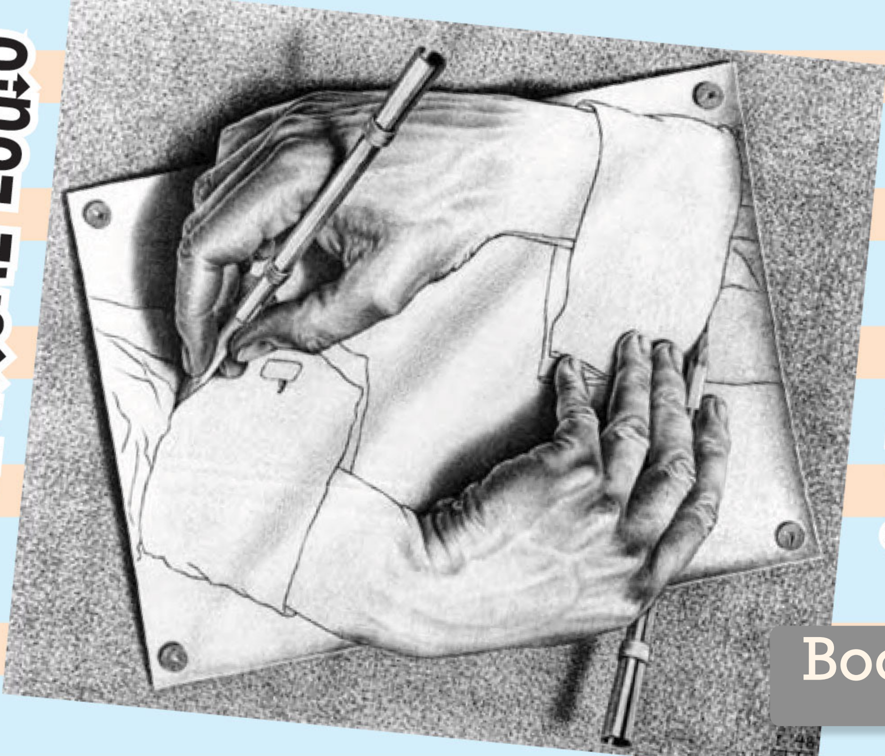
MATHEMATICAL LOGIC & SET THEORY



Book 3

Trivial examples: Fix $x_0 \in X$. Define $\mu(A) = \begin{cases} 0 & \text{if } x_0 \notin A \\ 1 & \text{if } x_0 \in A. \end{cases}$

A **measurable cardinal** is a uncountable cardinal κ which admits a nontrivial ~~countably additive~~ two-valued measure.

Does such a κ exist? If so then any larger cardinal satisfies this condition.

Given $\kappa < \kappa'$, μ nontrivial countably additive two-valued measure on κ, lift it to one on κ': $\iota : \kappa \to \kappa'$ injection. Define (for $B \subseteq \kappa'$)

$$\mu'(B) = \mu(\iota^{-1}(B)).$$

Theorem (Ulam) If there exists a nontrivial countably additive two-valued measure on an uncountable set X then let X be a smallest example. Then X has a nontrivial κ-additive two-valued measure for all $\kappa \leq |X|$.

μ is κ-additive if

$$\mu\Big(\bigsqcup_{\alpha \in I} A_\alpha\Big) = \sum_{\alpha \in I} \mu(A_\alpha)$$

for every collection of $|I| < \kappa$ sets $(A_\alpha \subseteq X)$.

A **measurable cardinal** is an uncountable cardinal κ having a κ-additive two-valued measure.

$$[0,1] = \bigsqcup_{\alpha \in [0,1]} \{\alpha\}$$

Do they exist? And who cares?

$$\begin{tikzcd}[column sep=small, row sep=small]
& \underset{\text{open}}{\Sigma^0_1} \arrow[dr, phantom, "\subset"] & & \overset{\text{ctble unions of closed sets}}{\Sigma^0_2} \arrow[dr, phantom, "\subset"] & & \\
\underset{\text{clopen}}{\Delta^0_1} \arrow[ur, phantom, "\subset"] \arrow[dr, phantom, "\subset"] & & \Delta^0_2 \arrow[ur, phantom, "\subset"] \arrow[dr, phantom, "\subset"] & & \Delta^0_3 \arrow[ur, phantom, "\subset"] \arrow[dr, phantom, "\subset"] & \cdots \\
& \underset{\text{closed}}{\Pi^0_1} \arrow[ur, phantom, "\subset"] & & \underset{\text{ctble intersections of open sets}}{\Pi^0_2} \arrow[ur, phantom, "\subset"] & &
\end{tikzcd}$$

Projective Hierarchy Σ^1_n, Π^1_n, $\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$

$$\begin{array}{ccccccc} & & \Sigma^1_1 & & & & \Sigma^1_2 \cap \\ \Delta^1_0 & \subset & & \supset & \Delta^1_1 = \Sigma^1_1 \cap \Sigma^1_1 & \subset & \\ \text{Borel sets} & \subset & \Pi^1_1 & \supset & & \cap & \Pi^1_2 \subset \end{array}$$

Σ^1_1 = {analytic sets in X} $A \in \Sigma^1_1$ iff A is a continuous image of a Borel set under $f: Y \to X$ (f continuous, Y Polish space)

Π^1_1 = {coanalytic sets in X} = {complements of analytic sets}

Σ^1_2 = {continuous images of coanalytic sets}

If there exist measurable cardinals, then every Σ^1_2-set is Lebesgue measureable.

Coming to: an application a large cardinal to the finite world.

Non-associative algebra : Keis, Quandles, Racks, Shelves, ... (see Sam Nelson, Quandles)

A **kei** is a set S with a binary operation $\triangleright$ satisfying : for all $x, y, z \in S$,

(1) $x \triangleright x = x$ (every element is idempotent)

(2) $(x \triangleright y) \triangleright y = x$ ($x \mapsto x \triangleright y$ is involutory)

(3) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ ("$\triangleright$" is right-distributive over itself)

If $(S, \triangleleft)$ satisfies (3), it is a <u>shelf</u> (or <u>self distributive system</u>). If it satisfies (1) and (3), it is a <u>rack</u>.

If $(S, \triangleleft)$ satisfies (1), (3) and (2′) it is a <u>quandle</u>.

(2′): For all y, the map $S \to S$, $x \mapsto x \triangleright y$ is injective.

(1) $x \triangleright x = x$

(2) $(x \triangleright y) \triangleright y = x$

(3) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

The kei axioms are equivalent to the Reidemeister moves I, II, III.

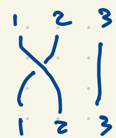
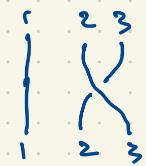
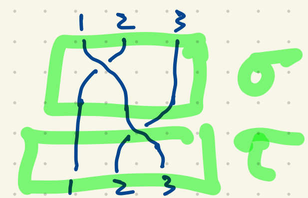
<u>Examples</u>: Fix $c \in \mathbb{R}$ and define $x \triangleright y = cx + (1-c)y$ for $x, y \in \mathbb{R}$. This gives a rack (satisfying (1), (3)). (?) It's a kei if $c = \pm 1$.

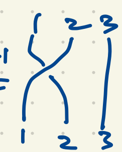
More generally let V be a vector space and $R \in GL(V)$ invertible linear transformation. For $u, v \in V$, $u \triangleright v = Ru + (I - R)v$. This is an Alexander quandle. (Sometimes a kei).

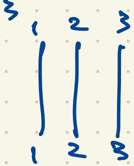
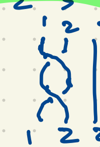
<u>Example</u> Let G be a group (multiplicative). Fix $n \in \mathbb{Z}$. For $a, b \in G$, $a \triangleright b = b^n a b^{-n}$ (n-fold conjugation of a by b). This is a rack, sometimes a quandle.

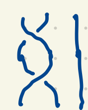
<u>Example</u> The Braid group B_n

eg. in B_3,

$\sigma =$  , $\tau =$  , $\sigma\tau =$ 

$\sigma^{-1} =$ 

$\iota =$  $\neq \sigma^2 =$ 

$\sigma\sigma^{-1} =$  $= \iota$ 

$S_n = \mathrm{Sym}\{1, 2, \cdots, n\}$

$|S_n| = n!$

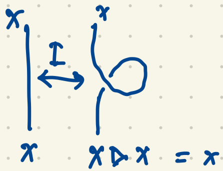
$B_n \twoheadrightarrow S_n$ epimorphism

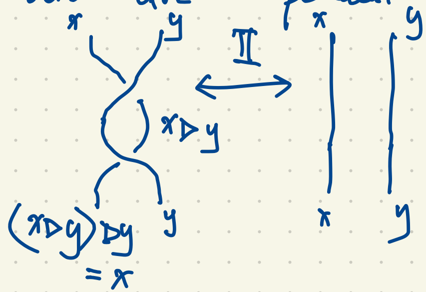
$|B_n| = \aleph_0$.

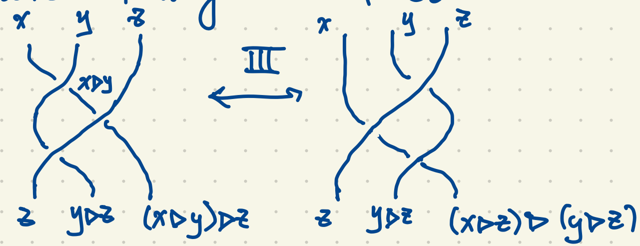
Kei colorings of braids

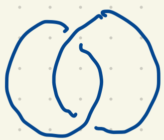
Given a braid $\sigma \in B_n$ and a Kei $(K, \triangleright)$ we color the arcs in a braid diagram of σ (i.e. label the arcs using elements of K) such that

This is the same as requiring that if we label the tops of the n strands, the labels on the bottom are independent of the choice of diagram used for the braid σ.









A right shelf satisfies right-distributivity $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

.. left .. .. left - . . . $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$

~~If~~ $(K, \triangleright)$ is left-distributive ~~then~~ $\Longleftrightarrow$ $(K, \triangleleft)$ is right-distributive where

$x \triangleleft y = y \triangleright x$ (transpose the "multiplication table")

Switch to studying left shelves. Example found by Richard Laver (set theorist in Boulder)

$A_n = \{1, 2, 3, \dots, N = 2^n\}$ (integers mod N) Note: 0 is written as N mod N.

Theorem There is a unique left shelf on A_n satisfying $a \triangleright 1 = a + 1$. for all $a \in A_n$.

Eg. $n = 2$, $N = 4$, $A = \{1, 2, 3, 4\}$ = integers mod 4

$\triangleright$	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

$4 \triangleright 2 = 4 \triangleright (1 \triangleright 1) = (4 \triangleright 1) \triangleright (4 \triangleright 1) = 1 \triangleright 1 = 2$

$4 \triangleright 3 = 4 \triangleright (2 \triangleright 1) = (4 \triangleright 2) \triangleright (4 \triangleright 1) = 2 \triangleright 1 = 3$

$4 \triangleright 4 = 4 \triangleright (3 \triangleright 1) = (4 \triangleright 3) \triangleright (4 \triangleright 1) = 3 \triangleright 1 = 4$

$3 \triangleright 2 = 3 \triangleright (1 \triangleright 1) = (3 \triangleright 1) \triangleright (3 \triangleright 1) = 4 \triangleright 4 = 4$

$2 \triangleright 2 = 2 \triangleright (1 \triangleright 1) = (2 \triangleright 1) \triangleright (2 \triangleright 1) = 3 \triangleright 3 = 4$

$2 \triangleright 3 = 2 \triangleright (2 \triangleright 1) = (2 \triangleright 2) \triangleright (2 \triangleright 1) = 4 \triangleright 3 = 3$

$2 \triangleright 4 = 2 \triangleright (3 \triangleright 1) = (2 \triangleright 3) \triangleright (2 \triangleright 1) = 3 \triangleright 3 = 4$

$1 \triangleright 2 = 1 \triangleright (1 \triangleright 1) = (1 \triangleright 1) \triangleright (1 \triangleright 1) = 2 \triangleright 2 = 4$

$1 \triangleright 3 = 1 \triangleright (2 \triangleright 1) = (1 \triangleright 2) \triangleright (1 \triangleright 1) = 4 \triangleright 2 = 2$

Fact: The left-distributive law holds in all cases although we haven't checked this here.

A_0	1
1	1

A_1	1	2
1	2	2
2	1	2

A_2	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

Figure 2: Multiplication tables for the first four Laver tables

<u>Conjecture</u> As $n \to \infty$ $(N \to \infty)$ the period of the first row of the table $\to \infty$.

The conjecture holds if there exist a Laver cardinal (a certain kind of large cardinal). No one knows how to prove this in ZFC.

We have an inverse system of left shelves

$$\cdots \longrightarrow A_4 \longrightarrow A_3 \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0$$

Let X be any set and let $M = \{$ injective maps $X \to X \}$.
(1-to-1)

Then M is a monoid under composition. (A group iff X is finite).

Let A be a set of sentences over some language L, and let $M, N \models A$. (models of A i.e. L-structures which satisfy all the sentences in A)

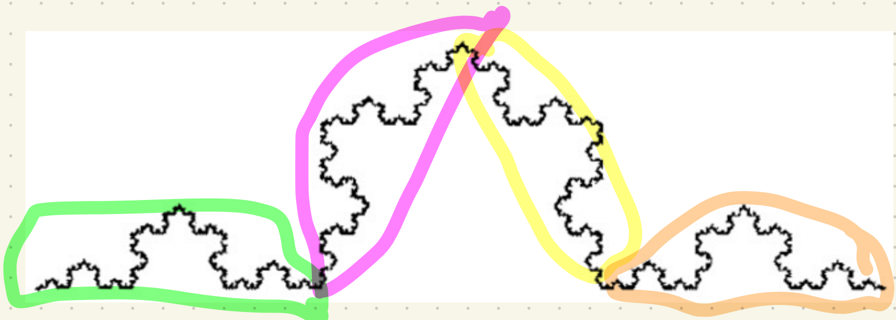
eg. A: axioms for a ring

L: $+, -, \times$

$\mathbb{Z}, \mathbb{Q} \models A$ and $\mathbb{Z}$ is a submodel of $\mathbb{Q}$ (there is a 1-to-1 map $\mathbb{Z} \xrightarrow{\iota} \mathbb{Q}$ preserving the operations. But $\mathbb{Z}$ is not elementarily embedded in $\mathbb{Q}$ because
(elementary embedded)
there are sentences ϕ over L
such that $\mathbb{Z} \models \phi$, $\mathbb{Q} \models \neg\phi$ (or the other way around) e.g.

eg. $\phi : (\exists x)(\forall y)(\neg(y+y=x))$.

We say $\iota : M \to N$ $(M, N \models A)$ is an elementary embedding if ι is injective and for every sentence ϕ, $\underbrace{\iota(M) \subseteq N}_{\text{submodel}}$ where $\iota(M)$ is elementarily equivalent to N:
For all ϕ, $\iota(M) \models \phi$ iff $N \models \phi$.



A portion of the Koch snowflake curve illustrating self-similarity.

There are many embeddings of $\mathbb{C}$ in itself. Pick such an embedding $\iota:\mathbb{C}\to\mathbb{C}$
$\mathbb{C},\ \iota(\mathbb{C})\subset\mathbb{C}$ are models of the field axioms A. $\iota(\mathbb{C})$ is an elementary submodel of $\mathbb{C}$ i.e. $\iota:\mathbb{C}\to\mathbb{C}$ is an elementary embedding i.e. $\mathbb{C}$ is an elementary extension of $\iota(\mathbb{C})$.

Note: $\iota:\mathbb{C}\to\mathbb{C}$ preserves $0, 1, +, \times, -$ but not the topology.

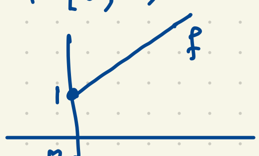
For models of ZFC $(L : \in)$ a Laver cardinal is a (inaccessible) cardinal κ such that the V_κ admits an elementary embedding $\iota: V_\kappa \longrightarrow V_\kappa$ which is not surjective.
This ι generates a left shelf under the following:

If $f, g: X\to X$ are injective then $f \triangleright g : X \to X$ is

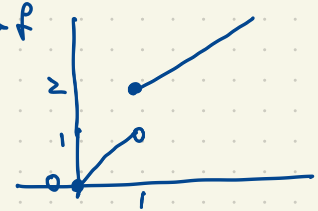
$$(f\triangleright g)(x) = \begin{cases} fgf^{-1}(x) & , \text{ if } x\in f(X) \\ x & , \text{ if } x\notin f(X) \end{cases}$$

$$f(X) = \{f(x) : x\in X\} \subset X$$

eg. $f:[0,\infty)\longrightarrow[0,\infty)$, $x\longmapsto x+1$



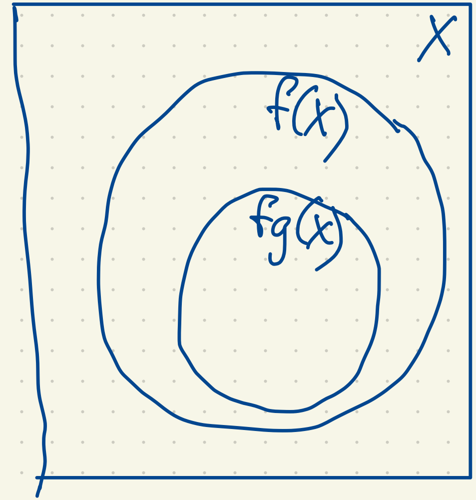
$f\triangleright f$



why is $\triangleright$ a left shelf?

$((f \triangleright g) \triangleright (f \triangleright h))(x)$

$= (f \triangleright (g \triangleright h))(x)$ Check three cases



If $x \in fg(X)$ then $x = fg(y)$ so

$(g \triangleright h)(x) =$

$\iota : V_\kappa \longrightarrow V_\kappa$ is an elementary embedding but not surjective.

It generates a left shelf under "$\triangleright$". This is the free shelf on one generator $\mathcal{F}_1$

$\mathcal{F}_1 = \{ \iota, \iota \triangleright \iota, (\iota \triangleright \iota) \triangleright \iota, \iota \triangleright (\iota \triangleright \iota), \dots \}$ These combinations of ι under $\triangleright$

are distinct except when required by the left shelf axiom e.g. $(\iota \triangleright \iota) \triangleright (\iota \triangleright \iota) = \iota \triangleright (\iota \triangleright \iota)$

$\mathcal{F}_1$ is a countably infinite left shelf; moreover $\mathcal{F}_1 = \varprojlim A_n$

Let X be an infinite set. A <u>filter</u> on X is a collection $\mathcal{F}$ of subsets of X such that

(i) $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$ (Sets in $\mathcal{F}$ are <u>large</u> subsets of X.)

(ii) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$ then $B \in \mathcal{F}$.

(iii) If $A, A' \in \mathcal{F}$ then $A \cap A' \in \mathcal{F}$.

By Zorn's lemma, every $\mathcal{F}$ filter extends to an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$ on X which is a filter satisfying

(iv) For all $A \subseteq X$, either A or $X \smallsetminus A$ is in $\mathcal{U}$.

$\mathcal{U}$ gives a two-valued finitely additive probability measure on X.

To get a nonprincipal ultrafilter on X, we start with the Fréchet filter consisting of all cofinite subsets of X (complements of finite subsets of X) and take $\mathcal{U} \supseteq \mathcal{F}$ a maximal filter containing $\mathcal{F}$. $\mathcal{U}$ is nonprincipal: $\mathcal{U}$ contains no finite sets.

We take $\mathcal{U}$ to be a nonprincipal ultrafilter on $\omega = \{0,1,2,3,\dots\}$ and consider the ring $\mathbb{R}^\omega = \{(a_0, a_1, a_2, a_3, \dots) : a_i \in \mathbb{R}\}$ with coordinatewise operations. $\mathbb{R}^\omega$ is a commutative ring with identity, not a field; eg. $(1,0,1,0,\dots)(0,1,0,1,\dots) = (0,0,0,0,\dots) = 0 \in \mathbb{R}^\omega$.

<u>Now</u> identify two sequences $a = (a_0, a_1, a_2, \dots)$, $b = (b_0, b_1, b_2, \dots)$ if they agree almost everywhere with respect to $\mathcal{U}$ i.e. if $\{i \in \omega : a_i = b_i\} \in \mathcal{U}$.

In the case $a = (1,0,1,0,1,0,\dots)$ we have $a_i = 0$ whenever $i \in \{1,3,5,7,\dots\}$; $b_i = 0$ whenever $i \in \{0,2,4,6,\dots\}$
$b = (0,1,0,1,0,1,\dots)$

If $\{1,3,5,7,\dots\} \in \mathcal{U}$ then $a \sim (0,0,0,0,0,\dots)$ and $b \sim (1,1,1,1,1,\dots)$.

If $\{0,2,4,6,\dots\} \in \mathcal{U}$ then $a \sim (1,1,1,1,1,\dots)$ and $b \sim (0,0,0,0,0,\dots)$.

Identify two sequences in $\mathbb{R}^\omega$ whenever they agree almost everywhere w.r.t. $\mathcal{U}$. Then we get a quotient ring $\mathbb{R}^\omega/\sim \; = {}^*\mathbb{R}$ denoted $\hat{\mathbb{R}}$ in the handout.
This is the field of nonstandard reals or hyperreals.
${}^*\mathbb{R}$ has the same first order theory (an ordered field and it's a real closed field, e.g. every poly $f(x) \in {}^*\mathbb{R}[x]$ of odd degree has a root in ${}^*\mathbb{R}$). In fact we have an elementary embedding of $\mathbb{R}$ in ${}^*\mathbb{R}$. The main difference between $\mathbb{R}$ and ${}^*\mathbb{R}$ is that $\mathbb{R}$ has no infinite or infinitesmal elements but ${}^*\mathbb{R}$ does.

The Archimedean property says that if $a>0$ then $\underbrace{a+a+a+\cdots+a}_{n} = na > 1$ for some n.

$(\forall a)(a>0 \rightarrow (a+a>1 \;\vee\; a+a+a>1 \;\vee\; a+a+a+a>1 \;\vee\; \ldots))$

This property is not expressible in the first order theory of fields.
$\mathbb{R}$ satisfies this property, ${}^*\mathbb{R}$ does not.

Eg. $\varepsilon = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots) \in \mathbb{R}^\omega$, up to equivalence mod $\mathcal{U}$, defines an infinitesmal in ${}^*\mathbb{R}$.

$n\varepsilon = (n, \frac{n}{2}, \frac{n}{3}, \frac{n}{4}, \cdots) \in \mathbb{R}^\omega$, $n\varepsilon < 1$ since this holds for all but the first n terms of the sequence.

$\frac{1}{\varepsilon} = (1, 2, 3, 4, 5, \cdots) \in \mathbb{R}^\omega$ defines an infinite element of ${}^*\mathbb{R}$.

Every structure M has a enlargement ${}^*M$.

<u>Łoś' Theorem</u> If $M_0, M_1, M_2, \ldots \models \mathcal{A}$ (statements over a first-order language over L) then the ultraproduct

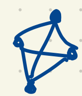
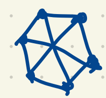
$$\left(\prod_{i\in\omega} M_i\right)/\mathcal{U} \models \mathcal{A}.$$

Eg. $\mathcal{A}$ = axioms for fields, $M_i = \mathbb{R}$ for all i. $\prod_{i\in\omega} M_i = \{(m_0, m_1, m_2, \ldots) : m_i \in M_i\}$.

Eg. L = language of a single binary relation '$\sim$'

$\mathcal{A}$ = axioms for ordinary graphs of degree 3

A model of $\mathcal{A}$, $\Gamma \models \mathcal{A}$, is an ordinary graph of degree 3.

For each $i \in \omega$, take $\Gamma_i \models \mathcal{A}$ eg. $\Gamma_0 =$ , $\Gamma_1 =$ , $\Gamma_2, \Gamma_3, \ldots$

$$\prod_{i\in\omega} \Gamma_i = \Gamma_0 \times \Gamma_1 \times \Gamma_2 \times \cdots = \{(v_0, v_1, v_2, v_3, \ldots) : v_i \in \Gamma_i\}$$

$\mathcal{U}$ a nonprincipal ultrafilter on ω — i.e. v_i is a vertex in Γ_i.

Now $\left(\prod_{i\in\omega} \Gamma_i\right)/\mathcal{U}$ is the set of equiv. classes of sequences $v = (v_0, v_1, v_2, \ldots)$.

If $v, w \in \underbrace{\left(\prod_{i\in\omega} \Gamma_i\right)/\mathcal{U}}_{\Gamma =}$ then $v \sim w$ iff $v_i \sim w_i$ for almost all i i.e. $\{i \in \omega : v_i \sim w_i\} \in \mathcal{U}$.

This graph Γ has degree 3. If Γ_i has order $\leq n$ for some n then Γ is a graph of order $\leq n$