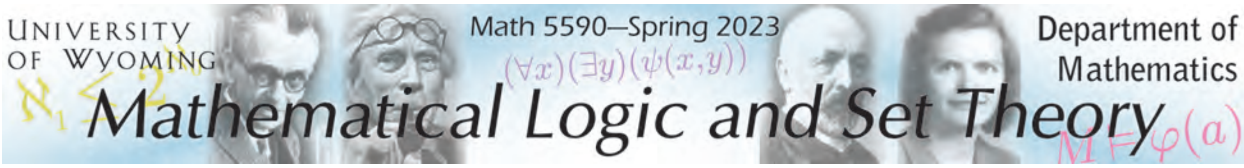

HW Problems

Instructions: The following is a list of problems of varying difficulty, to which I will add over the coming days. Submit solutions to problems of your choice on WyoCourses, as you are able. There is no need to complete all problems, but you are welcome to submit additional solutions as separate pdf documents as you are able. As usual, problems may be discussed with other students, but submitted solutions should be your own work.

1. (*Extensionality*) The *axiom of extensionality* (AE) is used in ZFC set theory to assert that two sets having the same elements, must be the same set. This language has a single binary relation symbol '$\in$' (not an actual relation, but a symbol whose intended interpretation is 'is an element of'). The axiom is as follows:

$$\text{AE:}\quad (\forall x)(\forall y)[((\forall z)((z\in x)\leftrightarrow(z\in y)))\rightarrow(x=y)]$$

Which of the following structures satisfy the axiom AE?

(a) $\mathbb{R}$ with the usual '$<$' relation. The question here is, if we take $\mathbb{R}$ as the underlying set of the structure, and interpret '$\in$' as the ordinary relation of 'strictly less than', is the axiom AE satisfied?

(b) $\mathbb{R}$ with the usual '$\leqslant$' relation

(c) $\mathbb{R}$ with the usual '$>$' relation

(d) $\mathbb{N}=\{1,2,3,4,\ldots\}$ with the usual divisibility relation '$\mid$'. Here $a\mid b$ iff there exists c such that $b=ac$.

2. (*The Order Relation is Algebraically Definable*) In class we pointed out that the usual order relation '$\leqslant$' on $\mathbb{R}$ may be defined algebraically; that is,

$$\mathbb{R}\models(\forall x)(\forall y)((x\leqslant y)\leftrightarrow(\exists z)(x+z^2=y))$$

where z^2 is an abbreviation for zz. Thus it makes very little difference whether we regard $\mathbb{R}$ (with its usual addition, multiplication, etc.) as a field (with the algebraic axioms for $+$, $-$, $\times$, 0, 1) or as an ordered field (with additional axioms for its order relation). An important consquence of this is that $\mathbb{R}$ has no nontrivial algebraic automorphisms. (That is, if $\theta:\mathbb{R}\rightarrow\mathbb{R}$ is any bijection which preserves addition and multiplication, then θ must also preserve order, and consequently θ is the identity map.) Show that the order relation on $\mathbb{Z}$ can also be defined algebraically. That is, find a formula $\varphi(x, y)$ in the first order theory of rings, such that

$$\mathbb{Z} \models (\forall x)(\forall y)((x \leqslant y) \leftrightarrow \varphi(x, y)).$$

The point is that $\phi(x, y)$ is expressed using quantifiers (over $\mathbb{Z}$) together with the usual symbols of propositional logic ($\neg$, $\vee$, $\wedge$) and the ring symbols 0, 1, $+$, $-$, $\times$.

3. (*First Order Properties of Graphs*) We consider graphs as structures defined over a language with one binary relation symbol '$\sim$', having just two axioms, IR ('irreflexive') and SY ('symmetric'), thus:

$$\begin{aligned} &\text{IR:} \quad (\forall x)(\neg(x \sim x)) \\ &\text{SY:} \quad (\forall x)(\forall y)((x \sim y) \rightarrow (y \sim x)) \end{aligned}$$

Which of the following graph properties are expressible in first order logic? Explain.

(a) The requirement that a graph contains a triangle.

(b) The requirement that a graph contains no triangles.

(c) The requirement that a graph is connected. (A graph is connected if for any two vertices x and y, there is a path $x \sim v_1 \sim v_2 \sim \cdots \sim v_k \sim y$.)

(d) The requirement that a graph is not connected.

(e) The requirement that a graph has diameter 3. (This says that the maximum distance between any two vertices is 3. The distance between two vertices is 3 if there is a path of three edges from one vertex to the other, and there is no shorter path than this between the two given vertices.)

(f) The requirement that a graph is connected but its diameter is infinite. (This means that any two vertices are joined by a path, but there is no upper bound for the distance between two vertices.)

4. (*More Examples of $\aleph_0$-Categorical Graphs*) Examples of $\aleph_0$-categorical graphs considered in class include the countable complete graph, and the countable random graph R (the Erdős-Rényi graph, also known as the Rado graph). These examples have diameter 1 and 2, respectively. Here we look for other examples. See #3 where graphs are described as structures over a language with a single relation symbol '$\sim$' having axioms IR and SY. Find an example of an $\aleph_0$-categorical graph of diameter 3. (In your answer, you should give an example of a graph Γ having diameter 3, with a countably infinite number of vertices, such that $\Gamma \models T$ for some set T consisting of sentences in first order graph theory, for which every countable model of T is isomorphic to Γ. Here T may include a statement asserting that the diameter is 3; but T

may include other properties as well. Clearly, there exist non-isomorphic countable graphs of diameter 3, so T cannot simply consist of one statement asserting that the diameter is 3.)

5. (*Application of the Compactness Theorem*) There exist functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x+y) = f(x) + f(y)$ for all x, y, with $f(1) = 0$ and $f(\alpha) = 1$ where $\alpha = \sqrt{2}$. In fact there are *many* such functions, all of them discontinuous everywhere. Here is an outline of the usual proof: Recall that $\mathbb{R}$ is an infinite-dimensional vector space over $\mathbb{Q}$. Since α is irrational, $\{1, \alpha\}$ is a linearly independent set of size 2 in this space. By Zorn's Lemma (see the handout on our course website), we may extend $\{1, \alpha\} \subset \mathcal{B}$ where $\mathcal{B}$ is a basis for $\mathbb{R}$ over $\mathbb{Q}$. Finally, given any $a \in \mathbb{R}$, denote by $f(a)$ the coefficient of α in the unique expression of a as a $\mathbb{Q}$-linear combination of the elements of $\mathcal{B}$.

   Give *an alternative proof, using the Compactness Theorem for First Order Logic*, using the following proof outline. Start with a language $\mathcal{L}$ containing

   - Four binary function symbols $+$, $-$, $\times$, $/$ for the four basic operations. However we simply abbreviate $x \times y$ as xy. The values $a/0$ can be disregarded since a/b is never used unless $b \neq 0$.
   - A collection of constant symbols c_a, one for each $a \in \mathbb{R}$. We will abbreviate $c_0 = 1$, $c_1 = 1$, $c_{\sqrt{2}} = \alpha$.
   - Another collection of constant symbols f_a, one for each $a \in \mathbb{R}$.

   Let $\mathcal{A}$ be the following set of axioms:

   - The usual field axioms. Here we include the usual commutative, associative and distributive laws, and the axioms for 1 and 0. I won't list them all, but the list would include
   $$\neg(0 = 1);$$
   $$(\forall x)((1x = x) \wedge (0 + x = x));$$
   $$(\forall x)(\forall y)(\forall z)(x(y+z) = xy + xz);$$
   $$(\forall x)(\forall y)(\neg(y = 0) \to (x/y)y = x);\text{ etc.}$$
   - Axiom Schema: For every pair of real numbers $a, b \in \mathbb{R}$, list the axioms $c_a + c_b = c_{a+b}$ and $c_a c_b = c_{ab}$. Yes, this is a huge collection of $2^{\aleph_0}$ separate axioms; but the size of our list of axioms is not a problem. Including them is an easy way to 'hardwire' every model to contain a copy of the reals. (Models don't have to be isomorphic to the reals, but they will necessarily be extension fields of $\mathbb{R}$, i.e. every model will have a subfield isomorphic to $\mathbb{R}$.)
   - Axiom Schema for the values of f: For every pair of real numbers $a, b \in \mathbb{R}$, list a separate axiom $f_a + f_b = f_{a+b}$. Also list the axioms $f_1 = 0$ and $f_\alpha = 1$.

We need to show that the set of axioms $\mathcal{A}$ is consistent, i.e. that it has a model. By the Compactness Theorem, it suffices to show that every finite subset of the axioms $\mathcal{A}_0 \subset \mathcal{A}$ has a model. The point is, such a subset only contains finitely many axioms of the form $f_a + f_b = f_{a+b}$. The values of a and b appearing in these axioms in $\mathcal{A}_0$, together with the values 1 and $\sqrt{2}$, lie in a finite dimensional subspace of $\mathbb{R}$ over $\mathbb{Q}$. One can argue as above, except that Zorn's Lemma is no longer needed since we are in a finite dimensional vector space. Fill in the missing details of this argument! Please pay attention to why it is that the axiom schema for the f-values (a list of $2^{\aleph_0}$ separate axioms) cannot be replaced by a single quantified formula $(\forall a)(\forall b)(f_a + f_b = f_{a+b})$; nor can we replace the set of constant symbols f_a by a function symbol f with the axiom $(\forall x)(\forall y)(f(x{+}y) = f(x) + f(y))$.

6. (*Ramsey's Theorem*) Given a set V, the *complete graph* on V is the graph K_V whose vertices are the elements of V, in which every pair $\{x, y\}$ of vertices $x \neq y$ in V is an edge. In particular if $|V| = n$ then we have the complete graph K_n of order n, with $\binom{n}{2}$ edges; and the countable complete graph $K_{\mathbb{N}}$ with vertex set $\mathbb{N} = \{1, 2, 3, \ldots\}$. We fix a set of k distinct colors, which we use to color the edges of a given graph Γ (each of the edges of Γ being assigned arbitrarily one of the k colors). The following are two versions of Ramsey's Theorem.

   (*Infinite Ramsey Theorem*) Given any k-coloring of the edges of $K_{\mathbb{N}}$, there exists an infinite subset $A \subseteq \mathbb{N}$ such that the subgraph $K_A \subseteq K_{\mathbb{N}}$ is monochromatic, i.e. every edge having both its endpoints in A has the same color.

   (*Finite Ramsey Theorem*) Given any positive integers k and n, there exists a positive integer $R = R_{k,n}$ (depending on both k and n) such that for every $N \geqslant R$ and every k-coloring of the edges of K_N, there is a monochromatic n-clique in K_N, i.e. a complete subgraph with n vertices in K_N, all of whose edges have the same color.

   First show that the Infinite Ramsey Theorem follows easily (almost trivially) from the Finite Ramsey Theorem. Then, using the Compactness Theorem, show that the Finite Ramsey Theorem follows also from the Infinite Ramsey Theorem.

   Really? Shouldn't it be the other way around? No, we stated the problem correctly, because the equivalence of the two theorems is most readily understood in contrapositive form. The *negation* of the Infinite Ramsey Theorem would say that there exists a k-coloring of the edges of $K_{\mathbb{N}}$ for which none of the infinite subcliques $K_A \subseteq K_{\mathbb{N}}$ are monochromatic. This would mean that for every n, there exist arbitrarily large integers N such that the edges of K_N can be k-colored without allowing any monochromatic n-cliques. That would give us the negation of the Finite Ramsey Theorem.

Conversely, you will want to assume the negation of the Finite Ramsey Theorem, and conclude from this the negation of the Infinite Ramsey Theorem. Write down a set of axioms which describe a k-coloring of the edges of $K_{\mathbb{N}}$ for which none of the infinite subcliques $K_A \subseteq K_{\mathbb{N}}$ are monochromatic. Use your hypothesis (the negation of the Finite Ramsey Theorem) to say that every finite subset of these axioms can be satisfied. Then use the Compactness Theorem to conclude that the entire list of axioms can be satisfied; and this gives the negation of the Infintie Ramsey Theorem as the desired conclusion.

7. (*Compactness in Topology and in Logic*) Finish the discussion, started in class, explaining why the the Compactness Theorem of First Order Logic can be viewed as a statement of topological compactness. Here is an outline, which you can use as a starting point. A collection $\mathcal{C}$ of subsets of X has the *finite intersection property* (*f.i.p.*) if $C_1 \cap \cdots \cap C_k \neq \varnothing$ for every finite list of sets $C_1, \ldots, C_k \in \mathcal{C}$. A topological space X is *compact* iff $\bigcap \mathcal{C} \neq \varnothing$ whenever $\mathcal{C}$ is a collection of closed sets in $\mathcal{C}$ with the finite intersection property. You should verify, using De Morgan's Laws, that this condition is equivalent to the usual definition of compactness (namely that every open cover of X has a finite subcover). Now let $\mathcal{L}$ be a first order language, and let X be a collection of $\mathcal{L}$-structures. (Strictly, we should ask X to be a *set* of $\mathcal{L}$-structures. If X were the collection of *all* $\mathcal{L}$-structures, that would typically be a proper class rather than a set; and then strictly speaking, we wouldn't call X a topological space. But we could call it a 'large topological space' and the same idea would go through.) For every $\mathcal{L}$-sentence θ, let $K_\theta \subseteq X$ be the set of all $\mathcal{L}$-structures $M \in X$ such that $M \models \theta$. Denote by $\mathcal{K}$ the collection of all such subsets $K_\theta \subseteq X$. Argue that $\mathcal{K}$ is 'closed' under finite unions, since $K_{\theta_1} \cup \cdots \cup K_{\theta_k} = K_{\theta_1 \vee \cdots \vee \theta_k}$. Show that the collection of all intersections of sets in $\mathcal{K}$ (i.e. sets of the form $\bigcap \mathcal{C} \subseteq X$ where $\mathcal{C} \subseteq \mathcal{K}$) is 'closed' under finite union and arbitrary intersection, and therefore X may be understood as a topological space whose closed sets are the subsets of the form $\bigcap \mathcal{C} \subseteq X$ (and $\mathcal{K}$ is a collection of basic closed sets for this topology). Explain why the Compactness Theorem for First Order Logic, in this new interpretation, says simply that X is a compact topological space.

8. (*Transfinite Induction*) It is easy to partition the point set $\mathbb{R}^3$ into Euclidean lines. There are many ways to do this, such as by taking all lines parallel to a fixed line ℓ. It is much trickier to partition the 'punctured 3-space' $X = \mathbb{R}^3 \setminus \{O\}$ into lines, where O is a single point of $\mathbb{R}^3$, so X consists of all points of $\mathbb{R}^3$ with one point removed. I have posted a solution to this on the course website (see 'Transfinite Induction'). As an example of something quite similar, show that it is possible to partition the points of $\mathbb{R}^3$ into Euclidean circles.

9. (*Cantor's Diagonal Argument*) This is presented in all the textbooks on Set Theory, including Cameron's book. It is so important that you should make sure you can explain it yourself. I suggest you first think about how you would try to prove it on your own; then refer to the hints below without reading the proof in a book. Here $\mathcal{P}A$ denotes the power set of A (the set of all subsets of A). Prove that $|A| < |\mathcal{P}A|$. By repeating this argument, one argues that there is no largest set.

   Hint: First find an injection $A \to \mathcal{P}A$; this shows that $|A| \leqslant |\mathcal{P}A|$. In order to prove strict inequality, suppose there is a bijection $f : A \to \mathcal{P}A$, and obtain a contradiction. For this, consider $B = \{a \in A : a \notin f(a)\}$. Since $B \in \mathcal{P}A$, we must have $B = f(b)$ for some $b \in A$, so ...

10. (*Cantor-Bernstein-Schröder Theorem*) This is even more basic than #9 and if you have not thought about it yourself, then you owe it to yourself to not wait any longer to figure it out! Once again, you can find the proof in many books including Cameron's. Try to figure out as much as you can without peeking in the book; but if you really find yourself stuck, then by all means read the details in the book and then restate the argument yourself. The theorem justifies the linear ordering of sets according to their size.

    (a) Given sets A and B, we say that $|A| \leqslant |B|$ if there is an injection $A \to B$. Show that if $|A| \leqslant |B|$ and $|B| \leqslant |C|$, then $|A| \leqslant |C|$.

    (b) If $f : A \to B$ is injective and $g : B \to A$ is injective, find a bijection $h : A \to B$. (*Hint*: Draw a picture showing some elements of A and B, and add arrows to indicate the action of f and g. This starts to look like a directed graph. Using the arrows for f and g, find a new set of arrows for h defining a bijection.)

    Now it makes sense to define $|A| = |B|$ whenever there is a bijection $A \to B$. Part (b) says that if $|A| \leqslant |B|$ and $|B| \leqslant |A|$, then $|A| = |B|$. Make sure you appreciate the subtlety of this statement: the proof is not too hard, but neither is it trivial. Finally, we say that $|A| < |B|$ if there is an injection $A \to B$ but no bijection (i.e. $|A| \leqslant |B|$ but $|A| \neq |B|$).

    The argument in (b) is quite constructive, not requiring the Axiom of Choice. However, in order to prove that our order relation is a total order (the 'trichotomoy law': given any two sets A, B, either $|A| < B$, or $|A| = |B|$, or $|A| > |B|$) requires the Axiom of Choice.

11. (*Lower Cardinals*) The following are also fairly standard results. Try to come up with your own explanations for them; but if you find yourself looking up the arguments, at least make sure you can explain them clearly yourself.

(a) Explain why $|\mathbb{R}| = |\mathcal{P}\mathbb{N}|$ where $\mathcal{P}A$ denotes the power set of A, and $\mathbb{N} = \{1, 2, 3, \ldots\}$. This justifies writing $|\mathbb{R}| = 2^{\aleph_0}$.

(b) Show that $|\mathbb{R}^2| = |\mathbb{R}|$. So by induction, $|\mathbb{R}^n| = 2^{\aleph_0}$. (More generally, if A is any infinite set, then $|A^n| = |A|$ for every positive integer n. However, that more general argument requires the Axiom of Choice. In the case of the set $\mathbb{R}$, an easier argument is available using an explicit bijection, not requiring the Axiom of Choice.)

(c) Show that $|\mathbb{R}^\omega| = 2^{\aleph_0}$. Here $\mathbb{R}^\omega$ denotes the set of all countable real sequences $(a_0, a_1, a_2, a_3, \ldots)$ where $a_i \in \mathbb{R}$ for all i.

(d) Show that $|\mathbb{R}^\mathbb{R}| = |\mathcal{P}\mathcal{P}\mathbb{N}| = 2^{2^{\aleph_0}} > |\mathbb{R}|$. Here $\mathbb{R}^\mathbb{R}$ denotes the set of all functions $\mathbb{R} \to \mathbb{R}$.

(e) Show that the set of all continuous functions $\mathbb{R} \to \mathbb{R}$ has cardinality $2^{\aleph_0}$. (*Hint*: Every continuous function $\mathbb{R} \to \mathbb{R}$ is uniquely determined by its restriction to $\mathbb{Q}$. Use (c) to complete the proof.)