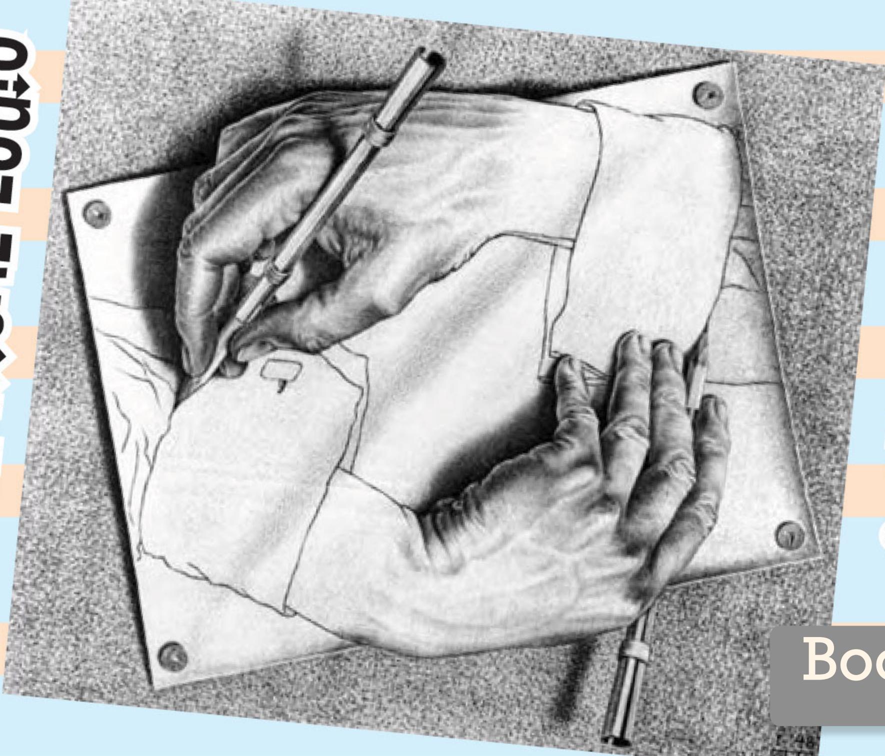
MATHEMATICAL LOGIC & SET THEORY



Book 2

Łoś-Vaught Test assures us that $Th(ACF_0)$ is complete. This uses: the theory has no finite models; and the theory is $2^{\aleph_0}$-categorical.

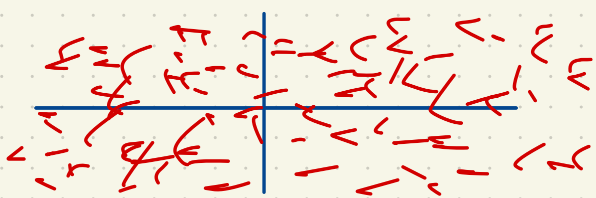
Ł ł Jerzy Łoś, Robert Vaught (1954)

C-words

Algebra	Topology		Logic
closed	(Cauchy) complete	No	(model) complete
	continuous		compact
No	convergent	yes	categorical
	connected		
	compact		
	closed		

Let L be a language and let X be the collection of all L-structures.
For any set of sentences Σ over L, let K_Σ = set of of L-structures satisfying all the sentences in Σ (i.e. the set of models of Σ).
Then X is a top. space with K_Σ as its basic closed set.
This space is (topologically) compact. $\{K_\theta : \theta \text{ sentence over } L\}$ are sub-basic closed sets.

Eg. $K = \mathbb{Q}[\sqrt{2}] = \{a+b\sqrt{2} : a,b \in \mathbb{Q}\}$ has two field automorphisms, $\iota(a+b\sqrt{2}) = a+b\sqrt{2}$, $\tau(a+b\sqrt{2}) = a-b\sqrt{2}$.



$\mathbb{C}$ has uncountably many automorphisms but only two of them are continuous.
Where do we get this?

$$\mathbb{C} \subset \mathbb{C}[x] \subset \mathbb{C}(x) = K \subset \bar{K}$$

The polynomial ring $\mathbb{C}[x]$ has automorphisms $f(x) \longmapsto f(x+a)$

$K = \mathbb{C}(x) = \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in \mathbb{C}[x] \right\}$

is a field extension of $\mathbb{C}$ and it's not alg. closed.

$K[t]$ has irreducible polys eg. $t^2 - x \in K[t]$

$\bar{K}$ is an alg. closed field of char. 0, $|\bar{K}| = 2^{\aleph_0} = |\mathbb{C}|$

But there is only one alg. closed field of char. 0 for each uncountable cardinality (the theory of ACF_0 is uncountably categorical) so $\bar{K} \cong \mathbb{C}$.

$\bar{K}$ has lots of automorphisms i.e. $\mathbb{C}$ has lots of automorphisms.

$\mathbb{R}$ has only one automorphism, the identity $\iota(a) = a$.

Axioms for $\mathbb{R}$?

Field axioms

+ Order axioms

and axioms

Introduce a new binary relation symbol '$<$' ($a<b$ is a shorthand for $R(a,b)$)

$(\forall a)(\forall b)\left[\left((a<b) \vee (a=b) \vee (b<a)\right) \wedge \neg[(a<b) \wedge (b<a)] \wedge \neg[(a<b) \wedge (a=b)] \wedge \neg[(b<a) \wedge (a=b)]\right]$

$(\forall a)(\forall b)(\forall c)\left(((a<b) \wedge (b<c)) \rightarrow (a<c)\right)$

$(\forall a)(\forall b)(\forall c)\ \left((a<b)\to\left[(a+c<b+c)\wedge\ (c>0)\to(ac<bc)\right]\right)$

$\mathbb{R}$ is the unique ordered field which is (Cauchy)-complete and having $\mathbb{Q}$ as a dense subfield.

But we cannot state "Cauchy complete" in first order theory of fields.

How much of the theory of $\mathbb{R}$ can be captured in first order logic?

Ordered field axioms

- $(\forall a)(a\neq 0\to a^2>0)$
- $(\forall a)(a>0\to(\exists b)(b^2=a))$
- Every polynomial $f(x)\in\mathbb{R}[x]$ of odd degree has a root. Eg. for degree 3 $(\forall a)(\forall b)(\forall c)(\exists x)\ (x^3+ax^2+bx+c=0)$.

RCF

The first order theory of $\mathbb{R}$ is complete.

However the theory is not κ-categorical for any cardinality κ. (No models for κ finite; more than one for each infinite κ.)

Eg. for $\kappa=\aleph_0$: $\bar{\mathbb{Q}}\cap\mathbb{R}$

For $\kappa=2^{\aleph_0}$: $\mathbb{R}$; hyperreals ${}^*\mathbb{R}$

Any model of RCF is a real closed field.

Every real closed field is <u>elementarily equivalent</u> to $\mathbb{R}$ (i.e. has the same first order theory).

$\bar{\mathbb{Q}}$ and $\mathbb{C}$ are elementarily equivalent.

Emil Artin (1927) proved the Hilbert 17^{th} problem using mathematical logic.

Hilbert's 17th Problem

Let $f(x_1,\dots,x_n) \in \mathbb{R}[x_1,\dots,x_n]$, such that $f \geq 0$ (i.e. $f(x_1,\dots,x_n) \geq 0$ for all $x_1,\dots,x_n \in \mathbb{R}$). Is it necessary that $f = s_1^2 + \dots + s_k^2$ for some rational functions $s_i(x_1,\dots,x_n) \in \mathbb{R}(x_1,\dots,x_n)$? (Pfister: $k \leq 2^n$)

Motzkin's example: $n=2$. $f(x,y) = 1 - 3x^2y^2 + x^2y^4 + x^4y^2 \geq 0$ This is not expressible as a sum of squares of poly's but

$$f(x,y) = \left[\frac{x^2y(x^2+y^2-2)}{x^2+y^2}\right]^2 + \left[\frac{xy^2(x^2+y^2-2)}{x^2+y^2}\right]^2 + \left[\frac{xy(x^2+y^2-2)}{x^2+y^2}\right]^2 + \left[\frac{x^2-y^2}{x^2+y^2}\right]^2.$$

Note: $\frac{1+x^4y^2+x^2y^4}{3} \geq (1\cdot x^4y^2\cdot x^2y^4)^{\frac{1}{3}} = x^2y^2$ by the arithmetic-geometric mean inequality

so $f(x,y) \geq 0$ for all x,y.

If $f = s_1^2 + \dots + s_k^2$ for some $s_i(x,y) \in \mathbb{R}[x,y]$ then $\deg s_i \leq 3$, so $s_i(x,y)$ may have terms

$1, x, y, x^2, xy, y^2, \cancel{x^3}, \cancel{x^2y}, \cancel{xy^2}, \cancel{y^3}$

$s_i(x,y) = a_i + b_i x + c_i y + d_i xy + e_i x^2 + f_i y^2$

$s_i^2 = \underline{2d_i} xy + \cdots$

In $\mathbb{R}$, the positive elements are squares.
(Not true in $\mathbb{Q}$)

Consequence: $|\mathrm{Aut}\,\mathbb{R}| = 1$. If $\phi \in \mathrm{Aut}\,\mathbb{R}$ i.e. $\phi: \mathbb{R} \to \mathbb{R}$ is bijective and $\phi(a+b) = \phi(a) + \phi(b)$, $\phi(ab) = \phi(a)\phi(b)$ for all $a,b \in \mathbb{R}$

then $\phi(a) = a$ for all $a \in \mathbb{R}$. Why? $\phi(a^2) = \phi(a)^2$ so $\phi(a) > 0$ iff $a > 0$.

So $\phi(a) < \phi(b) \iff a < b$.

$\iff \phi(b) - \phi(a) > 0$

$\iff \phi(b-a) > 0$

$\iff b-a > 0$

$\iff a < b$.

$\phi(0) = 0$

$\phi(1) = 1$

$\phi(2) = \phi(1+1) = \phi(1) + \phi(1) = 1+1 = 2$

$\vdots$

$\phi(n) = n$

$\phi(a) = a$ for all $a \in \mathbb{Q}$

$\phi(a) = a$ for all $a \in \mathbb{R}$.

Compare: $\mathbb{Q}[\sqrt{2}]$ is also an ordered field but it has a nontrivial automorphism $\phi(a+b\sqrt{2}) = a - b\sqrt{2}$ for all $a,b \in \mathbb{Q}$.

Hilbert's 17th problem is true for $n=1$: every $f(x) \in \mathbb{R}[x]$ with $f(x) \geq 0$ for all x satisfies $f(x) = g(x)^2 + h(x)^2$ for some $g(x), h(x) \in \mathbb{R}[x]$. Why? Factor

$$f(x) = \lambda \prod_{i=1}^{m} (x - r_i)^2 \cdot \prod_{j=1}^{n} \left((x-s_i)^2 + t_i^2 \right) \quad \text{where } \lambda \geq 0,\ \lambda = a^2$$

$$(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$$

Proof of Hilbert's 17th Problem (Artin; Serre)

Let $f = f(x_1, \dots, x_n) \in \mathbb{R}[x_1, \dots, x_n]$. Suppose f is not a sum of squares of rational functions; we must show $f(a_1, \dots, a_n) < 0$ for some $a_1, \dots, a_n \in \mathbb{R}$.

$F = \mathbb{R}(x_1, \dots, x_n)$ = field of rational functions in $x_1, \dots, x_n$ with real coefficients.

$T = \{$ sums of squares of rational functions in $F \}$.
$= \{ s_1^2 + \cdots + s_k^2 : s_i \in F \}$. Note: $T + T \subseteq T$, $TT \subseteq T$, $a^2 \in T$ for all $a \in F$.

T defines a <u>preorder</u> on F, namely for $g, h \in F$, we say $g \leq h$ iff $h - g \in T$.
"$\leq$" is transitive but it's a partial order in general.

It's an order (total order) iff $T \cup (-T) = F$ and $T \cap (-T) = \{0\}$.

$-T = \{-g : g \in T\}$

We are assuming $f \notin T$.
Among all preorders containing T but not containing f, choose a maximal preorder P using Zorn's Lemma.
Let $\{P_\alpha : \alpha \in A\}$ be a totally ordered collection of preorders on F with $P_\alpha \supseteq T$, $f \notin P_\alpha$.
(i.e. for every $\alpha, \beta \in A$, either $P_\alpha \subseteq P_\beta$ or $P_\beta \subseteq P_\alpha$)
($\{P_\alpha\}$ is a chain) Then $P = \bigcup_{\alpha \in A} P_\alpha$ is an upper bound for the chain i.e. $P_\alpha \subseteq P$ for all $\alpha \in A$. Then P is a preorder ($P + P \subseteq P$, $PP \subseteq P$, $a^2 \in P$) and $P \supseteq T$, $f \notin P$.
By Zorn's Lemma there exists a maximal preorder P as above.

(i) Show $-1 \notin P$. If $-1 \in P$ then $f = \left(\frac{1+f}{2}\right)^2 + (-1)\left(\frac{1-f}{2}\right)^2 \in P$, a contradiction.

(ii) Show $-f \in P$. Suppose $-f \notin P$ and consider $\tilde{P} = P - Pf = \{a - bf : a, b \in P\}$ which is a preorder.

$$\tilde{P} + \tilde{P} = \{(a_1 - b_1 f) + (a_2 - b_2 f) = \underset{\substack{\Uparrow \\ \tilde{P}}}{(a_1 + a_2)} - \underset{\substack{\Uparrow \\ P}}{(b_1 + b_2)} f : a_i, b_i \in P\} \subseteq \tilde{P}$$

$\tilde{P}\tilde{P}$:

$$(a_1 - b_1 f)(a_2 - b_2 f)$$
$$= \underbrace{(\underbrace{a_1 a_2} + \underbrace{f^2} \cdot \underbrace{b_1 b_2})}_{\Uparrow\, P} - \underbrace{(a_1 b_2 + a_2 b_1)}_{\Uparrow\, P} f \in \tilde{P}$$

$\tilde{P} \supset P$ $\qquad -f \notin P$
$\qquad -f \in \tilde{P}$

By maximality of P, $f \in \tilde{P}$.

$f = a - bf$, some $a, b \in P$. $(1+b)f = a \Rightarrow f = \frac{a}{1+b} = \underset{\in P}{(1+b)a \cdot \frac{1}{(1+b)^2}}$

(iii) Given $g \in F$, show $g \in P$ or $-g \in P$.

Assume $g \notin P$; show $-g \in P$. WLOG $g \neq 0$.

Consider $\tilde{P} = P + Pg$. As in (ii) $\tilde{P}$ is a preorder, $\tilde{P} \supseteq P$, $\tilde{P} \supset P$ since $g \notin P$, $g \in \tilde{P}$. By maximality of P, we must have $f \in \tilde{P}$ so $f = a + bg$. some $a, b \in P$.

$$-bg = a - f \Rightarrow -g = \frac{a-f}{b} = b \cdot (a-f) \cdot \left(\frac{1}{b}\right)^2 \in P$$

(iv) $P \cap (-P) = \{0\}$ If $g \neq 0$, $g \in P$, $-g \in P$ then

$$-1 = g \cdot (-g) \cdot \left(\frac{1}{g}\right)^2 \in P,$$ contrary to (i).

$(F, \leq)$ is an ordered field where $a \leq b \Longleftrightarrow b - a \in P$.

It's an extension of $(\mathbb{R}, \leq)$

By the Tarski Transfer Principle, if $(x_1, \dots, x_n) \in F^n$ satisfies a statement in first order theory of ordered fields, then there is $(a_1, \dots, a_n) \in \mathbb{R}^n$ realizing this statement.

Here $-f \in P$ ie. $f < 0$ i.e. $f(x_1, \dots, x_n) < 0$ so $f(a_1, \dots, a_n) < 0$ for some $a_1, \dots, a_n \in \mathbb{R}$.

Indiscernibles ... coming soon

Axioms for projective plane geometry: Here we consider only points, lines and their incidences.

Objects: points and lines

Relations: $\underbrace{P(\cdot),\ L(\cdot)}_{\text{unary relation symbols}},\ \underbrace{I(\cdot,\cdot)}_{\text{binary relation symbol}}$

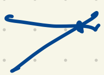
$$(\forall x)(P(x) \leftrightarrow (\neg L(x)))$$

$$(\forall x)(\forall y)(I(x,y) \rightarrow (P(x) \leftrightarrow L(y)))$$

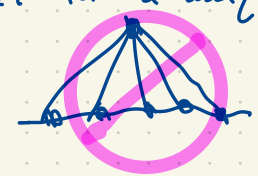
Axioms: (i) Any two distinct points are on a unique line.

$$(\forall x)(\forall y)(P(x) \wedge P(y) \wedge \neg(x=y) \rightarrow (\exists z)(I(x,z) \wedge I(y,z) \wedge (\forall w)((I(x,w) \wedge I(y,w)) \rightarrow (w=z))))$$

(ii) Any two distinct lines meet in a unique point.

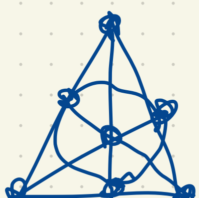


(iii) nondegeneracy axiom



There exist at least four points with no three of them collinear.

Models? There are some orders (sizes) for which models are unique up to isomorphism

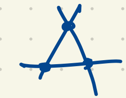


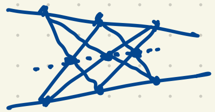
7 points
7 lines
3 points/line
3 lines/point

Finite projective planes:
n^2+n+1 points / lines
$n+1$ points / line
$n+1$ lines / point
n = order of the plane

Infinite planes:
For every infinite cardinal κ, there are many proj. planes of order κ (with cardinality κ).

Does there exist an infinite projective plane which is $\aleph_0$-categorical i.e. its theory has a unique countable model?

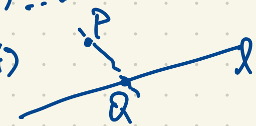




Generalized Quadrangles

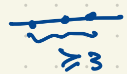
(i) . . . . Any two points are on at most line

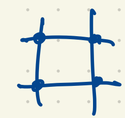
(ii)



If P is not on l then there is a unique Q on l joined to P.

(iii) nondegeneracy.





In every case

Can $s<\infty$, $t=\infty$?

If $s=2$ then $t\leq 4$ (easy).

If $s=3$ then $t\leq 9$ (4 pages)

If $s=4$ then $t\leq 16$ (Cherlin)

Let A be a set of first order sentences over a language L (i.e. a theory) and let $M \models A$ (a model of A).

A set of indiscernibles $S \subseteq M$ such that for every distinct $s_1, \dots, s_k \in S$ and $t_1, \dots, t_k \in S$ and every propositional function $\phi(x_1, \dots, x_k)$, $\phi(s_1, \dots, s_k)$ iff $\phi(t_1, \dots, t_k)$.

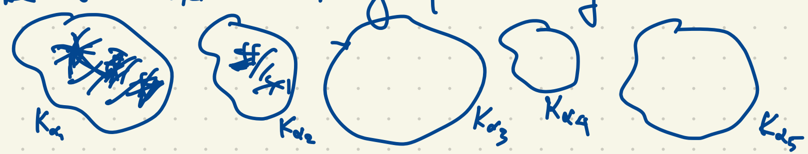
Eg. Let A be the axioms of field theory, $\mathbb{C} \models A$. Let S be any algebraically independent subset of $\mathbb{C}$. This means that for all distinct $s_1, \dots, s_k \in S$ and nonzero $f(x_1, \dots, x_k) \in \mathbb{Q}[x_1, \dots, x_k]$ then $f(s_1, \dots, s_k) \neq 0$.

eg. $\{\pi\}$, $\{e\}$. There are alg. ind. subsets of $\mathbb{C}$ of uncountable size!

Is $\{\pi, e\}$ alg. indep.?

Any set $S \subseteq \mathbb{C}$ which is alg. indep. is a set of indiscernibles.

Let A be the axioms of graph theory. Consider a graph $\Gamma \models A$ that looks like



where $\alpha_1, \dots, \alpha_5$ are infinite cardinals

Pick $s_1 \in K_{\alpha_1}, \dots, s_5 \in K_{\alpha_5}$.

$\{s_1, \dots, s_5\}$ is a set of indiscernibles.