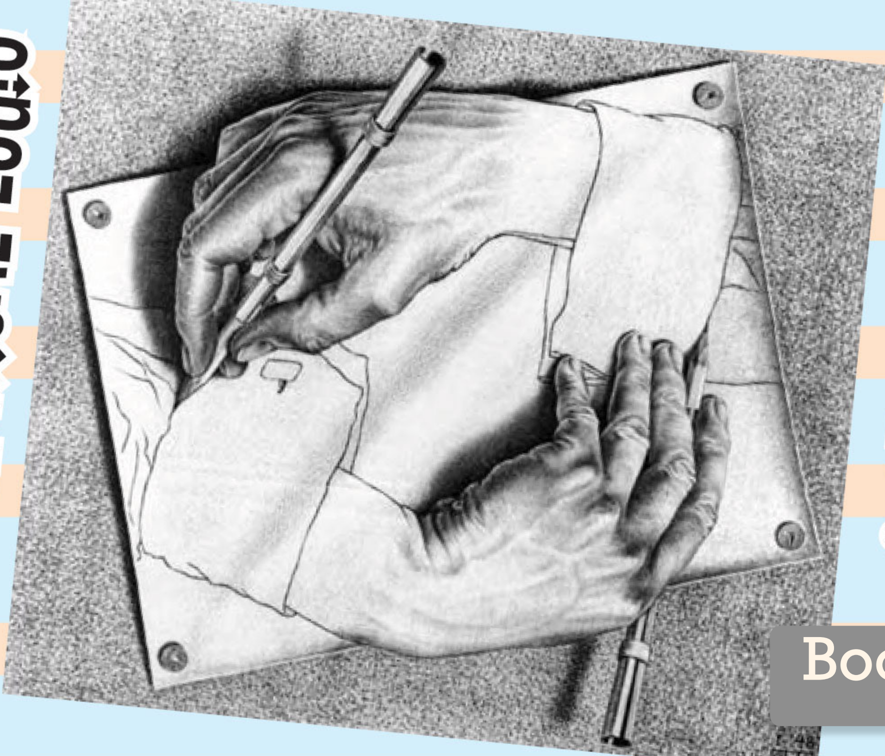
MATHEMATICAL LOGIC & SET THEORY



Book 3

Trivial examples: Fix $x_0 \in X$. Define $\mu(A) = \begin{cases} 0 & \text{if } x_0 \notin A \\ 1 & \text{if } x_0 \in A. \end{cases}$

A <u>measureable cardinal</u> is a uncountable cardinal κ which admits a nontrivial ~~countably additive~~ two-valued measure.

Does such a κ exist? If so then any larger cardinal satisfies this condition.

Given $\kappa < \kappa'$, μ nontrivial countably additive two-valued measure on κ, lift it to one on κ': $\iota : \kappa \to \kappa'$ injection. Define (for $B \subseteq \kappa'$)

$$\mu'(B) = \mu(\iota^{-1}(B)).$$

<u>Theorem</u> (Ulam) If there exists a nontrivial countably additive two-valued measure on an uncountable set X then let X be a smallest example. Then X has a nontrivial κ-additive two-valued measure for all $\kappa \leq |X|$.

μ is κ-additive if $\mu\left(\bigsqcup_{\alpha \in I} A_\alpha\right) = \sum_{\alpha \in I} \mu(A_\alpha)$ for every collection of $|I| < \kappa$ sets $(A_\alpha \subseteq X)$.

A <u>measurable cardinal</u> is an uncountable cardinal κ having a κ-additive two-valued measure.

Do they exist? And who cares?

$$[0,1] = \bigsqcup_{\alpha \in [0,1]} \{\alpha\}$$

$$\begin{array}{ccccccccc}
 & & \underset{\text{open}}{\Sigma^0_1} & & & & \underset{\text{ctble unions of closed sets}}{\Sigma^0_2} & & \\
 & \subset & & \supset & & \subset & & \supset & \\
\underset{\text{clopen}}{\Delta^0_1} & & & & \Delta^0_2 & & & & \Delta^0_3 \quad \cdots \\
 & \supset & & \subset & & \supset & & \subset & \\
 & & \underset{\text{closed}}{\Pi^0_1} & & & & \underset{\text{ctble intersections of open sets}}{\Pi^0_2} & &
\end{array}$$

<u>Projective Hierarchy</u> Σ^1_n, Π^1_n, $\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$

$$\Delta^1_0 \subset \Sigma^1_1 \supset \Delta^1_1 = \Sigma^1_1 \cap \Sigma^1_1 \subset \Sigma^1_2 \cap$$

Borel sets $\cap\ \Pi^1_1 \supset \qquad \cap\ \Pi^1_2 \subset$

$\Sigma^1_1 = \{$analytic sets in $X\}$ $\qquad A \in \Sigma^1_1$ iff A is a continuous image of a Borel set under $f: Y \to X$ (f continuous, Y Polish space)

$\Pi^1_1 = \{$coanalytic sets in $X\} = \{$complements of analytic sets$\}$

$\Sigma^1_2 = \{$continuous images of coanalytic sets$\}$

If there exist measurable cardinals, then every Σ^1_2-set is Lebesgue measureable.

Coming to: an application a large cardinal to the finite world. see

Non-associative algebra : Keis, Quandles, Racks, Shelves, ... (Sam Nelson, Quandles)

A <u>kei</u> is a set S with a binary operation $\triangleright$ satisfying : for all $x, y, z \in S$,

(1) $x \triangleright x = x$ (every element is idempotent)

(2) $(x \triangleright y) \triangleright y = x$ ($x \mapsto x \triangleright y$ is involutory)

(3) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ ("$\triangleright$" is right-distributive over itself)

If $(S, \triangleleft)$ satisfies (3), it is a <u>shelf</u> (or <u>self distributive system</u>). If it satisfies (1) and (3), it is a <u>rack</u>.

If $(S, \triangleleft)$ satisfies (1), (3) and (2′) it is a <u>quandle</u>.

(2′): For all y, the map $S \to S$, $x \mapsto x \triangleright y$ is injective.

(1) $x \triangleright x = x$

(2) $(x \triangleright y) \triangleright y = x$

(3) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

The kei axioms are equivalent to the Reidemeister moves I, II, III.

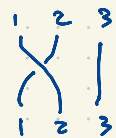
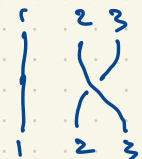
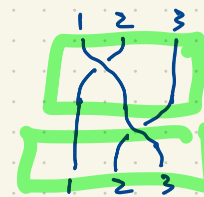
<u>Examples</u>: Fix $c \in \mathbb{R}$ and define $x \triangleright y = cx + (1-c)y$ for $x, y \in \mathbb{R}$. This gives a rack (?) (satisfying (1), (3)). It's a kei if $c = \pm 1$.

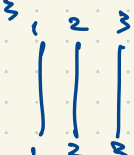
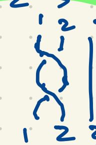
More generally let V be a vector space and $R \in GL(V)$ invertible linear transformation. For $u, v \in V$, $u \triangleright v = Ru + (I - R)v$. This is an Alexander quandle. (Sometimes a kei).

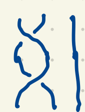
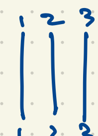
<u>Example</u> Let G be a group (multiplicative). Fix $n \in \mathbb{Z}$. For $a, b \in G$, $a \triangleright b = b^n a b^{-n}$ (n-fold conjugation of a by b). This is a rack, sometimes a quandle.

<u>Example</u> The Braid group B_n

eg. in B_3,

$\sigma =$  , $\tau =$  , $\sigma\tau =$  σ τ

$\sigma^{-1} =$  $\iota =$  $\neq \sigma^2 =$ 

$\sigma\sigma^{-1} =$  $= \iota$ 

$S_n = \mathrm{Sym}\{1, 2, \cdots, n\}$

$|S_n| = n!$

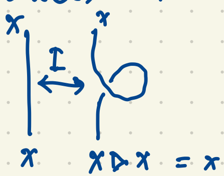
$B_n \twoheadrightarrow S_n$ epimorphism

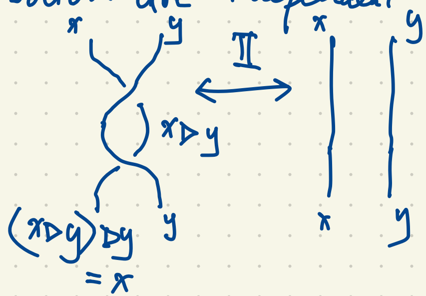
$|B_n| = \aleph_0$.

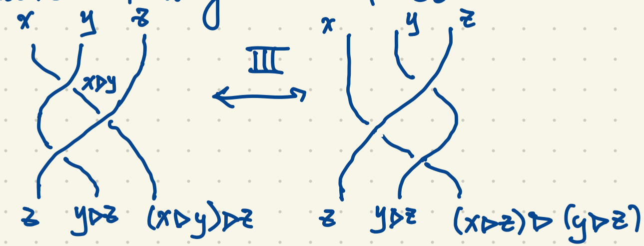
Kei colorings of braids

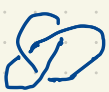
Given a braid $\sigma \in B_n$ and a Kei $(K, \triangleright)$ we color the arcs in a braid diagram of σ (i.e. label the arcs using elements of K) such that

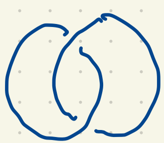
This is the same as requiring that if we label the tops of the n strands, the labels on the bottom are independent of the choice of diagram used for the braid σ.











A right shelf satisfies right-distributivity $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

" left " " left - - - - $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$

$(K, \triangleright)$ is left-distributive $\Longleftrightarrow$ $(K, \triangleleft)$ is right-distributive where $x \triangleleft y = y \triangleright x$ (transpose the "multiplication table")

Switch to studying left shelves. Example found by Richard Laver (set theorist in Boulder)

$A_n = \{1, 2, 3, \cdots, N = 2^n\}$ (integers mod N) Note: 0 is written as N mod N.

Theorem There is a unique left shelf on A_n satisfying $a \triangleright 1 = a + 1$. for all $a \in A_n$.

Eg. $n = 2$, $N = 4$, $A = \{1, 2, 3, 4\}$ = integers mod 4

$\triangleright$	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

$4 \triangleright 2 = 4 \triangleright (1 \triangleright 1) = (4 \triangleright 1) \triangleright (4 \triangleright 1) = 1 \triangleright 1 = 2$

$4 \triangleright 3 = 4 \triangleright (2 \triangleright 1) = (4 \triangleright 2) \triangleright (4 \triangleright 1) = 2 \triangleright 1 = 3$

$4 \triangleright 4 = 4 \triangleright (3 \triangleright 1) = (4 \triangleright 3) \triangleright (4 \triangleright 1) = 3 \triangleright 1 = 4$

$3 \triangleright 2 = 3 \triangleright (1 \triangleright 1) = (3 \triangleright 1) \triangleright (3 \triangleright 1) = 4 \triangleright 4 = 4$

$2 \triangleright 2 = 2 \triangleright (1 \triangleright 1) = (2 \triangleright 1) \triangleright (2 \triangleright 1) = 3 \triangleright 3 = 4$

$2 \triangleright 3 = 2 \triangleright (2 \triangleright 1) = (2 \triangleright 2) \triangleright (2 \triangleright 1) = 4 \triangleright 3 = 3$

$2 \triangleright 4 = 2 \triangleright (3 \triangleright 1) = (2 \triangleright 3) \triangleright (2 \triangleright 1) = 3 \triangleright 3 = 4$

$1 \triangleright 2 = 1 \triangleright (1 \triangleright 1) = (1 \triangleright 1) \triangleright (1 \triangleright 1) = 2 \triangleright 2 = 4$

$1 \triangleright 3 = 1 \triangleright (2 \triangleright 1) = (1 \triangleright 2) \triangleright (1 \triangleright 1) = 4 \triangleright 2 = 2$

Fact: The left-distributive law holds in all cases although we haven't checked this here.

A_0	1
1	1

A_1	1	2
1	2	2
2	1	2

A_2	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

Figure 2: Multiplication tables for the first four Laver tables

<u>Conjecture</u> As $n \to \infty$ $(N \to \infty)$ the period of the first row of the table $\to \infty$.
The conjecture holds if there exists a Laver cardinal (a certain kind of large cardinal). No one knows how to prove this in ZFC.

Let X be any set and let $M = \{\underset{(1\text{-to-}1)}{\text{injective}} \text{ maps } X \longrightarrow X\}$.

Then M is a monoid under composition. (A group iff X is finite).