Finitely Additive Probability Measures

Denote the power set $\mathcal{P}(\mathbb{Z}) = \{A : A \subseteq \mathbb{Z}\}$.

A f.a.p. (finitely additive probability) measure on $\mathbb{Z}$ is a map $\mu : \mathcal{P}(\mathbb{Z}) \to [0,1]$ such that $\mu(\varnothing) = 0$, $\mu(\mathbb{Z}) = 1$ and

$$\mu(A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n) = \mu(A_1) + \mu(A_2) + \cdots + \mu(A_n).$$

Such a measure is translation-invariant if for all $A \subseteq \mathbb{Z}$, $x \in \mathbb{Z}$,

$$\mu(A+x) = \mu(A) \quad \text{where} \quad A+x = \{a+x : a \in A\}.$$

Goal: "construct" a translation-invariant f.a.p. measure on $\mathbb{Z}$.

(The same can be done for $\mathbb{R}$ in place of $\mathbb{Z}$. Or for any amenable group.)

We cannot have $\mu(\{a\}) > 0$, otherwise

$$\mu(\{1,2,3,\cdots,n\}) = n\mu(\{a\}) > 1 \text{ for } n \text{ sufficiently large.}$$

So $\mu(\{a\}) = 0$ and $\mu(A) = 0$ whenever $|A| < \infty$.

Ultrafilters

Warm-up: Let S be an infinite set. Look for a f.a.p. measure $\lambda : \mathcal{P}(S) \to \{0,1\}$. (No translation-invariance; S is just a set.) Every $A \subseteq S$ is either an almost nowhere set ($\lambda(A) = 0$) or an almost everywhere set ($\lambda(A) = 1$). The set $\mathcal{U} = \{A \subseteq S : \lambda(A) = 1\}$ is an ultrafilter:

(1) $\emptyset \notin \mathcal{U}$, $S \in \mathcal{U}$

(2) $\mathcal{U}$ is closed under finite intersections and supersets.

(3) Whenever $S = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$, exactly one of $A_i \in \mathcal{U}$.

How do we construct an ultrafilter? Trivial ("principal") ultrafilter: Fix $s \in S$ and take $\mathcal{U} = \{A \subseteq S : s \in A\}$. AVOID THIS! We want a nonprincipal ultrafilter i.e. $\mathcal{U}$ contains no finite sets. So $\mathcal{U}$ contains every cofinite set $S - A$, $|A| < \infty$. The cofinite sets form a filter $\mathcal{F} \subseteq \mathcal{P}(S)$ satisfying (1), (2). By Zorn's Lemma, we extend $\mathcal{F} \subseteq \mathcal{U}$ where $\mathcal{U}$ is an ultrafilter, necessarily nonprincipal.

But an ultrafilter on $\mathbb{Z}$ cannot be translation-invariant. Every translation-invariant f.a.p. measure on $\mathbb{Z}$ has $\mu(\text{evens}) = \mu(\text{odds}) = \frac{1}{2}$.

A Better (?!) Notion of Limits

If (x_n) is a sequence in $[a,b]$, then (x_n) has at least one limit point (cluster point, accumulation point) in $[a,b]$. One of these (which one?) is the **λ-limit** of (x_n). Every bounded sequence (x_n) has a λ-limit satisfying

- If $x_n \to x$ then $\lambda\text{-}\lim x_n = \lim x_n = x$.
- $\lambda\text{-}\lim (x_n + y_n) = \lambda\text{-}\lim x_n + \lambda\text{-}\lim y_n$
- $\lambda\text{-}\lim (x_n y_n) = (\lambda\text{-}\lim x_n)(\lambda\text{-}\lim y_n)$
- If $x_n \leq y_n$ then $\lambda\text{-}\lim x_n \leq \lambda\text{-}\lim y_n$.

To define the λ-limit, choose a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

$\lambda\text{-}\lim x_n \leq L$ iff for all $\varepsilon > 0$, $\{n : x_n < L + \varepsilon\} \in \mathcal{U}$

$\lambda\text{-}\lim x_n \geq L$ iff for all $\varepsilon > 0$, $\{n : x_n > L - \varepsilon\} \in \mathcal{U}$.

There is a unique real number L satisfying both conditions.

A Translation-Invariant f.a.p. on $\mathbb{Z}$

Fix a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Given $A \subseteq \mathbb{Z}$, define

$$\mu(A) = \lambda\text{-}\lim_{n} \frac{|A \cap [-n,n]|}{2n+1}.$$

This is a translation-invariant f.a.p. measure on $\mathbb{Z}$.

The sequence of subsets $[-n,n] \subset \mathbb{Z}$ is a Følner sequence. Every amenable group has such a sequence.

Application

The Banach-Tarski Theorem (for balls in $\mathbb{R}^3$) has no analogue in $\mathbb{R}^1$ or $\mathbb{R}^2$.

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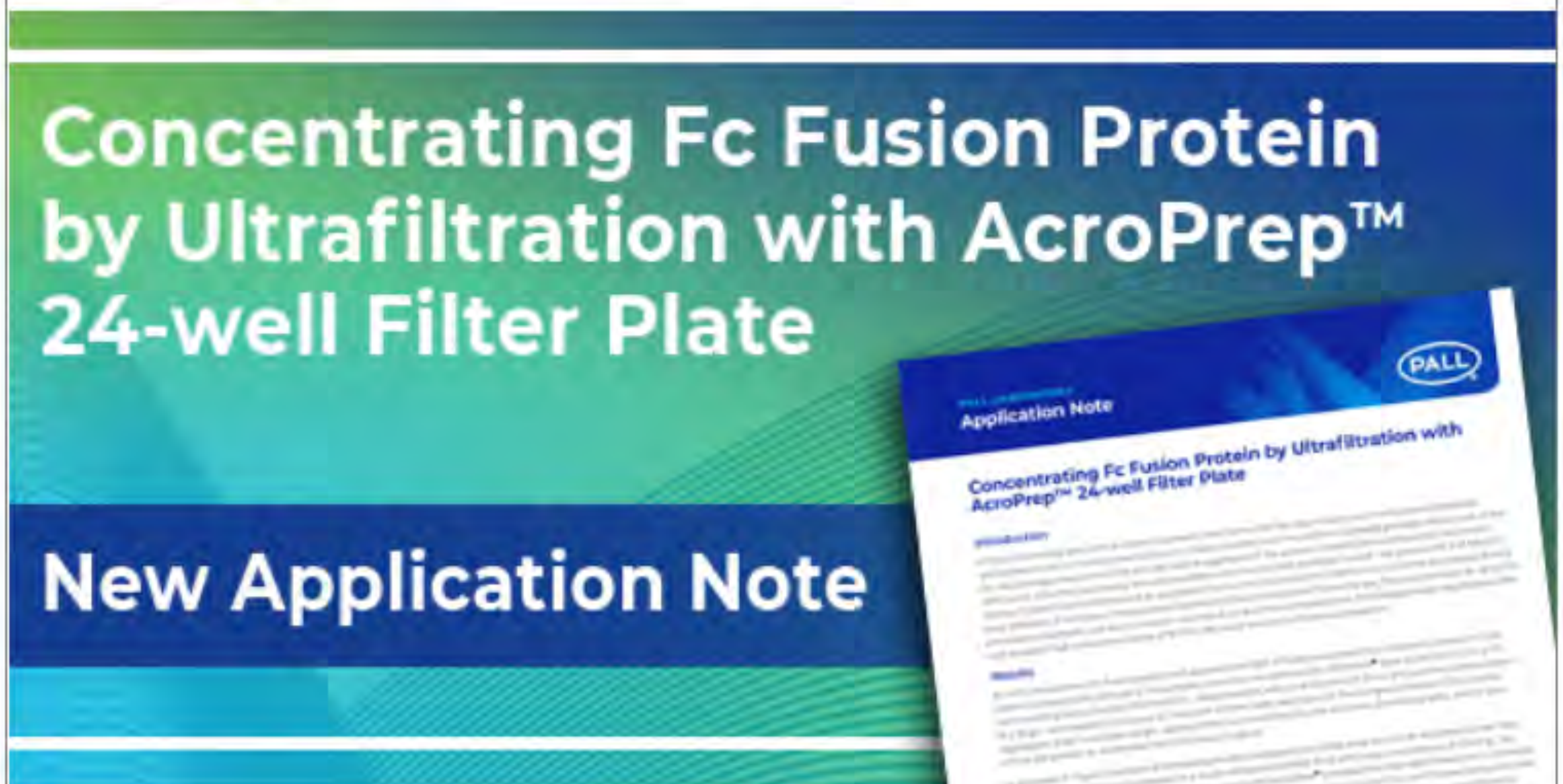

New Application Note

Concentration FC fusion protein ultrafiltration with AcroPrep® 24-well filter plate

Hi G,

This application note describes the concentration of an FC fusion protein with AcroPrep® 24-well filter plates with 30K molecular weight cut-off Omega™ ultrafiltration membrane. Processing was very reproducible with yields of 96-97%. The filter plates provided a reliable way of processing multiple samples in parallel. Download now to learn more.

Download Now

The Hyperreals

Consider the ring $\mathbb{R}^\infty = \{(a_1, a_2, a_3, a_4, \dots) : a_i \in \mathbb{R}\}$ with coordinatewise addition and multiplication. This is not a field, but if we identify two sequences whenever they agree almost everywhere (i.e. $(a_n) \simeq (b_n)$ whenever $\{n : a_n = b_n\} \in \mathcal{U}$) then we get the field extension $\hat{\mathbb{R}} \supset \mathbb{R}$ of hyperreal numbers.

If (a_n) is bounded then $\mathcal{U}$-$\lim a_n$ is the standard part of (a_n), which is the unique real number closest to (a_n).