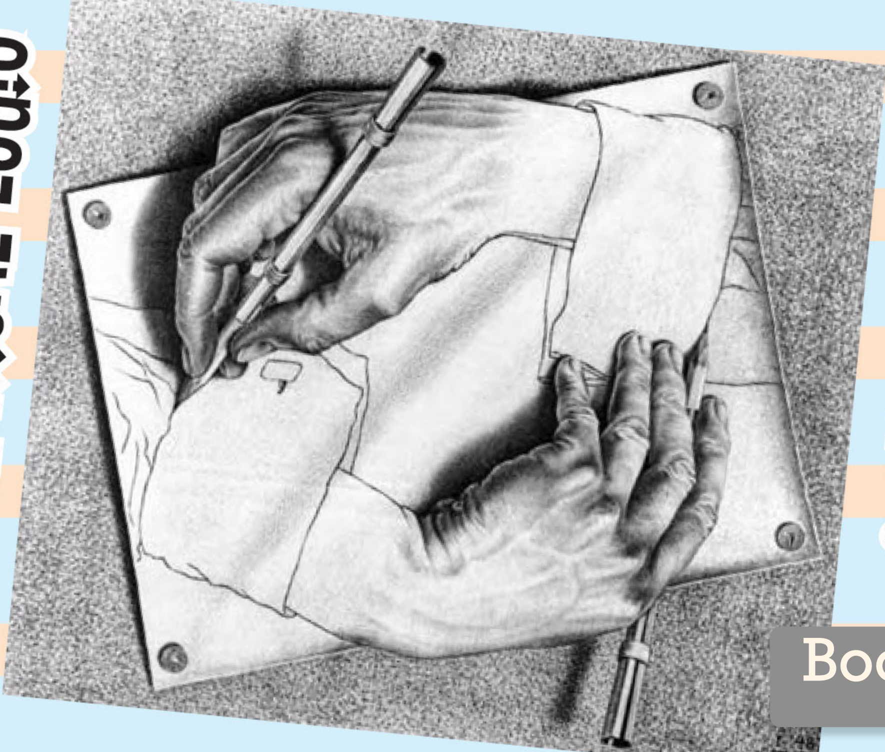
# MATHEMATICAL LOGIC & SET THEORY



Book 3

Trivial examples: Fix $x_0 \in X$. Define $\mu(A) = \begin{cases} 0 & \text{if } x_0 \notin A \\ 1 & \text{if } x_0 \in A. \end{cases}$

A **measureable cardinal** is a uncountable cardinal $\kappa$ which admits a nontrivial ~~countably additive~~ two-valued measure.

~~Does~~ such a $\kappa$ exist? If so then any larger cardinal satisfies this condition.

Given $\kappa < \kappa'$, $\mu$ nontrivial countably additive two-valued measure on $\kappa$, lift it to one on $\kappa'$: $\iota : \kappa \to \kappa'$ injection. Define (for $B \subseteq \kappa'$)

$$\mu'(B) = \mu(\iota^{-1}(B)).$$

**Theorem** (Ulam) If there exists a nontrivial countably additive two-valued measure on an uncountable set $X$ then let $X$ be a smallest example. Then $X$ has a nontrivial $\kappa$-additive two-valued measure for all $\kappa \leq |X|$.

$\mu$ is $\kappa$-additive if

$$\mu\Big(\bigsqcup_{\alpha \in I} A_\alpha\Big) = \sum_{\alpha \in I} \mu(A_\alpha)$$

for every collection of $|I| < \kappa$ sets $(A_\alpha \subseteq X)$.

A **measurable cardinal** is an uncountable cardinal $\kappa$ having a $\kappa$-additive two-valued measure.

$$[0,1] = \bigsqcup_{\alpha \in [0,1]} \{\alpha\}$$

~~Do~~ they exist? And who cares?

$$\begin{tikzcd}
 & \overset{\text{open}}{\Sigma^0_1} \arrow[dr, hook] & & \overset{\text{ctble unions of closed sets}}{\Sigma^0_2} \arrow[dr, hook] & & \\
\underset{\text{clopen}}{\Delta^0_1} \arrow[ur, hook] \arrow[dr, hook] & & \Delta^0_2 \arrow[ur, hook] \arrow[dr, hook] & & \Delta^0_3 \arrow[r, phantom, "\cdots"] & {} \\
 & \underset{\text{closed}}{\Pi^0_1} \arrow[ur, hook] & & \underset{\text{ctble intersections of open sets}}{\Pi^0_2} \arrow[ur, hook] & &
\end{tikzcd}$$

Projective Hierarchy $\Sigma^1_n$, $\Pi^1_n$, $\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$

$$\begin{array}{ccccccc} & & \Sigma^1_1 & & & & \Sigma^1_2 \\ \Delta^1_0 & \begin{smallmatrix}\subset \\ \subset\end{smallmatrix} & & \begin{smallmatrix}\supset \\ \supset\end{smallmatrix} \Delta^1_1 = \Sigma^1_1 \cap \Sigma^1_1 & \begin{smallmatrix}\subset \\ \subset\end{smallmatrix} & & \\ \text{Borel sets} & & \Pi^1_1 & & & & \Pi^1_2 \end{array}$$

$\Sigma^1_1$ = { analytic sets in $X$ } $A \in \Sigma^1_1$ iff $A$ is a continuous image of a Borel set under $f : Y \to X$ ($f$ continuous, $Y$ Polish space)

$\Pi^1_1$ = { coanalytic sets in $X$ } = { complements of analytic sets }

$\Sigma^1_2$ = { continuous images of coanalytic sets }

If there exist measurable cardinals, then every $\Sigma^1_2$-set is Lebesgue measureable.

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Coming to: an application a large cardinal to the finite world. see

Non-associative algebra : Keis, Quandles, Racks, Shelves, ... (Sam Nelson, Quandles)

A **kei** is a set $S$ with a binary operation $\triangleright$ satisfying : for all $x, y, z \in S$,

(1) $x \triangleright x = x$ (every element is idempotent)

(2) $(x \triangleright y) \triangleright y = x$ ($x \mapsto x \triangleright y$ is involutory)

(3) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ ("$\triangleright$" is right-distributive over itself)

If $(S, \triangleleft)$ satisfies (3), it is a shelf (or self distributive system). If it satisfies (1) and (3), it is a rack.

If $(S, \triangleleft)$ satisfies (1), (3) and (2') it is a quandle.

(2'): For all $y$, the map $S \to S$, $x \mapsto x \triangleright y$ is injective.

(1) $x \triangleright x = x$

(2) $(x \triangleright y) \triangleright y = x$

(3) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

The kei axioms are equivalent to the Reidemeister moves I, II, III.

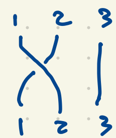
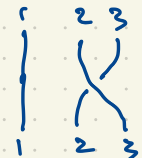
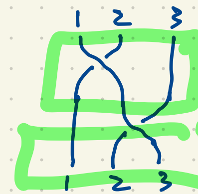
**Examples**: Fix $c \in \mathbb{R}$ and define $x \triangleright y = cx + (1-c)y$ for $x, y \in \mathbb{R}$. This gives a rack (?) (satisfying (1), (3)). It's a kei if $c = \pm 1$.

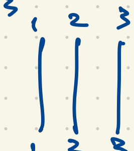
More generally let $V$ be a vector space and $R \in GL(V)$ invertible linear transformation. For $u, v \in V$, $u \triangleright v = Ru + (I - R)v$. This is an Alexander quandle. (Sometimes a kei).

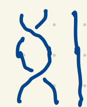
**Example** Let $G$ be a group (multiplicative). Fix $n \in \mathbb{Z}$. For $a, b \in G$, $a \triangleright b = b^n a b^{-n}$ ($n$-fold conjugation of $a$ by $b$). This is a rack, sometimes a quandle.

**Example** The Braid group $B_n$

eg. in $B_3$,

$\sigma =$  , $\tau =$  , $\sigma\tau =$ 


$\sigma^{-1} =$  $\iota =$  $\neq \sigma^2 =$ 

$\sigma\sigma^{-1} =$  $= \iota$ 

$S_n = \mathrm{Sym}\{1, 2, \cdots, n\}$

$|S_n| = n!$

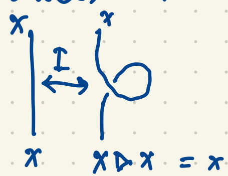
$B_n \twoheadrightarrow S_n$ epimorphism

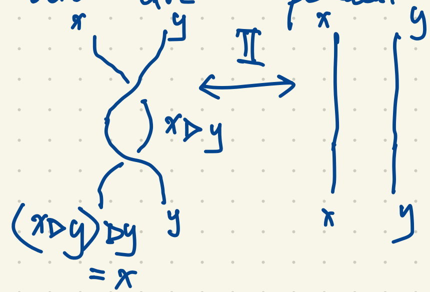
$|B_n| = \aleph_0$.

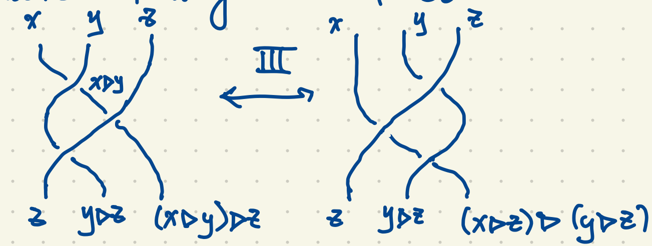
# Kei colorings of braids

Given a braid $\sigma \in B_n$ and a Kei $(K, \triangleright)$ we color the arcs in a braid diagram of $\sigma$ (i.e. label the arcs using elements of $K$) such that


This is the same as requiring that if we label the tops of the $n$ strands, the labels on the bottom are independent of the choice of diagram used for the braid $\sigma$.




A <u>right shelf</u> satisfies right-distributivity $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

.. <u>left</u> .. .. left - . . . $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$

~~If~~ $(K, \triangleright)$ is left-distributive ~~then~~ $\Longleftrightarrow$ $(K, \triangleleft)$ is right-distributive where

$x \triangleleft y = y \triangleright x$ (transpose the "multiplication table")

Switch to studying left shelves. Example found by Richard Laver (set theorist in Boulder)

$A_n = \{1, 2, 3, \cdots, N = 2^n\}$ (integers mod $N$) Note: $0$ is written as $N$ mod $N$.

<u>Theorem</u> There is a unique left shelf on $A_n$ satisfying $a \triangleright 1 = a+1$. for all $a \in A_n$.

<u>Eg.</u> $n = 2$, $N = 4$, $A = \{1, 2, 3, 4\}$ = integers mod 4

| $\triangleright$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 2 | 4 | 2 | 4 |
| 2 | 3 | 4 | 3 | 4 |
| 3 | 4 | 4 | 4 | 4 |
| 4 | 1 | 2 | 3 | 4 |

$4 \triangleright 2 = 4 \triangleright (1 \triangleright 1) = (4 \triangleright 1) \triangleright (4 \triangleright 1) = 1 \triangleright 1 = 2$

$4 \triangleright 3 = 4 \triangleright (2 \triangleright 1) = (4 \triangleright 2) \triangleright (4 \triangleright 1) = 2 \triangleright 1 = 3$

$4 \triangleright 4 = 4 \triangleright (3 \triangleright 1) = (4 \triangleright 3) \triangleright (4 \triangleright 1) = 3 \triangleright 1 = 4$

$3 \triangleright 2 = 3 \triangleright (1 \triangleright 1) = (3 \triangleright 1) \triangleright (3 \triangleright 1) = 4 \triangleright 4 = 4$

$2 \triangleright 2 = 2 \triangleright (1 \triangleright 1) = (2 \triangleright 1) \triangleright (2 \triangleright 1) = 3 \triangleright 3 = 4$

$2 \triangleright 3 = 2 \triangleright (2 \triangleright 1) = (2 \triangleright 2) \triangleright (2 \triangleright 1) = 4 \triangleright 3 = 3$

$2 \triangleright 4 = 2 \triangleright (3 \triangleright 1) = (2 \triangleright 3) \triangleright (2 \triangleright 1) = 3 \triangleright 3 = 4$

$1 \triangleright 2 = 1 \triangleright (1 \triangleright 1) = (1 \triangleright 1) \triangleright (1 \triangleright 1) = 2 \triangleright 2 = 4$

$1 \triangleright 3 = 1 \triangleright (2 \triangleright 1) = (1 \triangleright 2) \triangleright (1 \triangleright 1) = 4 \triangleright 2 = 2$

Fact: The left-distributive law holds in all cases although we haven't checked this here.

| $A_0$ | 1 |
|---|---|
| 1 | 1 |

| $A_1$ | 1 | 2 |
|---|---|---|
| 1 | 2 | 2 |
| 2 | 1 | 2 |

| $A_2$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 2 | 4 | 2 | 4 |
| 2 | 3 | 4 | 3 | 4 |
| 3 | 4 | 4 | 4 | 4 |
| 4 | 1 | 2 | 3 | 4 |

| $A_3$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 6 | 8 | 2 | 4 | 6 | 8 |
| 2 | 3 | 4 | 7 | 8 | 3 | 4 | 7 | 8 |
| 3 | 4 | 8 | 4 | 8 | 4 | 8 | 4 | 8 |
| 4 | 5 | 6 | 7 | 8 | 5 | 6 | 7 | 8 |
| 5 | 6 | 8 | 6 | 8 | 6 | 8 | 6 | 8 |
| 6 | 7 | 8 | 7 | 8 | 7 | 8 | 7 | 8 |
| 7 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| 8 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

Figure 2: Multiplication tables for the first four Laver tables

<u>Conjecture</u> As $n \to \infty$ ($N \to \infty$) the period of the first row of the table $\to \infty$.
The conjecture holds if there exists a Laver cardinal (a certain kind of large cardinal). No one knows how to prove this in ZFC.

We have an inverse system of left shelves

$$\cdots \longrightarrow A_4 \longrightarrow A_3 \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0$$

Let $X$ be any set and let $M = \{$ injective maps $X \to X \}$.
(1-to-1)

Then $M$ is a monoid under composition. (A group iff $X$ is finite).

Let $A$ be a set of sentences over some language $L$, and let $M, N \models A$. (models of $A$ i.e. $L$-structures which satisfy all the sentences in $A$)

eg. $A$: axioms for a ring

$L$: $+, -, \times$

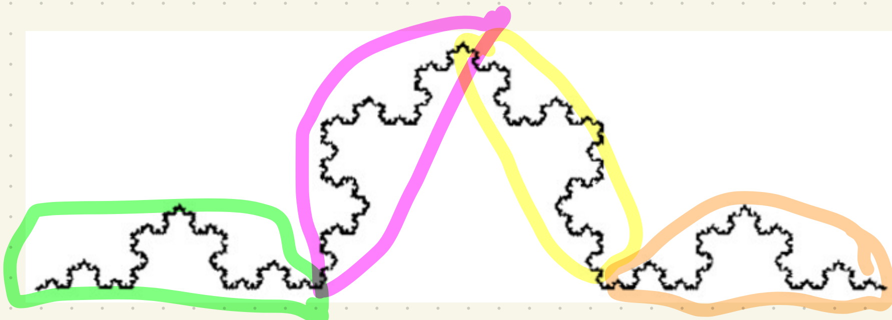
$\mathbb{Z}, \mathbb{Q} \models A$ and $\mathbb{Z}$ is a submodel of $\mathbb{Q}$ (there is a 1-to-1 map $\mathbb{Z} \xrightarrow{\iota} \mathbb{Q}$ preserving the operations. But $\mathbb{Z}$ is not elementarily embedded in $\mathbb{Q}$ because
(elementary embedded)

there are sentences $\phi$ over $L$ such that $\mathbb{Z} \models \phi$, $\mathbb{Q} \models \neg\phi$ (or the other way around) e.g.

eg. $\phi : (\exists x)(\forall y)(\neg(y+y=x))$.

We say $\iota : M \to N$ ($M, N \models A$) is an elementary embedding if $\iota$ is injective and for every sentence $\phi$, $\iota(M) \subseteq N$ where $\iota(M)$ is elementarily equivalent to $N$:
submodel

For all $\phi$, $\iota(M) \models \phi$ iff $N \models \phi$.



A portion of the Koch snowflake curve illustrating self-similarity.

There are many embeddings of $\mathbb{C}$ in itself. Pick such an embedding $\iota: \mathbb{C} \to \mathbb{C}$
$\mathbb{C}$, $\iota(\mathbb{C}) \subset \mathbb{C}$ are models of the field axioms $A$. $\iota(\mathbb{C})$ is an elementary submodel of $\mathbb{C}$ i.e. $\iota: \mathbb{C} \to \mathbb{C}$ is an elementary embedding i.e. $\mathbb{C}$ is an elementary extension of $\iota(\mathbb{C})$.

<u>Note: $\iota: \mathbb{C} \to \mathbb{C}$ preserves $0, 1, +, \times, -$</u> but not the topology.

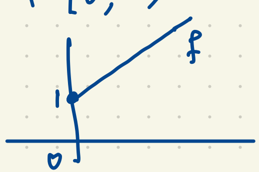
For models of ZFC $(L: \in)$ a <u>Laver cardinal</u> is a (inaccessible) cardinal $\kappa$ such that the $V_\kappa$ admits an elementary embedding $\iota: V_\kappa \longrightarrow V_\kappa$ which is <u>not</u> surjective.
<u>This $\iota$ generates a left shelf</u> under the following:

If $f, g: X \to X$ are injective then $f \triangleright g: X \to X$ is

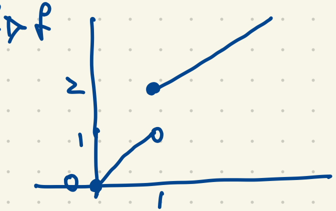
$$(f \triangleright g)(x) = \begin{cases} fgf^{-1}(x) & , \text{if } x \in f(X) \\ x & , \text{if } x \notin f(X) \end{cases}$$

$$f(X) = \{f(x) : x \in X\} \subset X$$

eg. $f: [0,\infty) \longrightarrow [0,\infty)$, $x \longmapsto x+1$



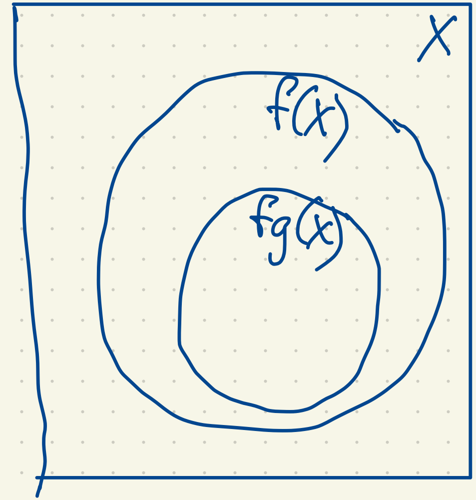

$f \triangleright f$



Why is $\triangleright$ a left shelf?

$((f \triangleright g) \triangleright (f \triangleright h))(x)$

$= (f \triangleright (g \triangleright h))(x)$ Check three cases




If $x \in fg(X)$ then $x = fg(y)$ so

$(g \triangleright h)(x) =$

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$\iota : V_\kappa \longrightarrow V_\kappa$ is an elementary embedding but not surjective.

It generates a left shelf under "$\triangleright$". This is the free shelf on one generator $\mathcal{F}_1$

$\mathcal{F}_1 = \{ \iota, \iota \triangleright \iota, (\iota \triangleright \iota) \triangleright \iota, \iota \triangleright (\iota \triangleright \iota), \dots \}$ These combinations of $\iota$ under $\triangleright$

are distinct except when required by the left shelf axiom e.g. $(\iota \triangleright \iota) \triangleright (\iota \triangleright \iota) = \iota \triangleright (\iota \triangleright \iota)$

$\mathcal{F}_1$ is a countably infinite left shelf; moreover $\mathcal{F}_1 = \varprojlim A_n$