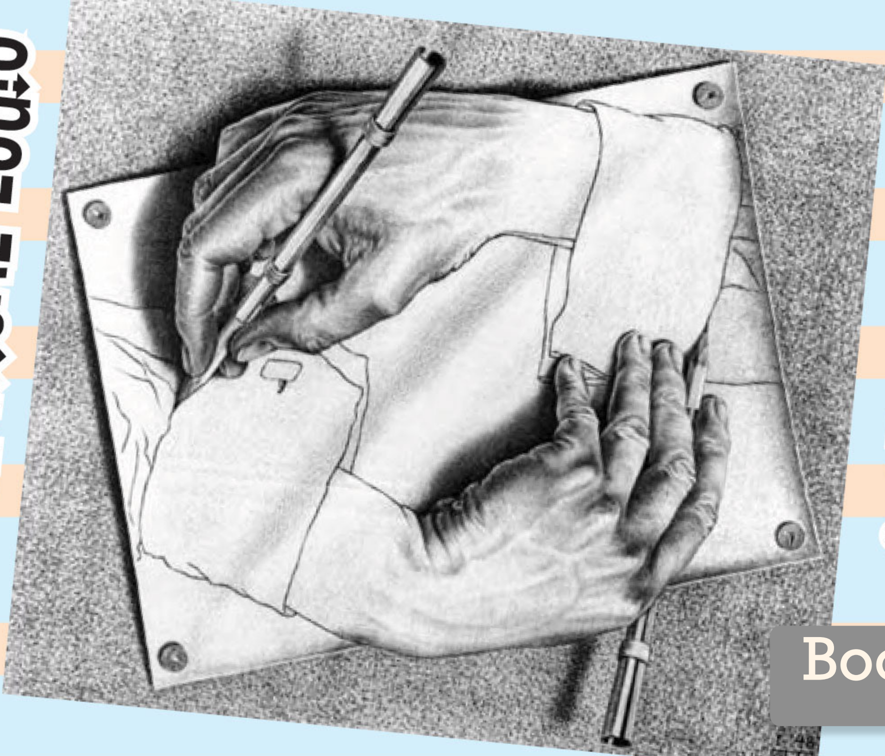
MATHEMATICAL LOGIC & SET THEORY



Book 3

Trivial examples: Fix $x_0 \in X$. Define $\mu(A) = \begin{cases} 0 & \text{if } x_0 \notin A \\ 1 & \text{if } x_0 \in A. \end{cases}$

A <u>measureable cardinal</u> is a uncountable cardinal κ which admits a nontrivial $\boxed{\cancel{\text{countably additive}}}$ two-valued measure.

Does such a κ exist? If so then any larger cardinal satisfies this condition.

Given $\kappa < \kappa'$, μ nontrivial countably additive two-valued measure on κ, lift it to one on κ': $\iota : \kappa \to \kappa'$ injection. Define (for $B \subseteq \kappa'$)

$$\mu'(B) = \mu(\iota^{-1}(B)).$$

<u>Theorem</u> (Ulam) If there exists a nontrivial countably additive two-valued measure on an uncountable set X then let X be a smallest example. Then X has a nontrivial κ-additive two-valued measure for all $\kappa \leq |X|$.

μ is κ-additive if

$$\mu\Big(\bigsqcup_{\alpha \in I} A_\alpha\Big) = \sum_{\alpha \in I} \mu(A_\alpha)$$

for every collection of sets $|I| < \kappa$ $(A_\alpha \subseteq X)$.

A <u>measurable cardinal</u> is an uncountable cardinal κ having a κ-additive two-valued measure.

$$[0,1] = \bigsqcup_{\alpha \in [0,1]} \{\alpha\}$$

Do they exist? And who cares?

$$\begin{array}{ccccccccc}
 & & \overset{\text{open}}{\Sigma^0_1} & & & & \overset{\text{ctble unions of closed sets}}{\Sigma^0_2} & & \\
\underset{\text{clopen}}{\Delta^0_1} & \subset & & \subset & \Delta^0_2 & \subset & & \subset & \Delta^0_3 \quad \cdots \\
 & & \underset{\text{closed}}{\Pi^0_1} & & & & \underset{\text{ctble intersections of open sets}}{\Pi^0_2} & &
\end{array}$$

Projective Hierarchy $\underset{\sim}{\Sigma}^1_n$, $\underset{\sim}{\Pi}^1_n$, $\underset{\sim}{\Delta}^1_n = \underset{\sim}{\Sigma}^1_n \cap \underset{\sim}{\Pi}^1_n$

$$\begin{array}{ccccccc} & & \underset{\sim}{\Sigma}^1_1 & & & & \underset{\sim}{\Sigma}^1_2 \\ \underset{\sim}{\Delta}^1_0 & \subset & & \supset\ \underset{\sim}{\Delta}^1_1 = \underset{\sim}{\Sigma}^1_1 \cap \underset{\sim}{\Sigma}^1_1 & \subset & & \cap \\ \text{Borel sets} & \cap & \underset{\sim}{\Pi}^1_1 & ? & & \cap\ \underset{\sim}{\Pi}^1_2 & \subset \end{array}$$

$\underset{\sim}{\Sigma}^1_1 = \{$analytic sets in $X\}$ $\quad A \in \underset{\sim}{\Sigma}^1_1$ iff A is a continuous image of a Borel set under $f: Y \to X$ (f continuous, Y Polish space)

$\underset{\sim}{\Pi}^1_1 = \{$coanalytic sets in $X\} = \{$complements of analytic sets$\}$

$\underset{\sim}{\Sigma}^1_2 = \{$continuous images of coanalytic sets$\}$

If there exist measurable cardinals, then every $\underset{\sim}{\Sigma}^1_2$-set is Lebesgue measureable.

Coming to: an application a large cardinal to the finite world.

Non-associative algebra : Keis, Quandles, Racks, Shelves, ... (see Sam Nelson, Quandles)

A <u>kei</u> is a set S with a binary operation $\triangleright$ satisfying : for all $x, y, z \in S$,

(1) $x \triangleright x = x$ (every element is idempotent)

(2) $(x \triangleright y) \triangleright y = x$ ($x \mapsto x \triangleright y$ is involutory)

(3) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ ("$\triangleright$" is right-distributive over itself)

If $(S, \triangleleft)$ satisfies (3), it is a <u>shelf</u> (or <u>self distributive system</u>). If it satisfies (1) and (3), it is a <u>rack</u>.

If $(S, \triangleleft)$ satisfies (1), (3) and (2') it is a <u>quandle</u>.

(2'): For all y, the map $S \to S$, $x \mapsto x \triangleright y$ is injective.

(1) $x \triangleright x = x$

(2) $(x \triangleright y) \triangleright y = x$

(3) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

The kei axioms are equivalent to the Reidemeister moves I, II, III.

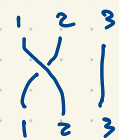
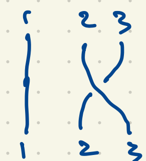
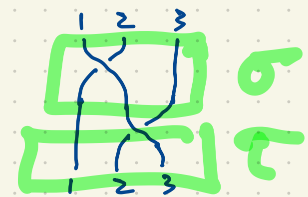
Examples: Fix $c \in \mathbb{R}$ and define $x \triangleright y = cx + (1-c)y$ for $x, y \in \mathbb{R}$. This gives a rack (satisfying (1), (3)). It's a kei if $c = \pm 1$. (?)

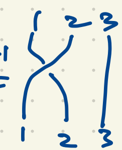
More generally let V be a vector space and $R \in GL(V)$ invertible linear transformation. For $u, v \in V$, $u \triangleright v = Ru + (I - R)v$. This is an Alexander quandle. (Sometimes a kei).

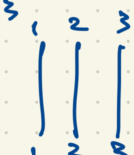
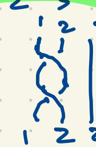
Example Let G be a group (multiplicative). Fix $n \in \mathbb{Z}$.
For $a, b \in G$, $a \triangleright b = b^n a b^{-n}$ (n-fold conjugation of a by b). This is a rack, sometimes a quandle.

Example The Braid group B_n

eg. in B_3,

$\sigma =$  , $\tau =$  , $\sigma\tau =$ 

$\sigma^{-1} =$ 

$\iota =$  $\neq \sigma^2 =$ 

$S_n = \mathrm{Sym}\{1, 2, \cdots, n\}$

$|S_n| = n!$

$B_n \twoheadrightarrow S_n$ epimorphism

$|B_n| = \aleph_0$.