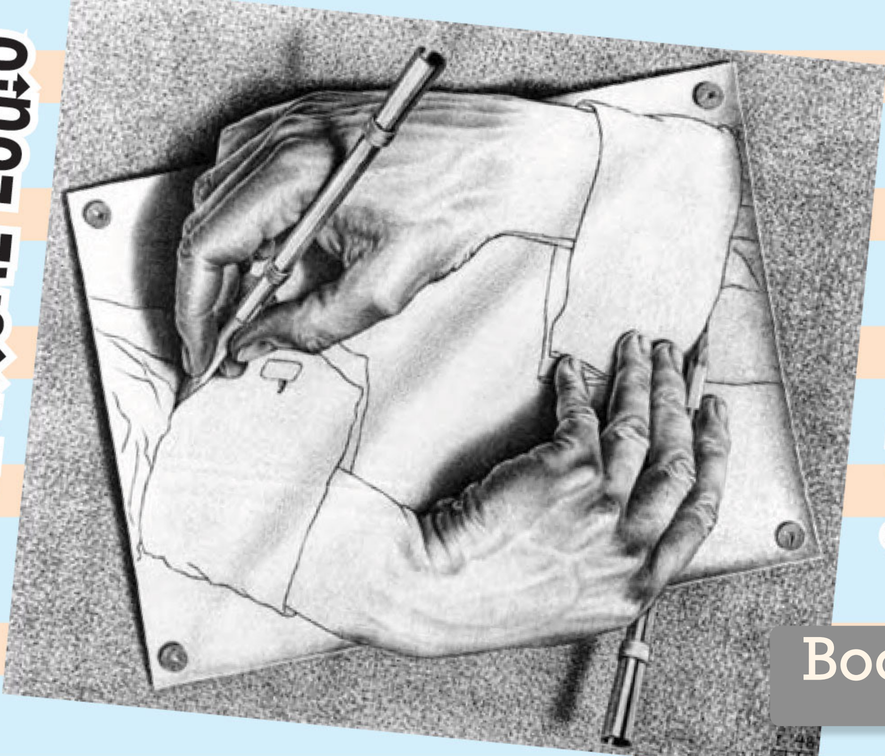
MATHEMATICAL LOGIC & SET THEORY



Book 3

Trivial examples: Fix $x_0 \in X$. Define $\mu(A) = \begin{cases} 0 & \text{if } x_0 \notin A \\ 1 & \text{if } x_0 \in A. \end{cases}$

A <u>measureable cardinal</u> is a uncountable cardinal κ which admits a nontrivial ~~countably additive~~ two-valued measure.

Does such a κ exist? If so then any larger cardinal satisfies this condition.

Given $\kappa < \kappa'$, μ nontrivial countably additive two-valued measure on κ, lift it to one on κ': $\iota : \kappa \to \kappa'$ injection. Define (for $B \subseteq \kappa'$)

$$\mu'(B) = \mu(\iota^{-1}(B)).$$

<u>Theorem</u> (Ulam) If there exists a nontrivial countably additive two-valued measure on an uncountable set X then let X be a smallest example. Then X has a nontrivial κ-additive two-valued measure for all $\kappa \leq |X|$.

μ is κ-additive if

$$\mu\Big(\bigsqcup_{\alpha \in I} A_\alpha\Big) = \sum_{\alpha \in I} \mu(A_\alpha)$$

for every collection of $|I| < \kappa$ sets ($A_\alpha \subseteq X$).

A <u>measurable cardinal</u> is an uncountable cardinal κ having a κ-additive two-valued measure.

$$[0,1] = \bigsqcup_{\alpha \in [0,1]} \{\alpha\}$$

Do they exist? And who cares?

$$\begin{array}{ccccccccc}
 & & \overset{\text{open}}{\Sigma^0_1} & & & & \overset{\text{ctble unions of closed sets}}{\Sigma^0_2} & & \\
 & \subset & & \supset & & \subset & & \supset & & \subset \\
\underset{\text{clopen}}{\Delta^0_1} & & & & \Delta^0_2 & & & & \Delta^0_3 & \cdots \\
 & \supset & & \subset & & \supset & & \subset & & \supset \\
 & & \underset{\text{closed}}{\Pi^0_1} & & & & \underset{\text{ctble intersections of open sets}}{\Pi^0_2} & &
\end{array}$$

Projective Hierarchy Σ^1_n, Π^1_n, $\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$

$$\begin{array}{ccccccc} & & \Sigma^1_1 & & & & \Sigma^1_2 \\ \Delta^1_0 & \subset & & \supset & \Delta^1_1 = \Sigma^1_1 \cap \Sigma^1_1 & \subset & \cap \\ \text{Borel sets} & \subset & \Pi^1_1 & \supset & & \subset & \Pi^1_2 \quad \subset \end{array}$$

$\Sigma^1_1 = \{\text{analytic sets in } X\}$ $\qquad$ $A \in \Sigma^1_1$ iff A is a continuous image of a Borel set under $f: Y \to X$
(f continuous, Y Polish space)

$\Pi^1_1 = \{\text{coanalytic sets in } X\} = \{\text{complements of analytic sets}\}$

$\Sigma^1_2 = \{\text{continuous images of coanalytic sets}\}$

If there exist measurable cardinals, then every Σ^1_2-set is Lebesgue measureable.

Coming to: an application a large cardinal to the finite world. see

Non-associative algebra : Keis, Quandles, Racks, Shelves, ... (Sam Nelson, Quandles)

A <u>kei</u> is a set S with a binary operation $\triangleright$ satisfying : for all $x, y, z \in S$,

(1) $x \triangleright x = x$ $\qquad$ (every element is idempotent)

(2) $(x \triangleright y) \triangleright y = x$ $\qquad$ ($x \mapsto x \triangleright y$ is involutory)

(3) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ $\qquad$ ("$\triangleright$" is right-distributive over itself)

If $(S, \triangleleft)$ satisfies (3), it is a <u>shelf</u> (or <u>self distributive system</u>). If it satisfies (1) and (3), it is a <u>rack</u>.

If $(S, \triangleleft)$ satisfies (1), (3) and (2′) it is a <u>quandle</u>.

(2′): For all y, the map $S \to S$, $x \mapsto x \triangleright y$ is injective.

(1) $x \triangleright x = x$

(2) $(x \triangleright y) \triangleright y = x$

(3) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

The kei axioms are equivalent to the Reidemeister moves I, II, III.

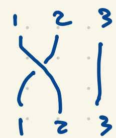
Examples: Fix $c \in \mathbb{R}$ and define $x \triangleright y = cx + (1-c)y$ for $x, y \in \mathbb{R}$. This gives a rack (?) (satisfying (1), (3)). It's a kei if $c = \pm 1$.

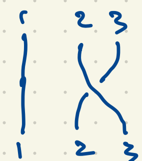
More generally let V be a vector space and $R \in GL(V)$ invertible linear transformation. For $u, v \in V$, $u \triangleright v = Ru + (I - R)v$. This is an Alexander quandle. (Sometimes a kei).

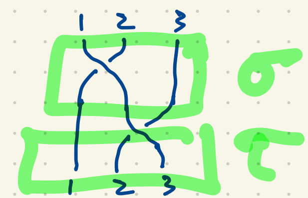
Example Let G be a group (multiplicative). Fix $n \in \mathbb{Z}$. For $a, b \in G$, $a \triangleright b = b^{n} a b^{-n}$ (n-fold conjugation of a by b). This is a rack, sometimes a quandle.

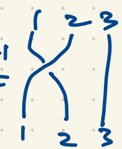
Example The Braid group B_n

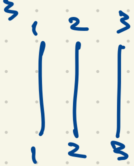
eg. in B_3,

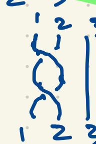
$\sigma =$ 

, $\tau =$ 

, $\sigma\tau =$ 

$\sigma^{-1} =$ 

$\iota =$ 

$\neq \sigma^2 =$ 

$\sigma\sigma^{-1} =$ 

$= \iota$ 

$S_n = \mathrm{Sym}\{1, 2, \cdots, n\}$

$|S_n| = n!$

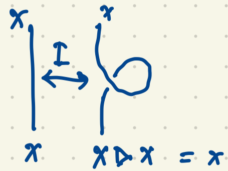
$B_n \twoheadrightarrow S_n$ epimorphism

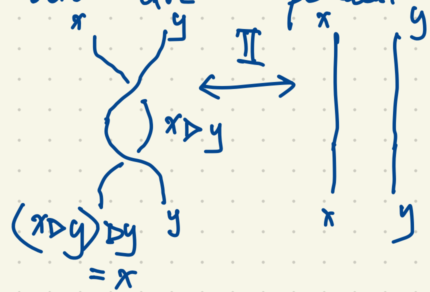
$|B_n| = \aleph_0$.

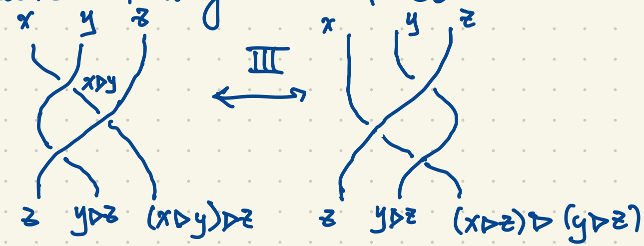
Kei colorings of braids

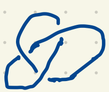
Given a braid $\sigma \in B_n$ and a Kei $(K, \triangleright)$ we color the arcs in a braid diagram of σ (i.e. label the arcs using elements of K) such that

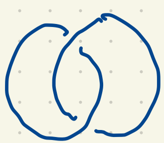
This is the same as requiring that if we label the tops of the n strands, the labels on the bottom are independent of the choice of diagram used for the braid σ.











A right shelf satisfies right-distributivity $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

" left " " left " " $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$

$(K, \triangleright)$ is left-distributive ~~then~~ $\Longleftrightarrow$ $(K, \triangleleft)$ is right-distributive where $x \triangleleft y = y \triangleright x$ (transpose the "multiplication table")

Switch to studying left shelves. Example found by Richard Laver (set theorist in Boulder)

$A_n = \{1, 2, 3, \cdots, N = 2^n\}$ (integers mod N) Note: 0 is written as N mod N.

Theorem There is a unique left shelf on A_n satisfying $a \triangleright 1 = a+1$. for all $a \in A_n$.

Eg. $n = 2$, $N = 4$, $A = \{1, 2, 3, 4\}$ = integers mod 4

$\triangleright$	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

$4 \triangleright 2 = 4 \triangleright (1 \triangleright 1) = (4 \triangleright 1) \triangleright (4 \triangleright 1) = 1 \triangleright 1 = 2$

$4 \triangleright 3 = 4 \triangleright (2 \triangleright 1) = (4 \triangleright 2) \triangleright (4 \triangleright 1) = 2 \triangleright 1 = 3$

$4 \triangleright 4 = 4 \triangleright (3 \triangleright 1) = (4 \triangleright 3) \triangleright (4 \triangleright 1) = 3 \triangleright 1 = 4$

$3 \triangleright 2 = 3 \triangleright (1 \triangleright 1) = (3 \triangleright 1) \triangleright (3 \triangleright 1) = 4 \triangleright 4 = 4$

$2 \triangleright 2 = 2 \triangleright (1 \triangleright 1) = (2 \triangleright 1) \triangleright (2 \triangleright 1) = 3 \triangleright 3 = 4$

$2 \triangleright 3 = 2 \triangleright (2 \triangleright 1) = (2 \triangleright 2) \triangleright (2 \triangleright 1) = 4 \triangleright 3 = 3$

$2 \triangleright 4 = 2 \triangleright (3 \triangleright 1) = (2 \triangleright 3) \triangleright (2 \triangleright 1) = 3 \triangleright 3 = 4$

$1 \triangleright 2 = 1 \triangleright (1 \triangleright 1) = (1 \triangleright 1) \triangleright (1 \triangleright 1) = 2 \triangleright 2 = 4$

$1 \triangleright 3 = 1 \triangleright (2 \triangleright 1) = (1 \triangleright 2) \triangleright (1 \triangleright 1) = 4 \triangleright 2 = 2$

Fact: The left-distributive law holds in all cases although we haven't checked this here.

A_0	1
1	1

A_1	1	2
1	2	2
2	1	2

A_2	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

Figure 2: Multiplication tables for the first four Laver tables

<u>Conjecture</u> As $n \to \infty$ ($N \to \infty$) the period of the first row of the table $\to \infty$.

The conjecture holds if there exist a Laver cardinal (a certain kind of large cardinal). No one knows how to prove this in ZFC.

We have an inverse system of left shelves

$$\cdots \longrightarrow A_4 \longrightarrow A_3 \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0$$

Let X be any set and let $M = \{\text{injective maps } X \to X\}$.
(1-to-1)

Then M is a monoid under composition. (A group iff X is finite).

Let A be a set of sentences over some language L, and let $M, N \models A$. (models of A i.e. L-structures which satisfy all the sentences in A)

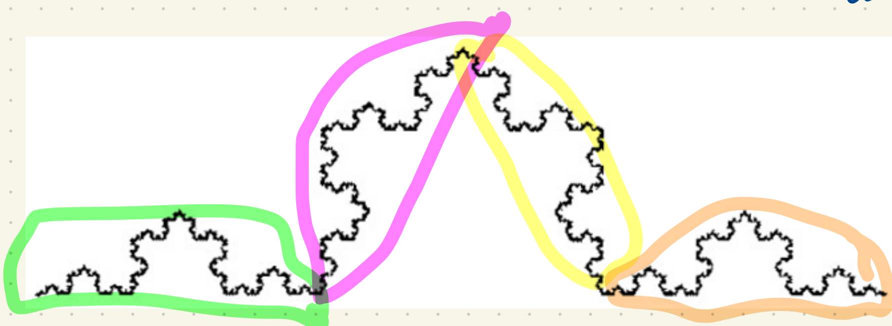
eg. A: axioms for a ring

L: $+, -, \times$

$\mathbb{Z}, \mathbb{Q} \models A$ and $\mathbb{Z}$ is a submodel of $\mathbb{Q}$ (there is a 1-to-1 map $\mathbb{Z} \xrightarrow{\iota} \mathbb{Q}$ preserving the operations. But $\mathbb{Z}$ is not elementarily embedded in $\mathbb{Q}$ because
(elementary embedded)
there are sentences ϕ over L such that $\mathbb{Z} \models \phi$, $\mathbb{Q} \models \neg\phi$ (or the other way around) e.g.

eg. $\phi: (\exists x)(\forall y)(\neg(y+y=x))$.

We say $\iota: M \to N$ ($M, N \models A$) is an elementary embedding if ι is injective and for every sentence ϕ, $\underset{\text{submodel}}{\iota(M) \subseteq N}$ where $\iota(M)$ is elementarily equivalent to N: For all ϕ, $\iota(M) \models \phi$ iff $N \models \phi$.



A portion of the Koch Snowflake curve illustrating self-similarity.

There are many embeddings of $\mathbb{C}$ in itself. Pick such an embedding $\iota : \mathbb{C} \to \mathbb{C}$.
$\mathbb{C}$, $\iota(\mathbb{C}) \subset \mathbb{C}$ are models of the field axioms $\mathcal{A}$. $\iota(\mathbb{C})$ is an elementary submodel of $\mathbb{C}$ i.e. $\iota : \mathbb{C} \to \mathbb{C}$ is an elementary embedding i.e. $\mathbb{C}$ is an elementary extension of $\iota(\mathbb{C})$.

Note: $\iota : \mathbb{C} \to \mathbb{C}$ preserves $0, 1, +, \times, -$ but not the topology.

For models of ZFC $(L : \in)$ a <u>Laver cardinal</u> is a (inaccessible) cardinal κ such that the V_κ admits an elementary embedding $\iota : V_\kappa \to V_\kappa$ which is <u>not</u> surjective.

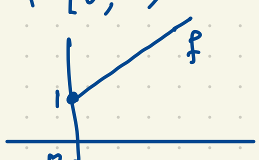
<u>This ι generates a left shelf under the following:</u>

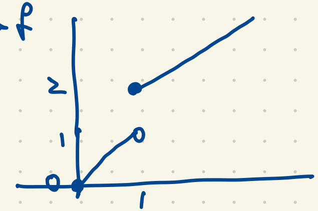
If $f, g : X \to X$ are injective then $f \triangleright g : X \to X$ is

$$(f \triangleright g)(x) = \begin{cases} f g f^{-1}(x) & , \text{ if } x \in f(X) \\ x & , \text{ if } x \notin f(X) \end{cases}$$

$f(X) = \{ f(x) : x \in X \} \subset X$

eg. $f : [0, \infty) \to [0, \infty)$, $x \mapsto x + 1$ $\qquad f \triangleright f$

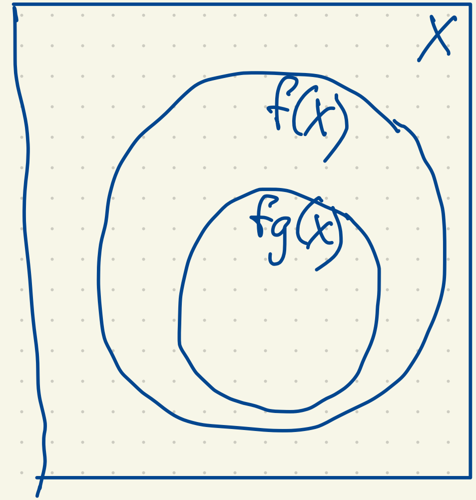




Why is $\triangleright$ a left shelf?

$((f \triangleright g) \triangleright (f \triangleright h))(x)$

$= (f \triangleright (g \triangleright h))(x)$ Check three cases



If $x \in fg(X)$ then $x = fg(y)$ so

$(g \triangleright h)(x) =$

$\iota : V_\kappa \longrightarrow V_\kappa$ is an elementary embedding but not surjective.

It generates a left shelf under "$\triangleright$". This is the free shelf on one generator $\mathcal{F}_1$

$\mathcal{F}_1 = \{ \iota, \iota \triangleright \iota, (\iota \triangleright \iota) \triangleright \iota, \iota \triangleright (\iota \triangleright \iota), \cdots \}$ These combinations of ι under $\triangleright$

are distinct except when required by the left shelf axiom e.g. $(\iota \triangleright \iota) \triangleright (\iota \triangleright \iota) = \iota \triangleright (\iota \triangleright \iota)$

$\mathcal{F}_1$ is a countably infinite left shelf; moreover $\mathcal{F}_1 = \varprojlim A_n$

Let X be an infinite set. A __filter__ on X is a collection $\mathcal{F}$ of subsets of X such that
(i) $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$ (Sets in $\mathcal{F}$ are __large__ subsets of X.)
(ii) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$ then $B \in \mathcal{F}$.
(iii) If $A, A' \in \mathcal{F}$ then $A \cap A' \in \mathcal{F}$.
By Zorn's lemma, every $\mathcal{F}$ filter extends to an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$ on X which is a filter satisfying
(iv) For all $A \subseteq X$, either A or $X \smallsetminus A$ is in $\mathcal{U}$.
$\mathcal{U}$ gives a two-valued finitely additive probability measure on X.
To get a nonprincipal ultrafilter on X, we start with the Fréchet filter consisting of all cofinite subsets of X (complements of finite subsets of X) and take $\mathcal{U} \supseteq \mathcal{F}$ a maximal filter containing $\mathcal{F}$. $\mathcal{U}$ is nonprincipal: $\mathcal{U}$ contains no finite sets.

We take $\mathcal{U}$ to be a nonprincipal ultrafilter on $\omega = \{0,1,2,3,\dots\}$ and consider the ring $\mathbb{R}^\omega = \{(a_0, a_1, a_2, a_3, \dots) : a_i \in \mathbb{R}\}$ with coordinatewise operations. $\mathbb{R}^\omega$ is a commutative ring with identity, not a field; eg. $(1,0,1,0,\dots)(0,1,0,1,\dots) = (0,0,0,0,\dots) = 0 \in \mathbb{R}^\omega$.
__Now__ identify two sequences $a = (a_0, a_1, a_2, \dots)$, $b = (b_0, b_1, b_2, \dots)$ if they agree almost everywhere with respect to $\mathcal{U}$ i.e. if $\{i \in \omega : a_i = b_i\} \in \mathcal{U}$.
In the case $a = (1,0,1,0,1,0,\dots)$ we have $a_i = 0$ whenever $i \in \{1,3,5,7,\dots\}$; $b_i = 0$ whenever $i \in \{0,2,4,6,\dots\}$
$b = (0,1,0,1,0,1,\dots)$
If $\{1,3,5,7,\dots\} \in \mathcal{U}$ then $a \sim (0,0,0,0,0,\dots)$ and $b \sim (1,1,1,1,1,\dots)$.
If $\{0,2,4,6,\dots\} \in \mathcal{U}$ then $a \sim (1,1,1,1,1,\dots)$ and $b \sim (0,0,0,0,0,\dots)$.

Identify two sequences in $\mathbb{R}^\omega$ whenever they agree almost everywhere w.r.t. $\mathcal{U}$.
Then we get a quotient ring $\mathbb{R}^\omega/\sim = {}^*\mathbb{R}$ denoted $\hat{\mathbb{R}}$ in the handout.
This is the field of nonstandard reals or hyperreals.
${}^*\mathbb{R}$ has the same first order theory (an ordered field and it's a real closed field, e.g. every poly $f(x) \in {}^*\mathbb{R}[x]$ of odd degree has a root in ${}^*\mathbb{R}$). In fact we have an elementary embedding of $\mathbb{R}$ in ${}^*\mathbb{R}$. The main difference between $\mathbb{R}$ and ${}^*\mathbb{R}$ is that $\mathbb{R}$ has no infinite or infinitesmal elements but ${}^*\mathbb{R}$ does.

The Archimedean property says that if $a > 0$ then $\underbrace{a+a+a+\cdots+a}_{n} = na > 1$ for some n.

$(\forall a)(a > 0 \to (a+a>1 \lor a+a+a>1 \lor a+a+a+a>1 \lor \dots))$

This property is not expressible in the first order theory of fields.
$\mathbb{R}$ satisfies this property, ${}^*\mathbb{R}$ does not.

Eg. $\varepsilon = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots) \in \mathbb{R}^\omega$, up to equivalence mod $\mathcal{U}$, defines an infinitesmal in ${}^*\mathbb{R}$.

$n\varepsilon = (n, \frac{n}{2}, \frac{n}{3}, \frac{n}{4}, \dots) \in \mathbb{R}^\omega$, $n\varepsilon < 1$ since this holds for all but the first n terms of the sequence.

$\frac{1}{\varepsilon} = (1, 2, 3, 4, 5, \dots) \in \mathbb{R}^\omega$ defines an infinite element of ${}^*\mathbb{R}$.

Every structure M has a enlargement $^*M$.

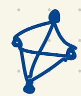
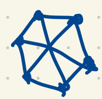
Łoś' Theorem If $M_0, M_1, M_2, \dots \models A$ (statements over a first-order language over L) then the ultraproduct

$$\Big(\prod_{i\in\omega} M_i\Big)/\mathcal{U} \models A.$$

Eg. A = axioms for fields, $M_i = \mathbb{R}$ for all i. $\prod_{i\in\omega} M_i = \{(m_0, m_1, m_2, \dots) : m_i \in M_i\}$.

Eg. L = language of a single binary relation '$\sim$'
A = axioms for ordinary graphs of degree 3

A model of A, $\Gamma \models A$, is an ordinary graph of degree 3.

For each $i \in \omega$, take $\Gamma_i \models A$ eg. $\Gamma_0 =$  , $\Gamma_1 =$  , $\Gamma_2, \Gamma_3, \dots$

$$\prod_{i\in\omega} \Gamma_i = \Gamma_0 \times \Gamma_1 \times \Gamma_2 \times \dots = \{(v_0, v_1, v_2, v_3, \dots) : v_i \in \Gamma_i\}$$

i.e. v_i is a vertex in Γ_i.

$\mathcal{U}$ a nonprincipal ultrafilter on ω

Now $\Big(\prod_{i\in\omega}\Gamma_i\Big)/\mathcal{U}$ is the set of equiv. classes of sequences $v = (v_0, v_1, v_2, \dots)$.

If $v, w \in \underbrace{\Big(\prod_{i\in\omega}\Gamma_i\Big)/\mathcal{U}}_{\Gamma =}$ then $v \sim w$ iff $v_i \sim w_i$ for almost all i i.e. $\{i\in\omega : v_i \sim w_i\} \in \mathcal{U}$.

This graph Γ has degree 3. If Γ_i has order $\leq n$ for some n then Γ is a graph of order $\leq n$. Why? Let θ be the first-order statement that Γ_i has at most n vertices; since $\Gamma_i \models \theta$, $\Gamma := \Big(\prod_{i\in\omega}\Gamma_i\Big)/\mathcal{U} \models \theta$.

You can take the "$*$" operation applied to any standard mathematical object, e.g.

$\mathbb{R} \xrightarrow{*} {}^*\mathbb{R}$, $\quad S \xrightarrow{*} {}^*S \quad ({}^*S = S$ if $|S| < \infty)$.

If $f : \mathbb{R} \to \mathbb{R}$, then ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$ "enlarges" f. (extends) How do we define ${}^*f(\alpha)$ for $\alpha \in \mathbb{R}^*$? α is represented by $(a_0, a_1, a_2, \ldots) \in \mathbb{R}^\omega$.

${}^*f(\alpha)$ is represented by $(f(a_0), f(a_1), f(a_2), \ldots) \in \mathbb{R}^\omega$. The equiv. class of this sequence is well-defined in ${}^*\mathbb{R}$.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Classically,

$$f'(a) = \lim_{t \to 0} \frac{f(a+t) - f(a)}{t}.$$

The nonstandard approach:

$$f'(a) = st\left[\frac{f(a+\varepsilon) - f(\varepsilon)}{\varepsilon}\right]$$ where ε is an infinitesmal

st : bounded hyperreals to reals. "$st(\alpha)$" is the standard part of α, ie. the unique real closest to α (infinitely close).

${}^*\mathbb{R}$ has the order topology which is not metrizable and not separable.

Integrals can be similarly defined in a nonstandard way: if f is Lebesgue integrable then

$$\int_a^b f(t)\,dt = \operatorname{st}\left[\frac{1}{N}\sum_{i=1}^{N} f(a+i\Delta x)\,\Delta x\right]$$

where N is an unbounded hypernatural number

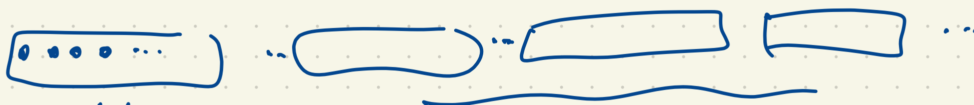
$$\Delta x = \frac{b-a}{N}$$

Hypernatural numbers ${}^*\mathbb{N} = \left(\prod_{i\in\omega}\mathbb{N}\right)/\mathcal{U}$

$\mathbb{N} = \{1, 2, 3, \dots\}$. Sequences $(n_0, n_1, n_2, \dots) \in \mathbb{N}^\omega$ mod $\mathcal{U}$ gives ${}^*\mathbb{N}$.

$\mathbb{N} \subset {}^*\mathbb{N}$

${}^*\mathbb{N}$ looks like



$\mathbb{N}$

" shifted copies of $\mathbb{Z}$ "

$$|{}^*\mathbb{N}| = |{}^*\mathbb{R}| = |\mathbb{R}| = 2^{\aleph_0}.$$

$$2^{\aleph_0} \overset{?}{\leq} |{}^*\mathbb{N}| \leq |\mathbb{N}^\omega| = \aleph_0^{\aleph_0} = 2^{\aleph_0}$$

Given $\alpha \in (0,1)$ (real) consider the sequence $u_\alpha = (\lceil\alpha\rceil, \lceil 2\alpha\rceil, \lceil 3\alpha\rceil, \lceil 4\alpha\rceil, \dots) \in \mathbb{N}^\omega$

If $\alpha < \beta$ in $(0,1)$ then $u_\alpha < u_\beta$ $\quad u_\alpha \not\equiv u_\beta$ mod $\mathcal{U}$.

An example of an elementary statement about $\mathbb{R}$ that has a (possibly) shorter nonstandard proof than standard proof:

<u>Theorem (Sierpinski)</u> If $a_1, \dots, a_k, b$ are positive reals then

$$\left| \left\{ (n_1, \dots, n_k) \in \mathbb{N}^k : \frac{a_1}{n_1} + \frac{a_2}{n_2} + \cdots + \frac{a_k}{n_k} = b \right\} \right| < \infty .$$

This statement was proved using elementary methods by Sierpinski.

A later nonstandard proof by Ross:

Suppose $S = \{ (a_1, \dots, n_k) \in \mathbb{N}^k : \frac{a_1}{n_1} + \cdots + \frac{a_k}{n_k} = b \}$ is infinite. Then

${}^*S$ contains a solution $(n_1, \dots, n_k)$ where not all $n_i \in \mathbb{N}$ (some n_i's are unbounded);

Say $n_1, \dots, n_r \in {}^*\mathbb{N} \setminus \mathbb{N}$; $n_{r+1}, \dots, n_k \in \mathbb{N}$; $1 \le r \le n$. There

$$\underbrace{\frac{a_1}{n_1} + \cdots + \frac{a_r}{n_r}}_{\substack{\text{positive} \\ \text{infinitesimal}}} = \underbrace{b - \frac{a_{r+1}}{n_{r+1}} - \cdots - \frac{a_k}{n_k}}_{\in \mathbb{R} \text{ (bounded)}} ,$$

Contradiction.

We have first-order axioms for group theory.

Axioms for the class of abelian groups:

- axioms of group theory
- $(\forall x)(\forall y)(xy = yx)$

Axioms for class of nonabelian groups

- axioms for group theory
- $(\exists x)(\exists y)(xy \neq yx)$.

There is no first-order axiomatization of the class of cyclic groups.

Cyclic: $(\exists g)(\forall x)(\underbrace{\exists n \in \mathbb{Z}})(x = g^n)$

Not permissible in first order group theory.

If there were a list of axioms A for the theory of cyclic groups then

$\left(\prod_{i\in\omega} C_{i+2}\right)/\mathcal{U}$ is a group of order $2^{\aleph_0}$, not cyclic.

C_{i+2}: cyclic of order $i+2$

$(C_2 \times C_3 \times C_4 \times C_5 \times \cdots)/\mathcal{U}$ is not cyclic.

A shorter argument that the class of cyclic groups is not first order axiomatizable:
Suppose $\mathcal{A}$ is a collection of statements in first order group theory such that
$G \models \mathcal{A}$ iff G is a cyclic groups. There exists an infinite model (additive $\mathbb{Z}$)
so by the Upward Loewenheim-Skolem Theorem, there exist models of arbitrarily
large cardinality. Take any uncountable model $G \models \mathcal{A}$; then G is not cyclic.

Let $\mathcal{A}$ be a set of statements in graph theory such that

$\Gamma \models \mathcal{A}$ iff Γ is a graph of degree 2.

Note: this equivalent to saying Γ is a disjoint union of cycles

Let $\mathcal{A}$ be the axioms for field theory (over the language $0, 1, +, -, \times$).
$\mathbb{F}_p \models \mathcal{A}$ is the field of prime order p; $\overline{\mathbb{F}}_p$ = algebraic closure of $\mathbb{F}_p$

$\overline{\mathbb{F}}_p$ is countably infinite of characteristic p $(\underbrace{1+1+\cdots+1}_{p}=0)$

Let $F = \left(\prod_p \overline{\mathbb{F}}_p\right)/\mathcal{U} = (\overline{\mathbb{F}}_2 \times \overline{\mathbb{F}}_3 \times \overline{\mathbb{F}}_5 \times \overline{\mathbb{F}}_7 \times \cdots)/\mathcal{U}$

Since $\overline{\mathbb{F}}_p \models \mathcal{A}$, ($\overline{\mathbb{F}}_p$ is a field) F is a field. What is it? $F \cong \mathbb{C}$.

$F = \left(\prod_{p \text{ prime}} \overline{\mathbb{F}}_p\right)/\mathcal{U}$ is a field of characteristic zero.

It is algebraically closed. (Each $\overline{\mathbb{F}}_p$ is alg. closed as we described in the first month.)

The theory of alg. closed fields of characteristic zero is uncountably categorical.

$|F| = 2^{\aleph_0}$ (look back four pages) so $F \cong \mathbb{C}$.

Now consider $F = \left(\prod_p \mathbb{F}_p\right)/\mathcal{U} = (\mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_5 \times \mathbb{F}_7 \times \mathbb{F}_{11} \times \cdots)/\mathcal{U}$.

This is a field. It's a subfield of $\mathbb{C}$ (up to isomorphism)

It has characteristic zero. $|F| = 2^{\aleph_0}$. $F \not\cong \mathbb{C}$ since F has irreducible poly's of every degree. (For every $n \geq 1$, there exists a poly. $f(x) \in F[x]$ of degree n which is irreducible. But so what, $\mathbb{Q}$ also has this property.)

$\mathbb{R}[x]$ has irred. poly's of degree 2 but they all give rise to $\mathbb{C}$:

$\mathbb{R}$ has a unique extension field of degree 2. $\mathbb{Q}$ has infinitely many extension fields of degree 2. $\mathbb{F}_p$ has a <u>unique</u> extension of each degree $n \geq 1$.

F is an uncountable field of char. 0 having a unique extension field of each degree $n \geq 1$.

Take a subset $S \subseteq \mathbb{N}^\omega = \{(n_0, n_1, n_2, \dots) : n_i \in \mathbb{N}\}$. Two players, Alice and Bob, take turns picking elements of $\mathbb{N} = \{1,2,3,4,\dots\}$ starting with Alice, resulting in a *play* $x = (a_0, b_0, a_1, b_1, a_2, b_2, \dots) \in \mathbb{N}^\omega$. If $x \in S$, then A wins. If $x \in \mathbb{N}^\omega - S$, B wins.

Eg. S is the set of eventually constant sequences. This has a winning strategy for Bob.

Eg. S is the set of eventually periodic sequences. Bob's advantage.

Eg. S is any countable collection of sequences, i.e. $S \subseteq \mathbb{N}^\omega$, $|S| = \aleph_0$.
Bob has a winning strategy. Enumerate $S = \{s_1, s_2, s_3, \dots\}$. On turn j, Bob chooses any $n \in \mathbb{N}$ which differs from the $2j$-indexed term in s_j.

Eg. S is the set of sequences having no '3,1,4,1,5,9' as subsequence. Alice has a winning strategy.

Eg. S is the set of 'universal' sequences in $\mathbb{N}^\omega$ (sequences containing every finite sequence of natural numbers appears as a consecutive subsequence). Bob can play 2, 2, 2, ... to win.

A *strategy* is a function : $\underbrace{\mathbb{N}^{<\omega}}_{\text{finite strings } (a_0, b_0, \dots)} \to \mathbb{N}$. A strategy for Alice (or Bob) is a *winning strategy* if the player in question is guaranteed to win if they follow that strategy.

a_n or b_n

Axiom of Determinacy (AD): For every $S \subseteq \mathbb{N}^\omega$, either Alice or Bob has a winning strategy for the game G_S.

Theorem (Gale, Stewart) Every open game is determined: either Alice or Bob has a winning strategy.

Topology of $\mathbb{N}^\omega$: $\mathbb{N}$ has the discrete topology : every subset of $\mathbb{N}$ is open.

A basic open set : Given $(x_0, x_1, \cdots, x_{n-1}) \in \mathbb{N}^{<\omega}$, $(x_0, x_1, \cdots, x_{n-1}) \times \mathbb{N}^\omega = \{(x_0, x_1, \cdots, x_{n-1}, x_n, x_{n+1}, \cdots) : x_n, x_{n+1}, x_{n+2}, \cdots \in \mathbb{N}\}$ is a <u>basic open set</u>.

An open set is an arbitrary union of basic open sets.

If $S \subseteq \mathbb{N}^\omega$ is open, then G_S is determined.

The condition that $S \subseteq \mathbb{N}^\omega$ is open means that all winning plays for Alice are determined after a finite number of moves.

An example of an open set is the set of all sequences $(x_0, x_1, x_2, \cdots) \in \mathbb{N}^\omega$ containing any phone number i.e. $x_n = 3077664394$ for some n.

It is $\bigcup_{n=0}^{\infty} U_n$: $U_n = \{(x_0, x_1, \cdots, x_{n-1}, 3077664394, x_{n+1}, x_{n+2}, \cdots) : x_i \in \mathbb{N}\}$

This set is open but not closed : its complement is not open.

Also, if $S \subseteq \mathbb{N}^\omega$ is closed, then G_S. More generally, if $S \subseteq \mathbb{N}^\omega$ is a Borel set, the game G_S is determined.

Consider $S = \{(a_0, b_0, a_1, b_1, \cdots) : b_0$ is odd <u>or</u> there exists $n \in \omega$ such that $b_n \neq b_0 - 2n$ and $(a_0, b_0, a_1, b_1, \cdots, b_{n-1}, a_n) = (a_0, b_0, b_0-1, b_0-2, \cdots, b_0-2n+1)\}$

Alice has a winning strategy.

Typical play: $\begin{array}{ccccccccccc} a_0 & b_0 & a_1 & b_1 & a_2 & b_2 & a_3 & & a_{23} & b_{23} & a_{24} \\ (1, & 48, & 47, & 46, & 45, & 44, & 43, & \cdots, & 3, & 2, & 1) \end{array}$

Alice has won; this game has value 0 ("0 moves remaining for Alice to win").

$(1, 48, 47, \cdots, 3, 2)$ has value 0.

$(1, 48, 47, \cdots, 4, 3)$ has value 1.

$(1, 48, 47, \cdots, 5, 4)$ " ... 1.

$\vdots$

$(1, 48, 47)$ has value 22.

$(1, 48)$ " " "

(1) has value $\omega = \sup\{0, 1, 2, \cdots\}$

$()$ has value $\omega + 1$.

In general, for every position of the game in which Alice has a winning strategy, we assign a value to that position which is an ordinal. '0' means Alice has won already, '1' means 1 move to reach a position of value 0, etc. Some positions will not have any value assigned; these are winning positions for Bob.

The value is defined recursively as follows:

Case I: It's Bob's turn. Position $(a_0, b_0, a_1, b_1, \cdots, a_n)$, $n \geq 0$.

Define the value of $(a_0, b_0, \cdots, a_n)$ to be the sup of the values of $(a_0, b_0, \cdots, a_n, b)$ for $b \in \mathbb{N}$ (if these sequences all have values).

Case II: It's Alice's turn. Position $(a_0, b_0, a_1, b_1, \cdots, a_{n-1}, b_{n-1})$, $n \geq 0$ (ie. $()$ if $n=0$).
This position has value $\alpha+1$ where α is the min value of $(a_0, b_0, a_1, b_1, \cdots, a_{n-1}, b_{n-1}, a)$, $a \in \mathbb{N}$
assuming there exist any such positions of value.

More generally, take any set X and consider games determined by $S \subseteq X^\omega$
(typically $X = \{0,1\}$ or $\mathbb{N}$).

AD is independent of ZF.
AD is inconsistent with AC i.e. $ZF \vdash (AD \rightarrow \neg AC)$.

In ZF + AD:

- Every $X \subseteq \mathbb{R}$ is Lebesgue measurable.
- Every uncountable $X \subseteq \mathbb{R}$ contains a perfect set.
- Every set $X \subseteq \mathbb{R}$ is "almost open": it differs from an open set by a meagre set.

Compare: in ZFC = ZF + AC there exist $X \subseteq \mathbb{R}$ such that X is not Lebesgue measurable.

Theorem (Woodin) $\mathrm{Con}(ZF + AD) \longleftrightarrow \mathrm{Con}(ZFC + \exists \infty$ many Woodin cardinals$)$

whereas: $\mathrm{Con}(ZF + AC) \longleftrightarrow \mathrm{Con}(ZF)$

Gabay, O'Connor 2004: grad students at Cornell

$C = \{(a_0, a_1, a_2, \cdots) : a_i \in \{0,1\}\} = 2^\omega$ Cantor space

is a graph in which $a = (a_0, a_1, a_2, \ldots)$, $b = (b_0, b_1, b_2, \ldots)$ are adjacent iff a, b differ in exactly one coordinate. $d(a,b)$ = no. of coords in which a, b differ $= |\{i \in \omega : a_i \neq b_i\}|$.

This graph has uncountably many connected components, each of which is countable.

(Hamming graph i.e. cube of infinite dimension)

Participants (in the countable case) number themselves using ω as index set.

They pick a vertex $\langle a \rangle \in [a] = \{b \in C : d(b,a) < \infty\}$ = connected component of $a \in C$.

Strategy: Participant $n \in \omega$ observes $[a]$ for the actual sequence and he/she picks the n^{th} coordinate in $\langle a \rangle$.

Hardin & Taylor, 2013: The mathematics of Coordinated Inference – A study of Generalized Hat Problems.

Hardin-Taylor Theorem: Want a predictor $\langle f \rangle_{t_0}$ for $f : \mathbb{R} \to \mathbb{R}$ which is unknown for $t \geq t_0$.
Desired properties:

(a) $\langle f \rangle_{t_0}$ agrees with f on $(-\infty, t_0)$

(b) $\langle f \rangle_{t_0}$ only depends on $f|_{(-\infty, t_0)}$

$\mathbb{R}^{\mathbb{R}} = \{\text{functions } \mathbb{R} \to \mathbb{R}\}$ (not just continuous functions)

(c) $\langle f \rangle_{t_0}$ agrees with f on $[t_0, t_0 + \varepsilon)$ for some $\varepsilon > 0$ depending on t_0.

__Theorem__: In ZFC there exists a predictor $\underset{t_0}{\mathbb{R}} \times \underset{f}{\mathbb{R}^{\mathbb{R}}} \to \mathbb{R}^{\mathbb{R}}$, $(t_0, f) \mapsto \langle f \rangle_{t_0}$

satisfying (a) and (b); and it satisfies (c) for all but countably many $t_0 \in \mathbb{R}$.
More precisely, the set of times $t_0 \in \mathbb{R}$ for which (c) fails, has no infinite descending sequences, (with respect to standard order). This set is countable, nowhere dense (its closure has empty interior) and Lebesgue measure 0.

$S \subseteq \mathbb{R}$ is nowhere dense if $\underset{\text{top. closure in } \mathbb{R}}{\bar{S}}$ does not contain any (a, b) with $a < b$.

eg. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\} \cup \{0\}$ is nowhere dense.

__Proof__ For each $t_0 \in \mathbb{R}$, we have an equiv. relation on $\mathbb{R}^{\mathbb{R}}$ given by $f \overset{t_0}{\equiv} g$ iff f, g agree on $(-\infty, t_0)$.
$[f]_{t_0} = \{g : g \overset{t_0}{\equiv} f\}$. Let $\preccurlyeq$ be a well-ordering of $\mathbb{R}^{\mathbb{R}}$. Define $\langle f \rangle_{t_0}$ = the $\preccurlyeq$-least element of $[f]_{t_0}$
$= \min_{\preccurlyeq} [f]_{t_0}$.

If $t_0 < t_1$ then $[f]_{t_1} \subseteq [f]_{t_0}$ so $\langle f \rangle_{t_1} \succcurlyeq \langle f \rangle_{t_0}$

$$[f]_{t_1} \xrightarrow{\min} \langle f \rangle_{t_1} \qquad [f]_{t_0} \xrightarrow{\min} \langle f \rangle_{t_0}$$

Define $S = \{t \in \mathbb{R} : \underbrace{\langle f \rangle_t \neq \langle f \rangle_{t'}}$ for all $t' > t\}$.
$S = S_f$

i.e. $\langle f \rangle_t \prec \langle f \rangle_{t'}$ for all $t' > t$.

Suppose S contains an infinite descending sequence $t_1 > t_2 > t_3 > \cdots$ in $\mathbb{R}$.
Then $\langle f \rangle_{t_1} \succ \langle f \rangle_{t_2} \succ \langle f \rangle_{t_3} \succ \cdots$ which is not possible since $\preccurlyeq$ is a well-ordering on $\mathbb{R}^{\mathbb{R}}$.

If $\overline{S} \supseteq (a,b)$ with $a<b$ then the intervals $(a + \frac{b-a}{n+1}, a + \frac{b-a}{n})$ $(n = 1, 2, 3, 4, \ldots)$ are mutually disjoint, pick $t_n \in S \cap (a + \frac{b-a}{n+1}, a + \frac{b-a}{n})$ giving an infinite decreasing sequence in S, contradiction. $\square$

Relationship with hat problems: Suppose we play the hat game with participants $0, 1, 2, 3, \ldots$ and hats are assigned from $\{R, B\}$. The winning strategy still works if every participant $n \in \omega = \{0, 1, 2, 3, \ldots\}$ can see the hats only for the participants $m > n$.

Stronger form for any infinite set of participants: Participants can agree beforehand on a well ordering of participants i.e. index themselves by some ordinal α. If each participant $\beta \in \alpha$ can see only the hats on the heads of participants "to the right" $(\beta < \gamma < \alpha)$. See handout.

Caleb's problem: There are three wizards. I put a real number on the forehead of each wizard. Each wizard can see only the numbers on the other two wizards' foreheads. Each wizard may list finitely guesses for the real number on his/her forehead. What strategy guarantees that at least one wizard has a correct guess?