)^2}}_{\in P}$

(iii) Given $g \in F$, show $g \in P$ or $-g \in P$.

Assume $g \notin P$; show $-g \in P$. WLOG $g \neq 0$.

Consider $\tilde{P} = P + Pg$. As in (ii) $\tilde{P}$ is a preorder, $\tilde{P} \supseteq P$, $\tilde{P} \supset P$ since $g \notin P$, $g \in \tilde{P}$. By maximality of P, we must have $f \in \tilde{P}$ so $f = a + bg$, some $a, b \in P$.

$$-bg = a - f \Rightarrow -g = \frac{a-f}{b} = b \cdot (a-f) \cdot \left(\frac{1}{b}\right)^2 \in P$$

(iv) $P \cap (-P) = \{0\}$. If $g \neq 0$, $g \in P$, $-g \in P$ then

$$-1 = g \cdot (-g) \cdot \left(\frac{1}{g}\right)^2 \in P, \text{ contrary to (i).}$$

$(F, \leq)$ is an ordered field where $a \leq b \Leftrightarrow b - a \in P$.

It's an extension of $(\mathbb{R}, \leq)$

By the Tarski Transfer Principle, if $(x_1, \dots, x_n) \in F^n$ satisfies a statement in first order theory of ordered fields, then there is $(a_1, \dots, a_n) \in \mathbb{R}^n$ realizing this statement.

Here $-f \in P$ i.e. $f < 0$ i.e. $f(x_1, \dots, x_n) < 0$ so $f(a_1, \dots, a_n) < 0$ for some $a_1, \dots, a_n \in \mathbb{R}$.

Indiscernibles ... coming soon

Axioms for projective plane geometry: Here we consider only points, lines and their incidences.

Objects: points and lines

Relations: $\underbrace{P(\cdot),\ L(\cdot)}_{\text{unary relation symbols}},\ \underbrace{I(\cdot,\cdot)}_{\text{binary relation symbol}}$

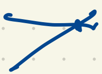
$$(\forall x)(P(x) \leftrightarrow (\neg L(x)))$$
$$(\forall x)(\forall y)(I(x,y) \rightarrow (P(x) \leftrightarrow L(y)))$$

Axioms: (i)

Any two distinct points are on a unique line.

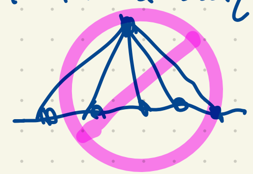
$$(\forall x)(\forall y)(P(x) \wedge P(y) \wedge \neg(x=y) \rightarrow (\exists z)(I(x,z) \wedge I(y,z) \wedge (\forall w)((I(x,w) \wedge I(y,w)) \rightarrow (w=z))))$$

(ii)



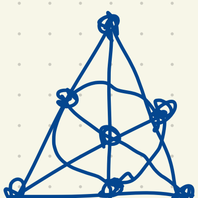
Any two distinct lines meet in a unique point.

(iii) nondegeneracy axiom



There exist at least four points with no three of them collinear.

Models? There are some orders (sizes) for which models are unique up to isomorphism

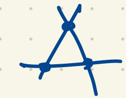


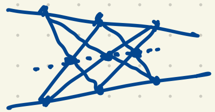
7 points
7 lines
3 points/line
3 lines/point

Finite projective planes:
n^2+n+1 points / lines
$n+1$ points / line
$n+1$ lines / point
n = order of the plane

Infinite planes:
For every infinite cardinal κ, there are many proj. planes of order κ (with cardinality κ).

Does there exist an infinite projective plane which is $\aleph_0$-categorical i.e. its theory has a unique countable model?

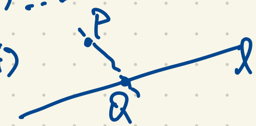




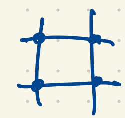
Generalized Quadrangles

(i) Any two points are on at most line

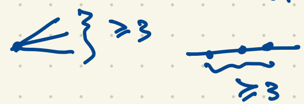
(ii)



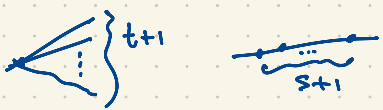
If P is not on l then there is a unique Q on l joined to P.



(iii) nondegeneracy.



In every case



Can $s<\infty$, $t=\infty$?

If $s=2$ then $t\le 4$ (easy).

If $s=3$ then $t\le 9$ (4 pages)

If $s=4$ then $t\le 16$ (Cherlin)

Let $\mathcal{A}$ be a set of first order sentences over a language L (i.e. a theory) and let $M \models \mathcal{A}$ (a model of $\mathcal{A}$).

A set of indiscernibles $S \subseteq M$ such that for every distinct $s_1, \dots, s_k \in S$ and $t_1, \dots, t_k \in S$ and every propositional function $\phi(x_1, \dots, x_k)$, $\phi(s_1, \dots, s_k)$ iff $\phi(t_1, \dots, t_k)$.

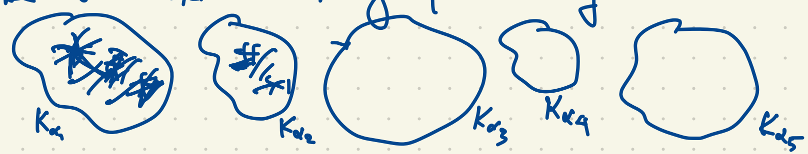
Eg. Let $\mathcal{A}$ be the axioms of field theory, $\mathbb{C} \models \mathcal{A}$. Let S be any algebraically independent subset of $\mathbb{C}$. This means that for all distinct $s_1, \dots, s_k \in S$ and nonzero $f(x_1, \dots, x_k) \in \mathbb{Q}[x_1, \dots, x_k]$ then $f(s_1, \dots, s_k) \neq 0$.

eg. $\{\pi\}$, $\{e\}$. There are alg. ind. subsets of $\mathbb{C}$ of uncountable size!

Is $\{\pi, e\}$ alg. indep.?

Any set $S \subseteq \mathbb{C}$ which is alg. indep. is a set of indiscernibles.

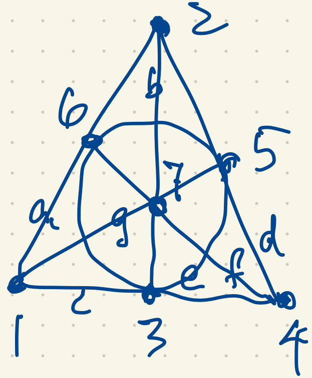
Let $\mathcal{A}$ be the axioms of graph theory. Consider a graph $\Gamma \models \mathcal{A}$ that looks like



where $\alpha_1, \dots, \alpha_5$ are infinite cardinals.

Pick $s_1 \in K_{\alpha_1}, \dots, s_5 \in K_{\alpha_5}$.

$\{s_1, \dots, s_5\}$ is a set of indiscernibles.



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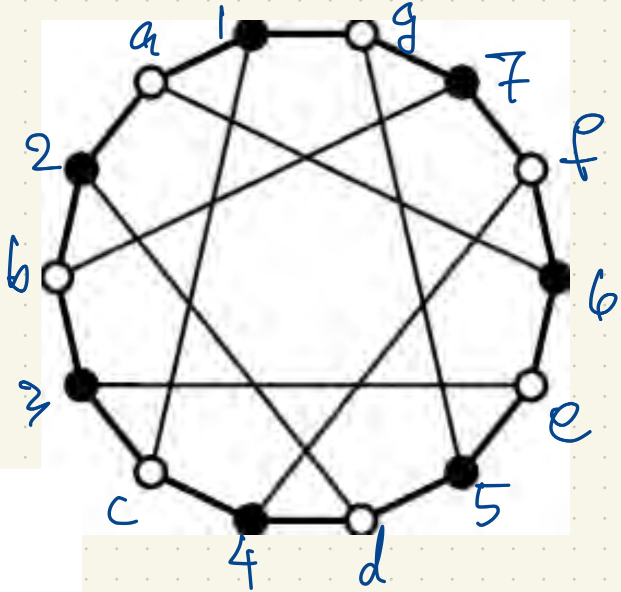
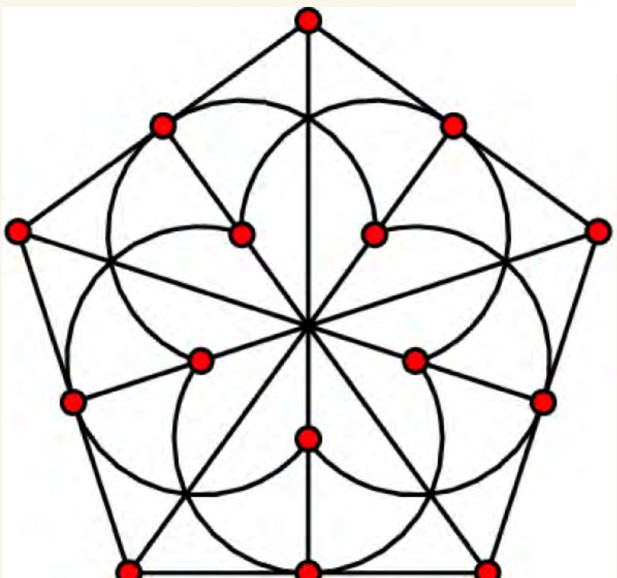


Proj. Plane $\longleftrightarrow$ bipartite graph of diameter 3 and girth 6 (shortest cycles have length 6)



generalized quadrangle $\longleftrightarrow$ bipartite graph of diameter 4 and girth 8.

Let $\mathcal{L}$ be a language and $\mathcal{A}$ a set of sentences over $\mathcal{L}$. Let $M \models \mathcal{A}$ be an $\mathcal{L}$-structure. A subset $S \subseteq M$ is a set of <u>indiscernibles</u> if for every $k \geq 1$ and $a_1, \cdots, a_k \in S$ distinct, also any $\phi(x_1, \cdots, x_n)$ ~~propositional~~ first ~~order~~ formula over $\mathcal{L}$, $b_1, \cdots, b_k \in S$ distinct,

$$M \models \phi(a_1, \cdots, a_k) \Leftrightarrow \phi(b_1, \cdots, b_k).$$

Eg. $\mathcal{L} = (\cdot, +, 0, 1)$ = language of rings with identity!
$\mathcal{A}$ = axioms of field theory
$M = \mathbb{C}$
$S \subset \mathbb{C}$ any algebraically independent set (i.e. for $a_1, \cdots, a_k \in S$ distinct, $f(x_1, \cdots, x_k) \in \mathbb{Q}[x_1, \cdots, x_k]$ nonzero poly., $f(a_1, \cdots, a_k) \neq 0$.)

Let $s, t \in S$. Eg. $\phi(x,y)$: $x^2 + xy + y^2 = 0$.
For all $s, t \in S$ $(s \neq t)$, $\phi(s,t)$ is false.
$\psi(x,y)$: $(\forall u)(\exists v)(ux + vy = 1)$.
$\psi(s,t)$ is true for all $s \neq t$ in S.

Dense Linear Order Without Endpoints

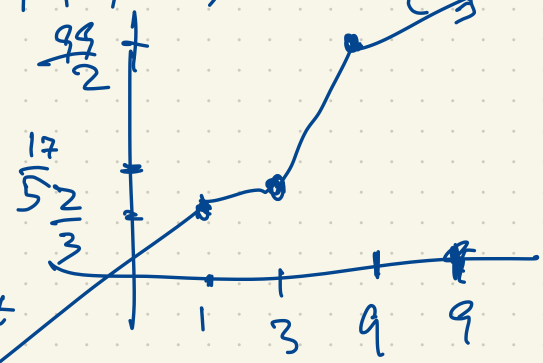
$\mathcal{L} = (<)$, A = axioms of DLO without endpoints, $M = (\mathbb{Q}, <)$ usual ordering on $\mathbb{Q}$. $M \models A$ (the unique countable model up to isomorphism). This structure has no indiscernible sets S with $|S| > 1$. If $s, t \in S$ with $s \neq t$ then (s,t), (t,s) are discernible
eg. $s < t \to \neg(t < s)$

A set of <u>order indiscernibles</u> in M is an ordered set $S = \{s_t : t \in \mathbb{Q}\}$ such that whenever $t_1 < \cdots < t_k$ in $\mathbb{Q}$ and $u_1 < \cdots < u_k$ in $\mathbb{Q}$ and $\phi(x_1, \dots, x_k)$ is a prop. formula over $\mathcal{L}$ we have

$$M \models \left(\phi(s_{t_1}, \dots, s_{t_k}) \leftrightarrow \phi(s_{u_1}, \dots, s_{u_k})\right).$$

Now $\mathcal{L} = (<)$, $M = (\mathbb{Q}, <)$, $S = \mathbb{Q}$. S is a set of order indiscernibles.

$$\{1, 3, 9\} \longrightarrow \left\{\frac{2}{3}, \frac{17}{5}, \frac{99}{2}\right\}$$



<u>Theorem</u> Let $\mathcal{A}$ be a collection of sentences over a language $\mathcal{L}$. If $\mathcal{A}$ has an infinite model $M \models \mathcal{A}$, then $\mathcal{A}$ has an infinite model with a set of order indiscernibles $S \subseteq M$, $S = \{s_t : t \in \mathbb{Q}\}$.

(Here we have chosen S having order type $(\mathbb{Q}, <)$ but you can choose any total order you want and get models of $\mathcal{A}$ with sets of order indiscernibles of the desired order type.)

<u>Remark</u>: The Upward Löwenheim-Skolem Theorem says: if $\mathcal{A}$ has an infinite model M then it also has models of every cardinality $\geq |M|$.

$|A| = |B|$ iff there is a bijection $A \longrightarrow B$.

$|A| \le |B|$ iff there is a bijection between A and a subset of B (i.e. an injection $A \to B$)
1:1 map

eg. $\mathbb{N} = \{1, 2, 3, \cdots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\} = \omega$. The map $x \mapsto x$, $\mathbb{N} \to \mathbb{N}_0$ is injective so $|\mathbb{N}| \le |\mathbb{N}_0|$. But $|\mathbb{N}| = |\mathbb{N}_0|$ since $x \mapsto x-1$ is a bijection $\mathbb{N} \longrightarrow \mathbb{N}_0$.

$|\mathbb{N}| = |\mathbb{N}_0| = |\mathbb{Q}| = |\mathbb{Z}| = |\mathbb{Q}^n| = \aleph_0$ $(n = 1, 2, 3, \ldots)$ countably infinite; $|\mathbb{R}| > \aleph_0$. why?

$\mathbb{N} \longrightarrow \mathbb{R}$, $x \mapsto x$ is an injection so $|\mathbb{N}| \le |\mathbb{R}|$. Cantor showed there is no bijection so $|\mathbb{N}| < |\mathbb{R}|$. More generally, if S is any set then $|S| < |\mathcal{P}(S)|$ where $\mathcal{P}(S)$ = power set of S = $\{$all subsets of $S\}$.

$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.

$$\begin{array}{l}
\qquad\qquad\qquad\qquad\qquad 0, 1, 2, 3, \ldots, \omega = \omega_0, \ \omega_1, \ \omega_2, \ \omega_3, \ \omega_4, \ldots, \omega_\omega, \ \omega_{\omega+1}, \ldots \\
\qquad\qquad\qquad\qquad\qquad \qquad\qquad\quad\ \ \| \quad\ \ \| \quad\ \ \| \quad\ \ \| \quad\ \ \|
\end{array}$$

~~Possible~~ Sizes (cardinalities) of sets: $0, 1, 2, 3, \ldots, \aleph_0, \aleph_1, \aleph_2, \aleph_3, \aleph_4, \ldots, \aleph_\omega, \aleph_{\omega+1}, \ldots$

Since $|\mathbb{R}| > \aleph_0$, we have $|\mathbb{R}| \ge \aleph_1$.

CH (Continuum Hypothesis): $|\mathbb{R}| = \aleph_1$, i.e. there is no set A with $|\mathbb{N}| < |A| < |\mathbb{R}|$.
"Conjecture"

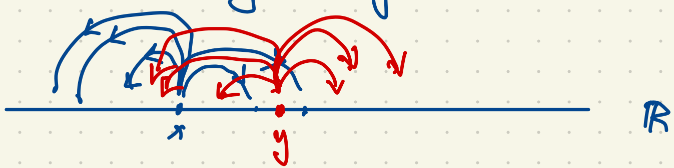
$\neg$CH : $|\mathbb{R}| \ge \aleph_2$ ie. there exists a set B with $|\mathbb{N}| < |B| < |\mathbb{R}|$.

By ZFC, every set S can be well-ordered. There is an order relation "$\trianglelefteq$" on S such that

- if $a \trianglelefteq b$ and $b \trianglelefteq c$ then $a \trianglelefteq c$
- if $a \trianglelefteq b$ and $b \trianglelefteq a$ then $a = b$. ($a \trianglelefteq b$ means $\underset{\sim}{a \triangleleft b}$ or $a = b$) ($a \triangleleft b$: $a \trianglelefteq b$ but $a \neq b$)
- Every nonempty subset of S has a least element. If $A \subseteq S$, $A \neq \emptyset$ then there exists $a \in A$ with $a \trianglelefteq x$ for all $x \in A$.

  In other words, there is no infinite decreasing sequence $a_1 \triangleright a_2 \triangleright a_3 \triangleright a_4 \triangleright \cdots$ in A.

The Axiom of Symmetry AS: x shoots at positions $A_x \subset \mathbb{R}$, $|A_x| \leq \aleph_0$, $x \notin A_x$



AS: There exist $x \neq y$ in $\mathbb{R}$ such that $x \notin A_y$, $y \notin A_x$.
(Neither of x, y hits the other.)

AS is very easily believable.

AS is equivalent to $\neg CH$.

Proof of CH implies ¬AS: Assuming CH, $|\mathbb{R}| = \aleph_1$, so well-order $(\mathbb{R}, \lhd)$ of type ω_1.

For every $x \in \mathbb{R}$, define $A_x = \{y \in \mathbb{R} : \underbrace{y \lhd x}_{y \in x}\}$. $x \in \mathbb{R}$ says $\underbrace{x \lhd \omega_1}_{x \in \omega_1}$ so x is a countable ordinal.

So $|A_x| \le \aleph_0$.

$$\left.\begin{array}{l} x \in A_y \iff x \lhd y \\ y \in A_x \iff y \lhd x \end{array}\right\}$$
Since $x \neq y$, one of these holds. This contradicts AS.

Proof of ¬CH → AS: Assuming there exists $B \subset \mathbb{R}$ with $\aleph_0 < |B| < |\mathbb{R}|$, say $|B| = \aleph_1$, $|\mathbb{R}| \geqslant \aleph_2$, and let $x \mapsto A_x$ be any assignment of countable subsets of $\mathbb{R}$ to the real numbers $x \in \mathbb{R}$.

$B_1 = \bigcup_{x \in B} A_x = \{$all points hit from B$\}$. $|B_1| \le \aleph_1$.

$B_2 = \bigcup_{x \in B_1} A_x$ $|B_2| \le \aleph_1$ etc. $B^* = B \cup B_1 \cup B_2 \cup B_3 \cup \cdots$ $|B^*| = \aleph_1$.

Since $|B^*| < |\mathbb{R}|$, we can pick $x \in \mathbb{R}$, $x \notin B^*$. We want to pick $y \in B^*$, $y \notin A_x$. Since $|A_x| = \aleph_0 < |B^*|$, such y exists.

Also $x \notin A_y$ since points $y \in B^*$ can only hit other points in B^*.

Thus AS holds.

Freiling c. 1986 introduced AS. But this was actually due to Sierpinski.

$AS = AS_1$

AS_2 says: Given any assignment $\{x,y\} \longmapsto A_{x,y} \subseteq \mathbb{R}$ (for $x \neq y$ in $\mathbb{R}$)

$|A_{x,y}| \leq \aleph_0$

there exist three distinct $x, y, z \in \mathbb{R}$ such that none of them are shot by the other two ie.

$$x \notin A_{y,z}$$
$$y \notin A_{x,z}$$
$$z \notin A_{x,y}.$$

AS_2 is equivalent to $|\mathbb{R}| \geq \aleph_3$.

AS_3 . - - - .. $|\mathbb{R}| \geq \aleph_4$

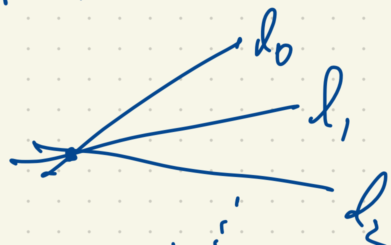
<u>Theorem</u> (Cherlin) Let Q be a generalized quadrangle with k points on every line, $k \in \{3,4,5\}$. Then Q is finite. (Actually known previously for $k = 3, 4$.)

Language: $I(x,y)$ binary relation "x is incident with y" i.e. 

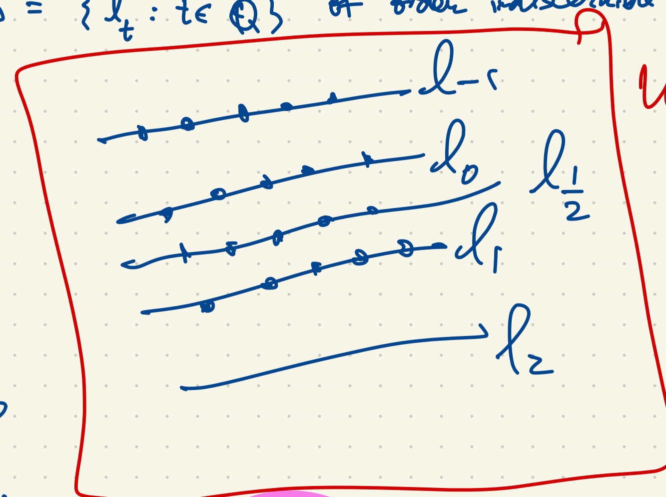
or 

$P(x)$, $L(y)$ unary relations.

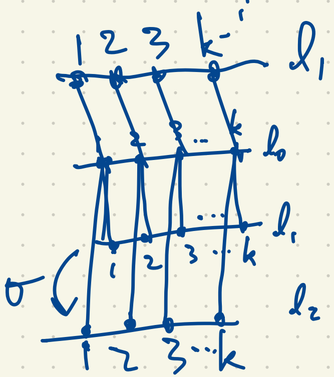
<u>Proof</u> Suppose the theory of GQ's with k points per line has an infinite model. Then it has an infinite model with a set $S = \{l_t : t \in \mathbb{Q}\}$ of order indiscernible lines.

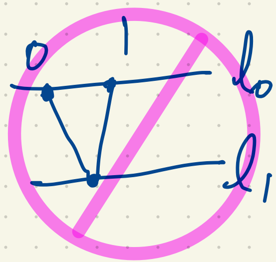


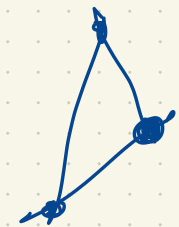
or



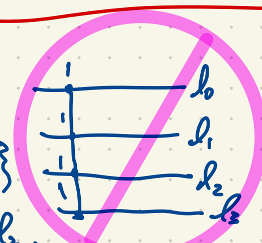
WLOG







There is a permutation $\sigma : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$ such that point i on l_1 is joined to point $\sigma(i)$ on l_2.



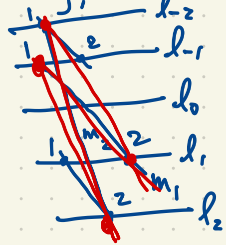
l_0, l_1, l_2 $\quad 0 < 1 < 2$

l_0, l_1, l_3 $\quad 0 < 1 < 3$

(σ is a <u>derangement</u> of $\{1, \ldots, k\}$.
σ is fixed-point-free i.e. $\sigma(i) \neq i$.

By order-indiscernibility, whenever $0 < s < t$, point i of ℓ_s is joined to point $\sigma(i)$ of ℓ_t.

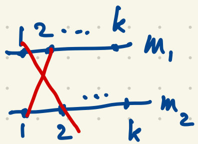
Re-index:



Suppose $\sigma(1) = 2$. (WLOG)

For each $t > 0$ $(t \in \mathbb{Q})$, let m_t be the line joining point 1 on ℓ_{-t} with point 2 on ℓ_t.

This gives from $\{\ell_t : t \in \mathbb{Q}\}$ a new set of lines $\{m_t : t \in \mathbb{Q},\ t > 0\}$. $m_t \cap m_{t'} = \emptyset$ for all $t \neq t'$ and the collection $\{m_t : t \in \mathbb{Q},\ t > 0\}$ of lines is again a collection of order indiscernibles.



If we replace the original $\{\ell_t\}_t$ with $\{m_t\}_t$ then the new σ is a derangement satisfying

$\sigma(1) = 2$, $\sigma(2) = 1$.

If $k = 3$ we have a contradiction!

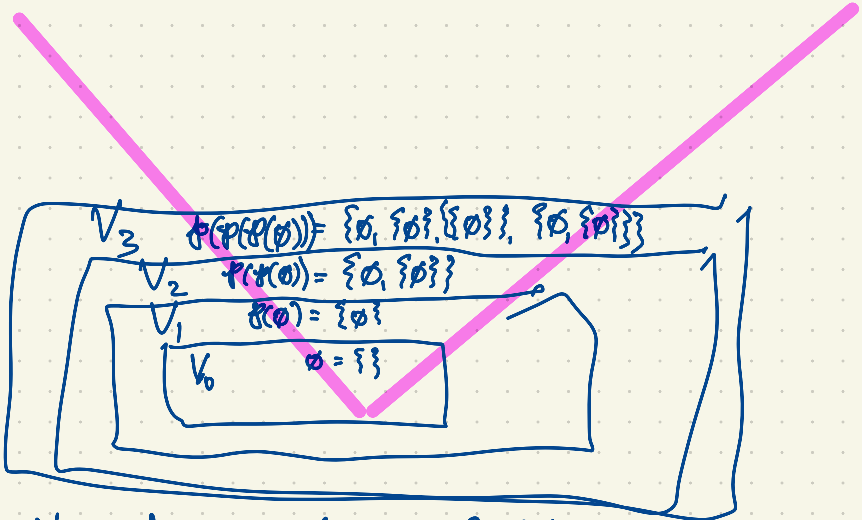
For $k = 4, 5$ we must work a little harder.

Set Theory ZFC axioms for first order set theory. See Cameron; Borcherds

Richard Borcherds YouTube $\to$ Zermelo-Fraenkel (≈ 9 videos)

Avoid Russel's Paradox!

If S has n elements then $\mathcal{P}(S)$ has 2^n elements



The Von Neumann Universe of Sets

Starting with $V_0 = \{\emptyset\}$, or $V_0 = \emptyset$??

recursively $V_{\alpha+1} = \mathcal{P}(V_\alpha)$

$V_\omega = V_0 \cup V_1 \cup V_2 \cup \cdots$

$V_{\omega+1} = \mathcal{P}(V_\omega)$

$V_{\omega+2} = \mathcal{P}(V_{\omega+1})$

$\vdots$

$V_{\omega\cdot 2} = V_{\omega+\omega} = V_0 \cup V_1 \cup \cdots \cup V_\omega \cup V_{\omega+1} \cup V_{\omega+2} \cup \cdots$

"Keep going"