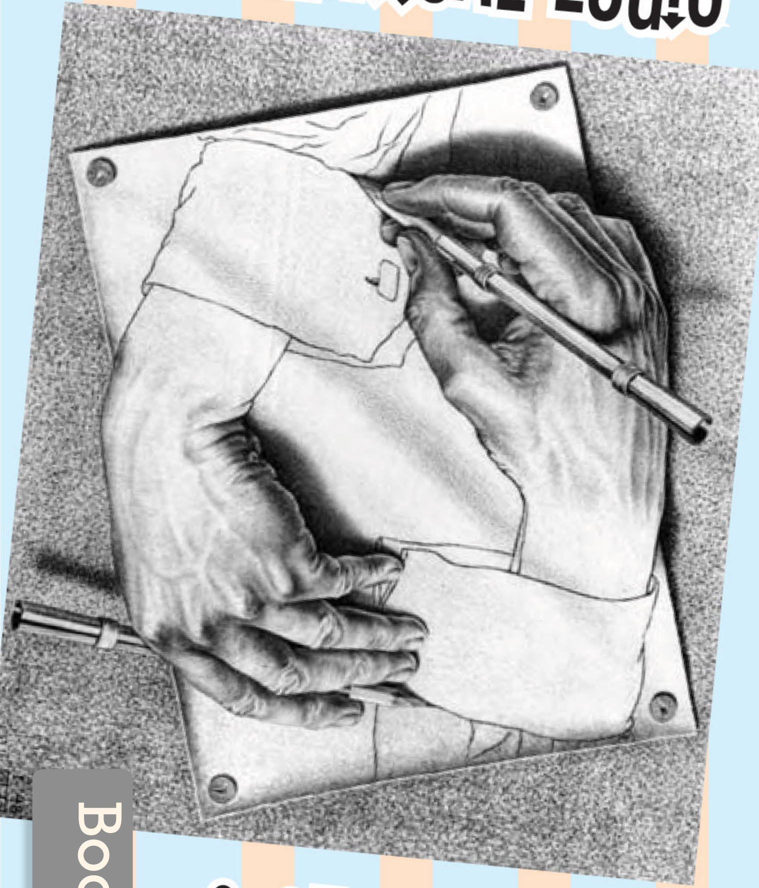
MATHEMATICAL LOGIC

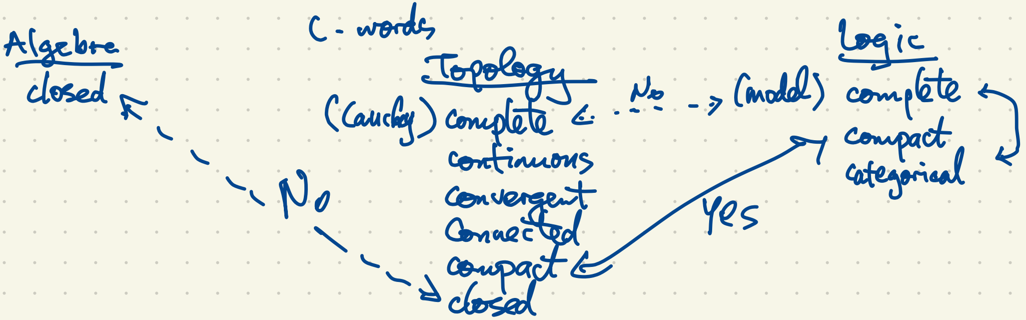


& SET THEORY

Book 2

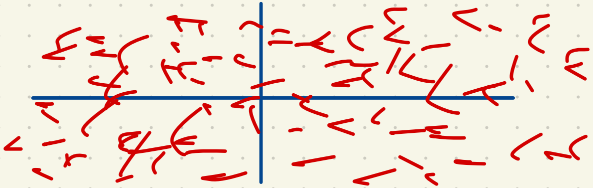
Łoś-Vaught Test assures us that $Th(ACF_0)$ is complete. This uses: the theory has no finite models; and the theory is $2^{\aleph_0}$-categorical.

Ł ł Jerzy Łoś, Robert Vaught (1954)



Let L be a language and let X be the collection of all L-structures.
For any set of sentences Σ over L, let K_Σ = set of of L-structures satisfying all the sentences in Σ (i.e. the set of models of Σ).
Then X is a top. space with K_Σ as its basis closed set.
This space is (topologically) compact. $\{K_\theta : \theta$ sentence over $L\}$ are sub-basic closed sets.

Eg. $K = \mathbb{Q}[\sqrt{2}] = \{a+b\sqrt{2} : a,b \in \mathbb{Q}\}$ has two field automorphisms, $\iota(a+b\sqrt{2}) = a+b\sqrt{2}$,
$\tau(a+b\sqrt{2}) = a-b\sqrt{2}$.



$\mathbb{C}$ has uncountably many automorphisms but only two of them are continuous.
Where do we get this?

$$\mathbb{C} \subset \mathbb{C}[x] \subset \mathbb{C}(x) = K \subset \bar{K}$$

The polynomial ring $\mathbb{C}[x]$ has automorphisms $f(x) \longmapsto f(x+a)$

$K = \mathbb{C}(x) = \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in \mathbb{C}[x] \right\}$
is a field extension of $\mathbb{C}$ and it's not alg. closed.

$K[t]$ has irreducible polys eg. $t^2 - x \in K[t]$

$\bar{K}$ is an alg. closed field of char. 0, $|\bar{K}| = 2^{\aleph_0} = |\mathbb{C}|$

But there is only one alg. closed field of char. 0 for each uncountable cardinality (the theory of ACF_0 is uncountably categorical) so $\bar{K} \cong \mathbb{C}$.

$\bar{K}$ has lots of automorphisms i.e. $\mathbb{C}$ has lots of automorphisms.

$\mathbb{R}$ has only one automorphism, the identity $\iota(a) = a$.

Axioms for $\mathbb{R}$?

Field axioms
+ Order axioms
and axioms

Introduce a new binary relation symbol '$<$' ($a<b$ is a shorthand for $R(a,b)$)

$(\forall a)(\forall b)[((a<b) \vee (a=b) \vee (b<a)) \wedge \neg[(a<b) \wedge (b<a)] \wedge \neg[(a<b) \wedge (a=b)] \wedge \neg[(b<a) \wedge (a=b)]]$

$(\forall a)(\forall b)(\forall c)((a<b) \wedge (b<c) \rightarrow (a<c))$

$(\forall a)(\forall b)(\forall c)\left((a<b)\to[(a+c<b+c)\wedge (c>0)\to(ac<bc)]\right)$

$\mathbb{R}$ is the unique ordered field which is (Cauchy)-complete and having $\mathbb{Q}$ as a dense subfield.

But we cannot state "Cauchy complete" in first order theory of fields.

How much of the theory of $\mathbb{R}$ can be captured in first order logic?

Ordered field axioms

- $(\forall a)(a\neq 0 \to a^2>0)$
- $(\forall a)(a>0 \to (\exists b)(b^2=a))$
- Every polynomial $f(x)\in\mathbb{R}[x]$ of odd degree has a root. Eg. for degree 3 $(\forall a)(\forall b)(\forall c)(\exists x)(x^3+ax^2+bx+c=0)$.

RCF

The first order theory of $\mathbb{R}$ is complete.

However the theory is not κ-categorical for any cardinality κ. (No models for κ finite; more than one for each infinite κ.)

Eg. for $\kappa=\aleph_0$: $\bar{\mathbb{Q}}\cap\mathbb{R}$

For $\kappa=2^{\aleph_0}$: $\mathbb{R}$; hyperreals ${}^*\mathbb{R}$

Any model of RCF is a real closed field.

Every real closed field is elementarily equivalent to $\mathbb{R}$ (i.e. has the same first order theory).

$\bar{\mathbb{Q}}$ and $\mathbb{C}$ are elementarily equivalent.

Emil Artin (1927) proved the Hilbert 17th problem using mathematical logic.

Hilbert's 17th Problem

Let $f(x_1,\dots,x_n)\in\mathbb{R}[x_1,\dots,x_n]$, such that $f\geq 0$ (i.e. $f(x_1,\dots,x_n)\geq 0$ for all $x_1,\dots,x_n\in\mathbb{R}$). Is it necessary that $f=s_1^2+\dots+s_k^2$ for some rational functions $s_i(x_1,\dots,x_n)\in\mathbb{R}(x_1,\dots,x_n)$? (Pfister: $k\leq 2^n$)

Motzkin's example: $n=2$. $f(x,y)=1-3x^2y^2+x^2y^4+x^4y^2\geq 0$ This is not expressible as a sum of squares of poly's but

$$f(x,y)=\left[\frac{x^2y\,(x^2+y^2-2)}{x^2+y^2}\right]^2+\left[\frac{xy^2\,(x^2+y^2-2)}{x^2+y^2}\right]^2+\left[\frac{xy\,(x^2+y^2-2)}{x^2+y^2}\right]^2+\left[\frac{x^2-y^2}{x^2+y^2}\right]^2.$$