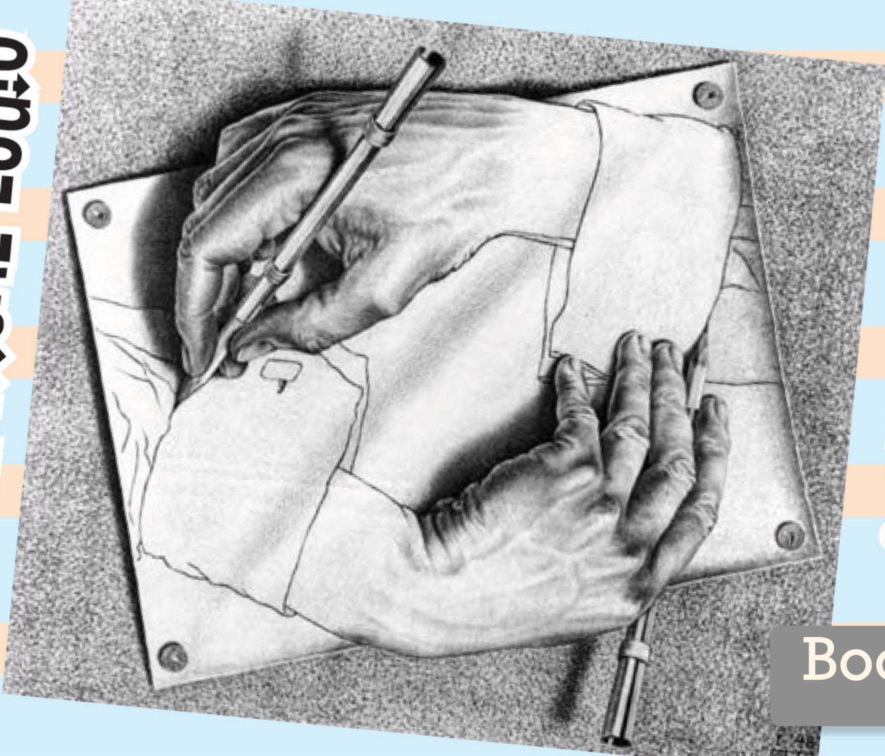
MATHEMATICAL LOGIC & SET THEORY



Book 1

Group Theory: an example of a first-order axiomatic system

An informal proof in group theory

<u>Theorem</u> If G is a (multiplicative) group of exponent 2, then G is abelian.

(G has exponent n if $g^n = 1$ for all $g \in G$.)

(Informal) <u>proof</u>: Let $a, b \in G$. Since $abab = (ab)^2 = 1$, multiplying on the left by "a" and on the right by "b" gives $aababb = a1b$, i.e. $ba = ab$. $\square$

Start with names for variables $x, y, z, \ldots$ (symbols)
Special symbols for first order logic: $\exists$, $\forall$, parentheses, $\wedge$, $\vee$, $\neg$, $\rightarrow$, ...
Symbols for constants: $1, \ldots$
Symbols for functions: $*, \ldots$ $x*y$ means $\mu(x,y)$
Symbols for relations: $=$

Axioms of Group Theory:

i.e. $\mu(\mu(x,y),z) = \mu(x,\mu(y,z))$

ID: $(\forall x)((x*1 = x) \wedge (1*x = x))$

ASSOC: $(\forall x)(\forall y)(\forall z)\ ((x*y)*z = x*(y*z))$

INV: $(\forall x)(\exists y)((x*y = 1) \wedge (y*x = 1))$

We happen to know some groups including C_n (cyclic group of order n), S_n (symmetric group of degree n), ...

GROUPS $= \{$ID, ASSOC, INV$\} = \{(\forall x)((x*1) = \ldots, \ldots, \ldots\}$ (the set consisting of our three axioms of group theory)

S_5 is a group, i.e. $S_5 \models$ GROUPS (S_5 is a <u>model</u> of GROUPS)

ABEL: $(\forall x)(\forall y)\ (x*y = y*x)$

ABEL-GPS $=$ GROUPS $\cup$ $\{$ABEL$\}$. S_5 is a non-abelian group; $S_5 \not\models$ ABEL; $S_5 \not\models$ ABEL-GPS.

A structure has an underlying set of elements, together with an interpretation of all the symbols for constants, functions, and relations.

How do we rewrite our informal proof (above) as a formal proof in first order logic?

Σ = GROUPS $\cup$ {EXP2} where EXP2 : $(\forall x)(x * x = 1)$

ABEL is a theorem in the theory of groups of exponent 2, i.e. $\Sigma \vdash$ ABEL.

A <u>theorem</u> is a sequence of steps

$$\begin{aligned} &\Sigma \vdash \boxed{} \\ &\Sigma \vdash \boxed{} \\ &\Sigma \vdash \boxed{} \\ &\quad \vdots \\ &\Sigma \vdash \boxed{} \end{aligned}$$

in which every step follows from previous steps by a statement in Σ, or an axiom of first order logic, or a rule of inference.

This is a formal (symbolic) proof!

An outline of a formal proof:

$\Sigma \vdash$ EXP2 since EXP2 $\in \Sigma$

$\Sigma \vdash$ (EXP2 $\rightarrow (\forall a)(a * a = 1))$ (A4) p.86

$\Sigma \vdash (\forall a)(a * a = 1)$ Modus Ponens (R1) p.86

$\vdots$

$\Sigma \vdash (\forall b)(b * b = 1)$

$\vdots$

$\Sigma \vdash (\forall a)(\forall b)((a*b) * (a*b) = 1)$

$\vdots$

$\Sigma \vdash (\forall a)(\forall b)\ ((a * ((a*b) * (a*b)) = a * 1)$

$\vdots$

$\Sigma \vdash (\forall a)(\forall b)(a*b = b*a)$

RICHARDS BORCHERDS

JOEL DAVID HAMKINS

$$\text{ORD3}: \underbrace{(\exists x)(\exists y)(\exists z)\big[(\forall g)((g=x) \vee (g=y) \vee (g=z))}_{\text{"there are at most three elements"}} \wedge \underbrace{(\overbrace{\neg(x=y)}^{x \neq y}) \wedge (\overbrace{\neg(x=z)}^{x \neq z}) \wedge (\overbrace{\neg(y=z)}^{y \neq z})}_{\text{"there are at least 3 elements"}}\big]$$

ABEL is independent of GROUPS (you cannot either prove or disprove that a general group is abelian). GROUPS $\nvdash$ ABEL and GROUPS $\nvdash$ $\neg$ABEL. This is because $C_3 \models$ GROUPS, $C_3 \models$ ABEL but $S_3 \models$ GROUPS, $S_3 \not\models$ ABEL.

In an arbitrary first-order theory, with axioms Σ, a statement θ is independent of Σ if $\Sigma \nvdash \theta$ and $\Sigma \nvdash \neg\theta$:

Soundness Theorem: If $\Sigma \vdash \theta$ then θ holds in every model of Σ i.e. $M \models \theta$ whenever $M \models \Sigma$.

Completeness Theorem: Converse holds: If θ holds in every model of Σ, then it is provable from Σ i.e. if $M \models \theta$ whenever $M \models \Sigma$, then $\Sigma \vdash \theta$.

Assume Σ is consistent

So: θ is independent of Σ iff there are models of Σ in which θ holds, and models of θ in which θ fails.

Σ is consistent if we cannot prove a contradiction from Σ, ie. $\Sigma \nvdash (\theta \wedge \neg\theta)$ for some θ.

Equivalently, Σ is consistent iff it has a model.

Eg. ABEL is independent of GROUPS.

ORD3 - - - - - - - .

GROUPS is consistent.

GROUPS $\cup$ {ORD3} is consistent since it has a model. In fact it has a unique model up to isomorphism: the cyclic group C_3 of order 3. The group C_3 (or its theory) is <u>categorical</u>.

GROUPS is not categorial. (There are models, but not a unique model.)

An alternative to INV: $(\forall x)(\exists y)((x*y=1) \wedge (y*x=1))$ is to add a function symbol $\iota(\cdot)$ to the language namely $(\forall x)((x*\iota(x)=1) \wedge (\iota(x)*x=1))$

We already have a binary function symbol $\mu(\cdot,\cdot)$, $\mu(x,y) = x*y$

A <u>theorem</u> of Σ is a statement that can be proved from Σ. A proof is a sequence of statements such....

The <u>theory</u> of Σ is $Th(\Sigma) = \{\text{statements provable from } \Sigma\} = \{\text{theorems of } \Sigma\}$.

First order theory of graphs has no symbols for constants or functions; there is only one relation symbol $R(\cdot,\cdot)$, for the binary relation of adjacency. We will abbreviate $R(x,y)$ as $x \sim y$.

Axioms of graph theory: two axioms to indicate that our relation is symmetric and irreflexive.

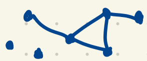
IRREFL: $(\forall x)(\neg(x \sim x))$

SYM: $(\forall x)(\forall y)((x \sim y) \rightarrow (y \sim x))$

GRAPHS = {IRREFL, SYM}

MIN7: $(\exists x_1)(\exists x_2)\cdots(\exists x_7)((\neg(x_1 = x_2)) \wedge \cdots \wedge (\neg(x_6 = x_7)))$

"there are at least 7 vertices"

 $\models$ GRAPHS

 $\not\models$ GRAPHS

MAX7: $(\exists x_1)(\exists x_2)\cdots(\exists x_7)(\forall y)((y = x_1) \vee \cdots \vee (y = x_7))$

"There are at most 7 vertices"

To say that Γ has exactly 7 vertices, we could write

ORD7: $(\exists x_1)(\exists x_2)\cdots(\exists x_7)\big[((\neg(x_1 = x_2)) \wedge \cdots \wedge (\neg(x_6 = x_7))) \wedge (\forall y)((y = x_1) \vee (y = x_2) \vee \cdots \vee (y = x_7))\big]$

GRAPHS $\cup$ {ORD7} : axioms for graphs with exactly 7 vertices.

Axioms for infinite graphs:

GRAPHS $\cup$ {MIN1, MIN2, MIN3, MIN4, ...}

In first order graph theory, we cannot express the condition that a graph is finite.

We can express the condition that a graph has at most 17 vertices.

- - - - - - - - - - - - is infinite.

We cannot express the condition that a graph is countably infinite.

The *diameter* of a graph is the max. distance between two vertices.

The *distance* between two vertices is the length of the shortest path between them.

eg. To say that a graph has diameter at most 2 in first order logic:

$$(\forall x)(\forall y)(\neg(x=y)) \rightarrow (\underbrace{(x \sim y)}_{dist(x,y)=1} \vee \underbrace{(\exists z)((x \sim z) \wedge (z \sim y))}_{dist(x,y) \leq 2})$$

Diameter 2:

(diameter at most 2) $\wedge (\exists x)(\exists y)((\neg(x \sim y)) \wedge \neg(x=y))$

In first order theory, we can express the condition that a graph has diameter 7 or diameter at most 7 but we cannot express the notion that a graph is connected.

Graphs of diameter ≤ 1 (i.e. cliques): $\text{GRAPHS} \cup \{(\forall x)(\forall y)((x=y) \vee (x \sim y))\} = \text{COMPL\_GRPHS}$

has models K_0, K_1, K_2, K_3, K_4, ...

For each cardinality κ (eg. $\kappa = 0, 5, \aleph_0, 2^{\aleph_0}, \ldots$) there is a model $K_\kappa \models \text{COMPL\_GRPHS}$

countably infinite $\quad |\mathbb{R}| = \text{continuum}$

and any two models of the same cardinality are isomorphic.

$\text{COMPL\_GRPHS} \cup \{\text{ORD4}\}$ has a unique model $K_4 =$  up to isomorphism.

$Th(K_4) = \{$ all statements in graph theory that hold in $K_4 \}$

K_4 (or $Th(K_4)$) is categorical: K_4 is the unique model (up to isomorphism) of

$\text{COMPL\_GRPHS} \cup \{\text{ORD4}\}$ or of $Th(K_4)$

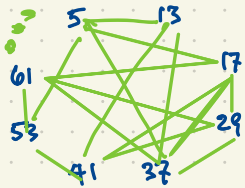
$\text{COMPL\_GRAPHS} \cup \{\underbrace{\text{MIN1}, \text{MIN2}, \ldots}_{\text{"there are inf. many vertices"}}\}$ has infinitely many models. But for each cardinality κ, there is only one model (up to isomorphism) of cardinality κ.

This theory is not categorical but it is κ-categorical.

Consider the graph with countably infinite vertex set $\{5, 13, 17, 29, 37, 41, 53, 61, \ldots\}$ (all primes $\equiv 1 \bmod 4$).

We say $p \sim q$ if p is a nonsquare mod q (iff q is a nonsquare mod p, by Quadratic Reciprocity).

eg. $5 \sim 13$ (0, 1, 4 are squares mod 5 but 2, 3 are nonsquares mod 5).

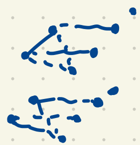


Let's call this graph $R \models \text{GRAPHS} \cup \{\text{INF}\} \cup \{\gamma_{m,n} : m,n \in \mathbb{N}\}$

Quadratic Reciprocity
Dirichlet's Theorem
Chinese Remainder Theorem

$$\gamma_{m,n}: (\forall x_1)(\forall x_2)\cdots(\forall x_m)(\forall y_1)(\forall y_2)\cdots(\forall y_n)\big((x_i, y_j \text{ distinct}) \to (\exists z)(z \sim x_1 \wedge \cdots \wedge z \sim x_m \wedge z \not\sim y_1 \wedge \cdots \wedge z \not\sim y_n)\big)$$

$$x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \cdots \wedge x_1 \neq y_1 \wedge \cdots \wedge y_{n-1} \neq y_n$$



$\gamma_{2,0}$, $\gamma_{1,1}$, $\gamma_{0,2}$

R = Random graph = Erdős-Rényi graph = Rado graph = Universal Graph

Take any countably infinite set V as vertices.
For all $x \neq y$ in V, flip a coin. Heads? join $x \sim y$. Tails? $x \not\sim y$ (unjoined).
With probability 1, $R \models \gamma_{m,n}$ for all m,n; even if the coin is biased.

<u>Theorem</u> Every countably infinite graph satisfying $\gamma_{m,n}$ for all m,n is isomorphic to R.

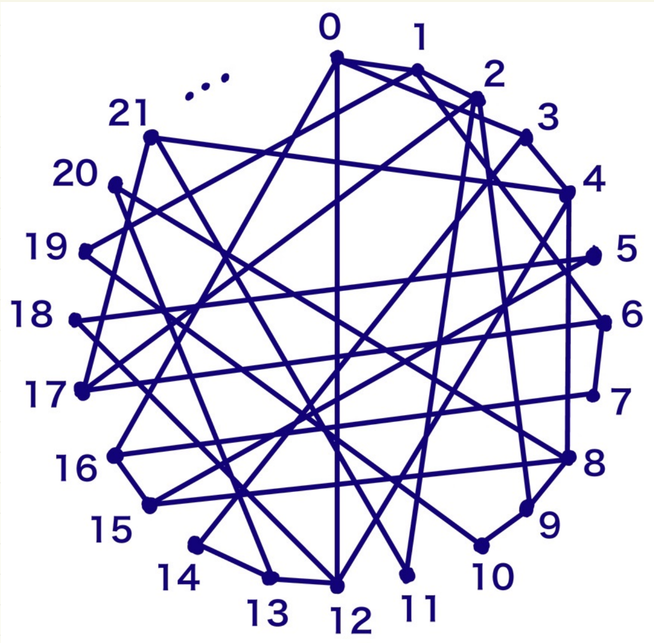
$\text{GRAPHS} \cup \{\text{INF}\} \cup \{\gamma_{m,n} : m,n \in \mathbb{N}\}$ has only one countable model. (up to isomorphism).

don't need this axiom; it follows from $\{\gamma_{m,n} : m,n \in \mathbb{N}\}$

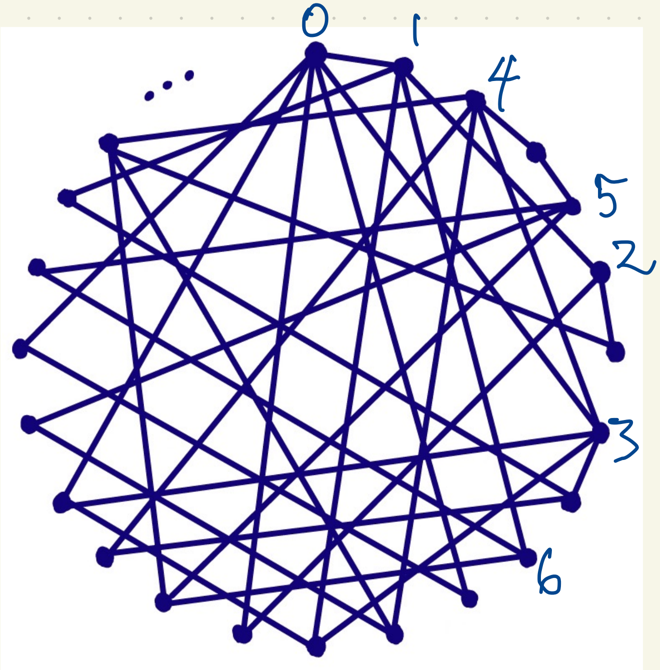
i.e. R is $\aleph_0$-categorical (countably categorical).

RANDOM :=

Proof First try, via greedy construction of a map $\Gamma \to \Gamma'$.
Suppose $\Gamma, \Gamma' \models$ RANDOM and Γ, Γ' have a countably infinite set of vertices.

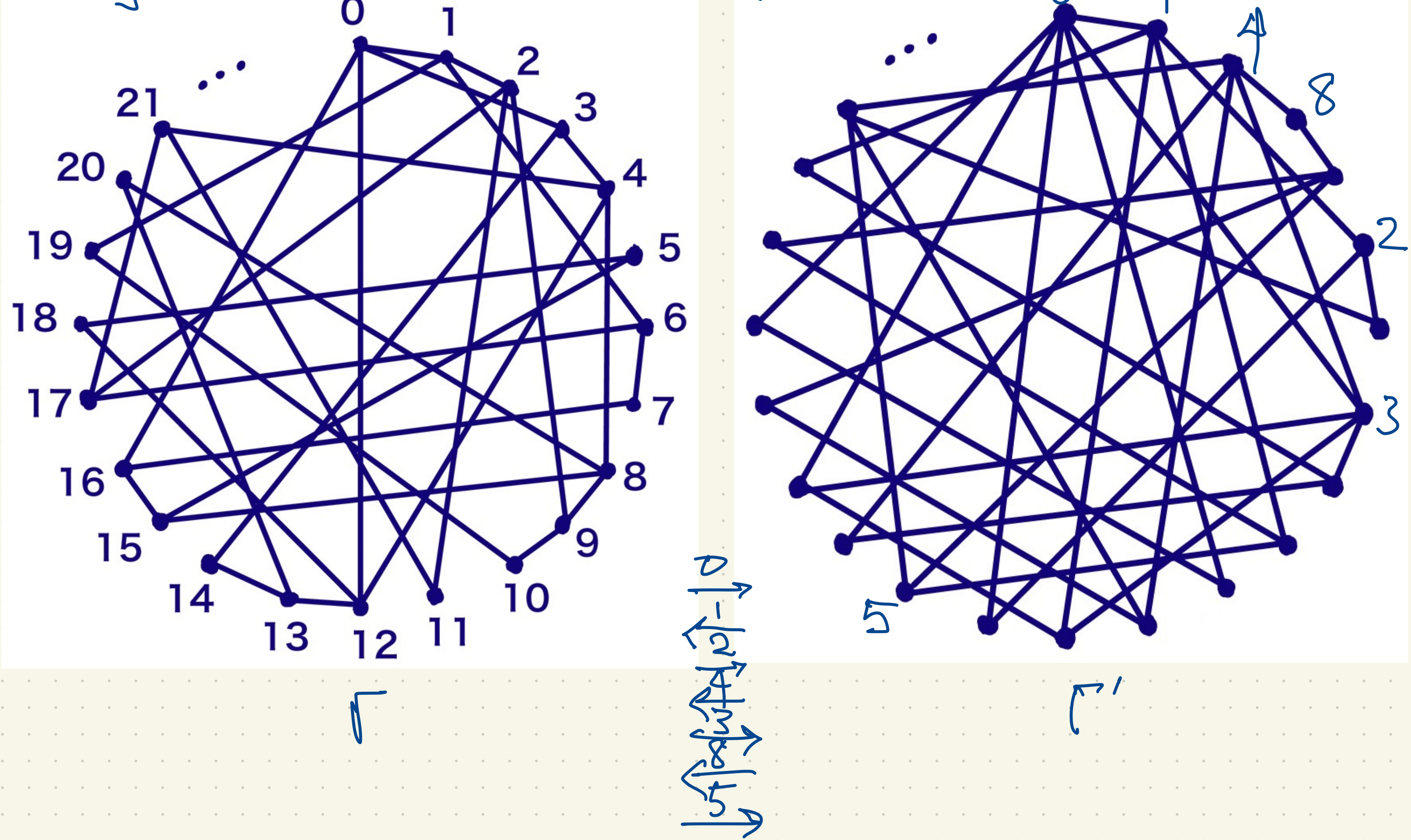


Γ



Γ'

Second try: Back-and-forth construction of isomorphism $\Gamma \to \Gamma'$

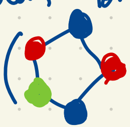


Question: Is there a universal random graph on $|\mathbb{R}| = 2^{\aleph_0}$ vertices?

Status of this problem is not fully known, but independent of ZFC, depends on CH; (Shelah)

Chromatic numbers of graphs:

Given a graph Γ, a proper (vertex) coloring of Γ is a coloring of the vertices so that no two vertices of the same color are joined. The chromatic number of Γ, $\chi(\Gamma)$, is the smallest number of colors for which Γ has a proper coloring. Eg.

$\chi\Big($  $\Big) = 3.$

<u>Theorem</u> (Appel-Haken) If Γ is a finite planar graph, then $\chi(\Gamma) \le 4$.

From this result, the generalization to infinite planar graphs holds:

If Γ is any planar graph, then $\chi(\Gamma) \le 4$.

First express the condition $\chi(\Gamma) \le k$ in first order logic:

Language in any first-order system has symbols for constants, r-ary functions, r-ary relations.

We are given a graph Γ and a positive integer k.

Introduce constants $v_1, v_2, \cdots$, one for each vertex of the graph. Also k many relations $C_1(\cdot), \cdots, C_k(\cdot)$.

Axioms: $(\forall x)\big((C_1(x) \vee C_2(x) \vee \cdots \vee C_k(x)) \wedge \neg((C_1(x)\wedge C_2(x)) \vee (C_1(x)\wedge C_3(x)) \vee \cdots \vee (C_{k-1}(x)\wedge C_k(x)))\big)$

For every pair of adjacent vertices i, j in Γ, and each l in $\{1,2,\dots,k\}$, include an axiom $\neg(C_l(v_i) \wedge C_l(v_j))$.

Let $\Sigma_{\Gamma,k}$ be the set of axioms listed here. A model of $\Sigma_{\Gamma,k}$, i.e. $M \models \Sigma_{\Gamma,k}$, is a proper k-coloring of Γ. Such a model exists iff $\chi(\Gamma) \le k$.

By the compactness theorem, $\Sigma_{\Gamma,k}$ has a model iff every finite subset of $\Sigma_{\Gamma,k}$ has a model i.e. iff every finite subgraph of Γ has chromatic number $\leq k$.

More generally, if Γ is any infinite graph, then $\chi(\Gamma) = k$ iff every finite subgraph $\Gamma_0 \subseteq \Gamma$ has $\chi(\Gamma_0) \leq k$; and $\chi(\Gamma_0) = k$ for some finite $\Gamma_0 \subseteq \Gamma$.

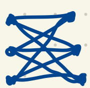
By the way, the compactness theorem follows easily from the completeness theorem. We won't prove the completeness theorem. Here's the argument in the case of graph coloring:

If $\Sigma_{\Gamma,k}$ has a model $M \models \Sigma_{\Gamma,k}$, then every finite subset $\Sigma_0 \subseteq \Sigma_{\Gamma,k}$ has a model $M \models \Sigma_0$.

Conversely, suppose every finite subset $\Sigma_0 \subseteq \Sigma_{\Gamma,k}$ has a model ("every finite subgraph $\Gamma_0 \subseteq \Gamma$ is properly k-colorable). Suppose $\Sigma_{\Gamma,k}$ does not have a model (Γ is not properly k-colorable). This says $\Sigma_{\Gamma,k}$ is inconsistent and we can derive a contradiction from $\Sigma_{\Gamma,k}$ by the completeness theorem i.e. $\Sigma_{\Gamma,k} \vdash (\theta \wedge (\neg\theta))$ for some θ. A proof of $\theta \wedge (\neg\theta)$ from $\Sigma_{\Gamma,k}$ only uses finitely many of our constants v_i, and relations C_j. These v_i's lie in a finite subgraph $\Gamma_0 \subseteq \Gamma$. This is a contradiction.

K_5

K_4



$K_{3,3}$

is not planar: it has K_5 as a minor.

Axioms for linear (total) order:

Language: single binary relation symbol $R(\cdot,\cdot)$. We denote $R(x,y)$ by $x<y$.

Axioms for linear order:

$(\forall x)(\forall y)\big((x=y) \vee (x<y) \vee (y<x)\big)$

$(\forall x)(\forall y)\big(\neg(x=y) \longleftrightarrow ((x<y) \vee (y<x))\big)$

$(\forall x)(\forall y)\big(\neg((x<y) \wedge (y<x))\big)$

$(\forall x)(\forall y)(\forall z)\big(((x<y) \wedge (y<z)) \rightarrow (x<z)\big)$

Nonempty axiom: $(\exists x)(x=x)$

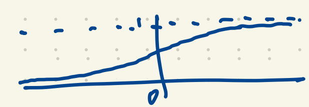
Dense linear order without endpoints:

axioms for linear order

$(\forall x)(\forall y)\big((x<y) \rightarrow (\exists z)((x<z) \wedge (z<y))\big)$

$(\forall x)(\exists y)(x<y)$

$(\forall x)(\exists y)(y<x)$



Models of "dense linear order without endpoints":

$(0,1) \subset \mathbb{R}$, with usual '<'
$\mathbb{R}$ with usual '<'. } isomorphic

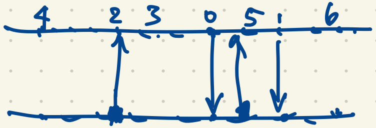
Here are three models, no two of which are isomorphic.

- $\mathbb{Q}$ with ordinary '<'
- $\mathbb{Q} \cup (0,1)$ with ordinary '<'

There are <u>many</u> uncountable models. For every uncountable cardinality κ, there are <u>many</u> models of cardinality κ.

Theorem (Cantor) $(\mathbb{Q}, <) \models$ "dense linear order without endpoints" is the unique countable model up to isomorphism.

Proof



$(\mathbb{Q}, <)$

Back-and-forth.

The theory of dense linear orders without endpoints is $\aleph_0$-categorical (countably categorical, ω-categorical).

Other $\aleph_0$-categorical theories: the theory of the complete graph; the theory of the random graph.

"Dense linear order without endpoints" is not κ-categorical for any uncountable cardinality.

Theorem (Morley) If a theory is κ-categorical for some uncountable cardinality κ, then it is κ-categorical for all uncountable κ (hence "uncountably categorial").

Linear Algebra: what are suitable axioms for vector spaces? Fix a field F and write down axioms for vector spaces over F. One way: All elements of the domain (underlying set of the model) are vectors. Implement scalar multiplication using unary functions $\mu_c(\cdot)$ $c \in F$ in addition to binary function "+" for adding vectors. We'll also use a constant symbol '0.

Axioms: $(\forall v)(\forall w)(v + w = w + v)$

$(\forall u)(\forall v)(\forall w)((u+v)+w = u + (v+w))$

$(\forall v)(0 + v = v)$

Axiom Schema

$(\forall v)(\mu_c(\mu_d(v)) = \mu_{cd}(v))$ (for every $c, d \in F$ we have such an axiom)

$(\forall v)(\mu_c(v) + \mu_d(v) = \mu_{c+d}(v))$

$(\forall v)(\forall w)(\mu_c(v+w) = \mu_c(v) + \mu_c(w))$

$(\forall v)(\mu_1(v) = v)$
$(\forall v)(\mu_0(v) = 0)$

Models of this theory are vector spaces over F.
There can be nonisomorphic models of the same cardinality.
Suppose $F = \mathbb{Q}$. For every uncountable κ, there is a unique model of cardinality κ up to isomorphism (the theory of rational vector spaces is uncountably categorical) but not for $\kappa = \aleph_0$: there are infinitely many vector spaces of cardinality $\aleph_0$.

$$\mathbb{Q}^1, \mathbb{Q}^2, \mathbb{Q}^3, \mathbb{Q}^4, \ldots, \quad \mathbb{Q}^\omega = \bigcup_{n=1}^{\infty} \mathbb{Q}^n$$

Think of $\mathbb{Q}^\omega = \mathbb{Q}[t]$ as the vector space of all polynomials in t with rational coefficients;

$\mathbb{Q}^n = \{ f(t) \in \mathbb{Q}[t] : \deg f(t) < n \}$ has basis $\{1, t, t^2, t^3, \ldots, t^{n-1}\}$

$\mathbb{Q}^\omega = \mathbb{Q}[t]$ has basis $\{1, t, t^2, t^3, \ldots\}$

Don't confuse with $\mathbb{Q}[[t]]$ = all power series in t with rational coefficients which has uncountable dimension.

Theorem of Engeler, Ryll-Nardzewski, Svenonius: $\aleph_0$-categoricity of M is equivalent to a "large" group of automorphisms of M: $\mathrm{Aut}\, M$ has only finitely many orbits on k-tuples of "points" (elements of M). Such a group is called <u>oligomorphic</u>. <u>Examples</u>

Any two k-sets of distinct rationals are in the same orbit of $\mathrm{Aut}(\mathbb{Q}, <)$.
For the countable random graph R, the number of orbits on k-sets of distinct vertices for $k = 1, 2, 3, 4, \ldots$ gives a sequence

$$1, 2, 4, 11, 34, \ldots$$

Nothing is better than a meal in a four-star restaurant.
A plain cafeteria meal is better than nothing.
Therefore a plain cafeteria meal is better than a meal in a four-star restaurant.

Why would we not allow the domain of a model (the underlying set) to be empty?

$\vdash (\forall x)(\phi(x)) \longrightarrow \phi(a)$

$\vdash \phi(a) \longrightarrow (\exists x)(\phi(x))$

$\vdash (\forall x)(\phi(x)) \rightarrow (\exists x)(\phi(x))$

Clarification: structure vs. model

We start with a language L (symbols for constants, relations, functions).
Any interpretation of the symbols in L on an underlying set (domain) is a structure for L (an L-structure). Given a language L and a set of sentences Σ over L,

a model of Σ is an L-structure M satisfying Σ i.e. every sentence $\theta \in \Sigma$ is satisfied by M; we denote $M \models \Sigma$.

How do we understand the statement that if κ is an uncountable cardinal, then a rational vector space of dimension κ is the same thing as a rational vector space of cardinality κ?

$|\mathbb{Q}| = \aleph_0 = |\mathbb{Z}| = |\mathbb{N}|$; $|\mathbb{R}| = 2^{\aleph_0} = |\mathcal{P}(\mathbb{N})|$ ($\mathcal{P}(A)$ = power set of A)

$|\mathbb{Q}^n| = |\mathbb{Q}|$ by induction. If $|A| = \aleph_0$ then $|A \times A| = \aleph_0$

$|\mathbb{Q}^\omega| = \aleph_0$ (union of countably many countable sets is countable).

$$\mathbb{Q}^\omega = \bigcup_{n=1}^{\infty} \mathbb{Q}^n$$

For κ, λ infinite cardinals, $\kappa + \lambda = \kappa\lambda = \max\{\kappa, \lambda\}$

i.e. if A, B are infinite sets then $|A \cup B| = |A \sqcup B| = |A \times B| = \max\{|A|, |B|\}$.

This uses AC (axiom of choice). (And its equivalent to AC.)

If κ is an infinite cardinal and V is a rational vector space, then $\dim V = \kappa \Rightarrow |V| = \kappa$.

Proof Let B be a basis of V. Then $|B| \le |V|$

$$V = \underbrace{\bigcup_{n=1}^{\infty} \underbrace{\bigcup_{v_1, \dots, v_n \in B}}\ \mathrm{Span}\{v_1, \dots, v_n\}}_{\kappa \cdot \aleph_0 = \kappa}$$

where $|\mathrm{Span}\{v_1, \dots, v_n\}| = |\{a_1 v_1 + \dots + a_n v_n : a_i \in \mathbb{Q}\}| \le \aleph_0$

The number of choices of $(v_1, \dots, v_n) \in B^n$ is $|B^n| = |\underbrace{B \times B \times \dots \times B}_{n \text{ times}}| = \kappa^n = \kappa$.

$|V| = \aleph_0 \cdot \kappa = \kappa$.

Thus: The theory of rational vector spaces is uncountably categorical but not countably categorical

Fields

Let F be a set containing distinct elements called 0 and 1 (thus $0 \neq 1$). Suppose addition, subtraction, multiplication and division are defined for all elements of F (except division by 0 is not defined).
Thus $a+b,\ a-b,\ ab,\ \frac{a}{d} \in F$ whenever $a, b, d \in F$ and $d \neq 0$.
Define $-a = 0 - a$.

If the following properties are satisfied by *all* elements $a, b, c, d \in F$ with $d \neq 0$, then F is a field.

$$a + b = b + a \qquad a + (b + c) = (a + b) + c \qquad ab = ba$$
$$a + 0 = a \qquad a(bc) = (ab)c \qquad 1a = a$$
$$a + (-a) = 0 \qquad a(b + c) = ab + ac \qquad \frac{a}{d}d = a$$
$$a + (-b) = a - b$$

A field F has characteristic p if $\underbrace{1+1+1+\cdots+1}_{p} = 0$. (This requires p to be prime)
If $1+1+1+1+1+1=0$ then $(1+1+1)(1+1)=0$. If there is no such p then F has characteristic zero.

If we want to force our field to have characteristic 5, say, add an axiom $1+1+1+1+1=0$.
If we want to work only with characteristic zero fields, then add infinitely many axioms

$$1+1\neq 0$$
$$1+1+1\neq 0$$
$$1+1+1+1\neq 0$$
$$1+1+1+1+1\neq 0$$
etc.

The field of complex numbers is the unique (up to isomorphism) algebraically closed field of characteristic zero having cardinality $2^{\aleph_0}$.

The theory of algebraically closed fields of characteristic zero is uncountably categorical.
(Not countably categorical.)

First order axioms for "algebraically closed":
For each $n\geq 1$ we add an axiom of the form

$$(\forall a_0)(\forall a_1)\cdots(\forall a_{n-1})(\exists z)\left(z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0=0\right)$$

$$\mathbb{C}\subset \mathbb{C}(x)\subset K = \text{alg. closure of } \mathbb{C}(x) \cong \mathbb{C}$$

$$|\mathrm{Aut}\,\mathbb{C}| = \text{infinite}$$