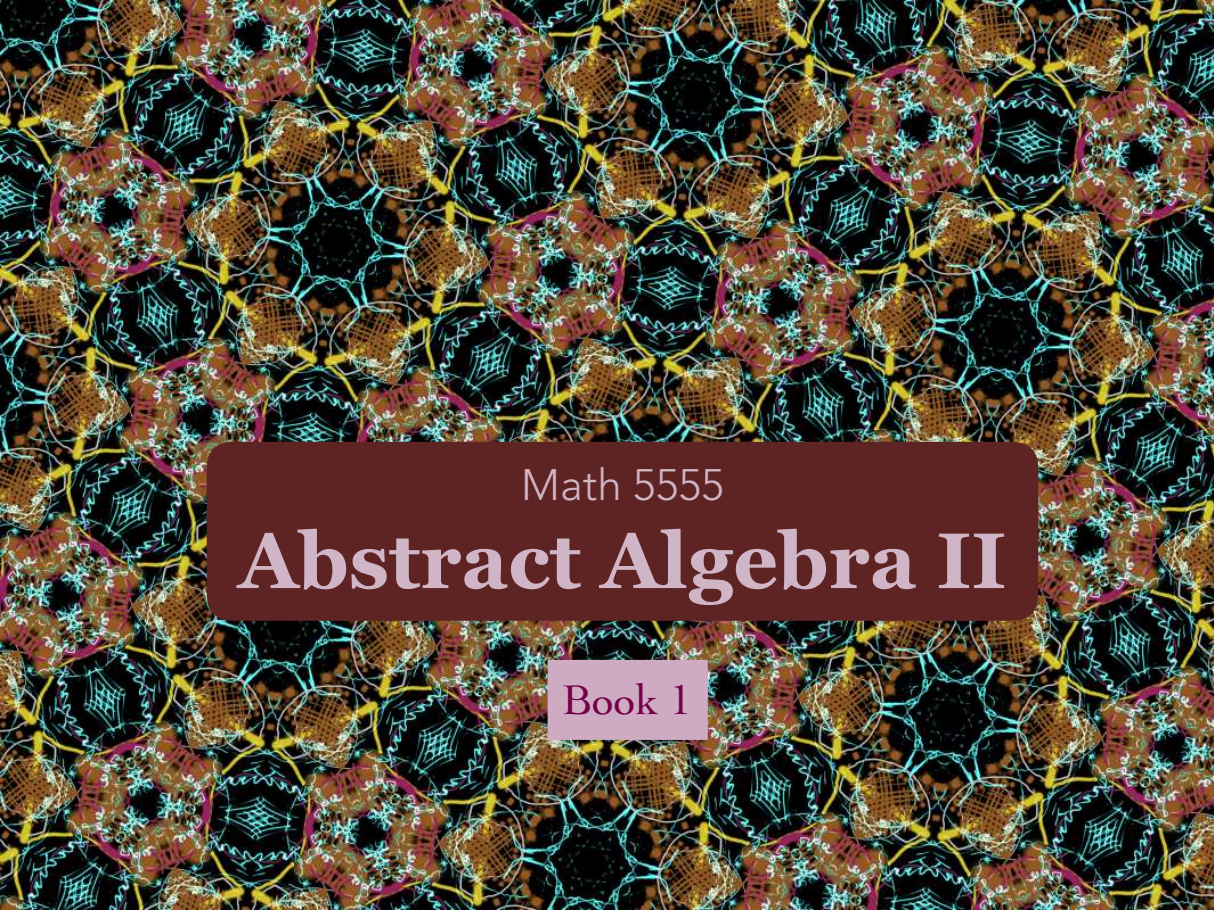




Math 5555

Abstract Algebra II

Book 1

In group theory we have

• permutation representations: homomorphisms $\pi: G \rightarrow S_n$ permutation representation of degree n
(if π is 1-to-1 then π is a faithful representation: then $\pi(G) \leq S_n$)

• linear representations: homomorphism $\pi: G \rightarrow GL_n(F)$ F : field
linear representation of degree n over F
If $F = \mathbb{C}$ (or $\mathbb{R}$ or ...) then π is an ordinary representation.

If $\text{char } F = p$ (prime) then π is a modular representation.

G : unless otherwise specified, G finite group. (Until later...)

Usually $F = \mathbb{C}$ (or $\mathbb{R}$) and $|G| < \infty$.

If $\pi_i: G \rightarrow GL_{n_i}(\mathbb{C})$ ($i=1,2$) then $\pi_1 \oplus \pi_2: G \rightarrow GL_{n_1+n_2}(\mathbb{C})$, $g \mapsto \left[\begin{array}{c|c} \pi_1(g) & 0 \\ \hline 0 & \pi_2(g) \end{array} \right]$ is a representation of degree $n_1 + n_2$

$\pi: G \rightarrow GL_n(\mathbb{C})$ is decomposable if there is a decomposition $\mathbb{C}^n = U \oplus V$ such that $U, V \neq 0$
($\dim U = n_1$, $\dim V = n_2$, n_1, n_2 positive integers, $n_1 + n_2 = n$)
 U, V invariant under all matrices $\pi(g)$, $g \in G$.

$G = S_3$ acting naturally on $\mathbb{C}^3$ by permuting the standard basis vectors $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

via $\sigma: e_i \mapsto e_{\sigma(i)}$

i.e.

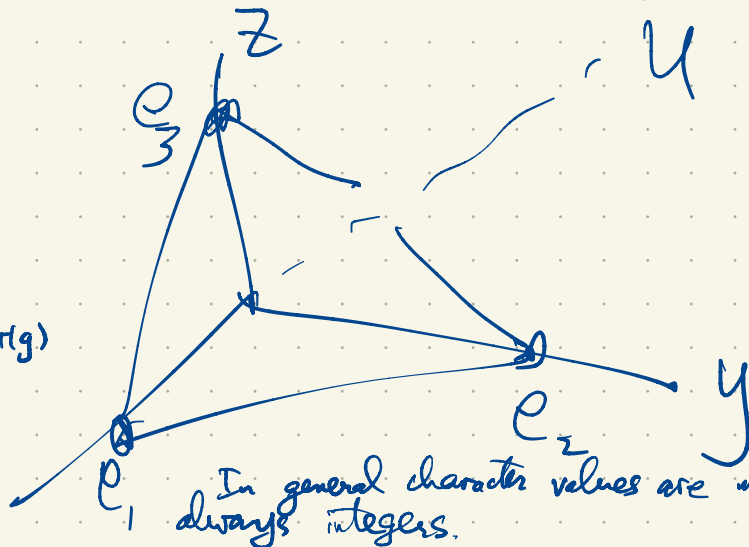
$$(12) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(123) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$(1) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is a faithful representation of degree 3.

It is decomposable: $\mathbb{C}^3 = U \oplus V$, $U = \langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rangle$, $V = U^\perp = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \rangle$
indecomposable.



The representation $\pi: S_3 \rightarrow GL_3(\mathbb{C})$ has (affords) character

$$\chi(g) = \text{tr } \pi(g)$$

$$\chi(1) = 3 = \text{degree of } \pi = \deg \chi$$

$$\chi((12)) = \chi((13)) = \chi((23)) = 1$$

$$\chi((123)) = \chi((132)) = 0$$

In general character values are not always integers.

For a permutation representation

$$\pi: G \rightarrow S_n \subset GL_n(\mathbb{C})$$

the associated character $\chi(g) = \text{tr } \pi(g)$
(called the permutation character)

$$\chi(g) = \text{no. of fixed points of } g$$

$$= |\{i: g(i) = i\}|, \quad 1 \leq i \leq n$$

Given a representation $\pi: G \rightarrow GL_n(\mathbb{C})$, the character of π is

$$\chi(g) = \text{tr } \pi(g) \in \mathbb{C}.$$

Character values $\chi(g) \in \mathbb{C}$ are always algebraic integers ($g \in G$, $|G| < \infty$) and character values of S_n are ordinary integers.

$\chi(g)$ depends only on the conjugacy class of g .

If $g, h \in G$ then $g \sim h'gh$ (conjugate in G) so

$$\pi(h'gh) = \pi(h')\pi(g)\pi(h) \sim \pi(g) \text{ (similar in } GL_n(\mathbb{C}) \text{ i.e. conjugate)}$$

$\uparrow$
inverses in $GL_n(\mathbb{C})$

$$\text{so } \text{tr } \pi(g) = \text{tr } (\pi(h')\pi(g)\pi(h)) = \text{tr } \pi(h'gh)$$

$$\text{tr } (AB) = \text{tr } (BA)$$

$$\text{tr } (B^{-1}AB) = \text{tr } (AB \cdot B^{-1}) = \text{tr } A$$

$$\chi(h'gh) = \chi(g).$$

If $\pi: G \rightarrow GL_n(\mathbb{C})$ is any representation i.e. homomorphism, and $B \in GL_n(\mathbb{C})$, then

$\tilde{\pi}(g) = B^{-1}\pi(g)B \in GL_n(\mathbb{C})$ is also a representation.

$\pi, \tilde{\pi}$ are equivalent (via a change of basis).

$$\begin{aligned} \tilde{\pi}(gh) &= B^{-1}\pi(gh)B = B^{-1}\pi(g)\pi(h)B = B^{-1}\pi(g)B \cdot B^{-1}\pi(h)B \\ &= \tilde{\pi}(g)\tilde{\pi}(h). \end{aligned}$$

They have the same character: the character of $\tilde{\pi}$ is

$$\tilde{\chi}(g) = \text{tr } \tilde{\pi}(g) = \text{tr } (B^{-1}\pi(g)B) = \text{tr } \pi(g) = \chi(g).$$

It's not obvious but the converse is true: χ determines π up to equivalence. Two representations have the same character iff they are equivalent.

Let $\pi: G \rightarrow GL_n(\mathbb{C})$ be a representation (i.e. homomorphism).

π is reducible if there exists nontrivial ^{proper} subspace $U \subset \mathbb{C}^n$ ($\dim U \in \{1, 2, \dots, n-1\}$) such that U is invariant under $\pi(g)$ for all $g \in G$ i.e. $\pi(g)U \subseteq U$ for all $g \in G$.

$$\text{i.e. } \pi(g) = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \begin{matrix} \left\{ \begin{matrix} k \\ n-k \end{matrix} \right\} \\ \text{for all } g \in G. \end{matrix}$$

$$k = \dim U.$$

when $|G| < \infty$ and $\pi: G \rightarrow GL_n(\mathbb{C})$
 $\text{char } F = 0$, π reducible $\Leftrightarrow \pi$ decomposable
 π irreducible $\Leftrightarrow \pi$ indecomposable

In general, π decomposable $\Rightarrow \pi$ reducible
 $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \Rightarrow \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$
 π indecomposable $\Leftarrow \pi$ irreducible.

The representation $\pi: \mathbb{R} \rightarrow GL_2(\mathbb{R})$, $\pi(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, $\pi(a+b) = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \pi(a)\pi(b)$
 additive group

is reducible: $\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$ is an invariant subspace. There is no complementary invariant subspace (in particular the complementary subspace $\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle$ is not invariant).

Maschke's Theorem Let G be a finite group and $F = \mathbb{C}$ (or more generally F is any field of characteristic not dividing $|G|$). Then any representation $\pi: G \rightarrow GL(F)$ is reducible iff it's decomposable (i.e. π is irreducible iff π is indecomposable; i.e. whenever every invariant subspace $U \leq F^n$ has a complementary subspace U' which is also invariant).

$U \leq V$ has a complementary subspace $U' \iff V = U \oplus U'$. If $\dim V = n$ and $\dim U = k$ then U' is a complement to U iff $\dim U' = n-k$ and $U + U' = V$
 iff $\dim U' = n-k$ and $U \cap U' = \{0\}$.

In this case every $v \in V$ is uniquely represented as $v = u + u'$, $u \in U$, $u' \in U'$.

In this, the projection $V \rightarrow U$ along U' is the map $P: v \mapsto Pv = u$. $P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \text{size } k \\ \text{size } n-k \end{matrix}$
 P has image $PV = U$; $\ker P = U'$. Note: $I-P$ is the projection from V onto U' along U .

A linear transformation $P: V \rightarrow V$ is a projection iff $P^2 = P$. In this case P is a root of $x^2 - x$ so the eigenvalues of P are in $\{0, 1\}$. Take $U = 1$ -eigenspace of $P = \ker(I-P)$, $U' = \ker P$.

Proof of Maschke's Theorem: (in the nontrivial direction)

Suppose $\pi: G \rightarrow GL(\mathbb{C})$ is a representation having an invariant subspace $U \leq \mathbb{C}^n$ i.e. $\pi(g)U \subseteq U$ for all $g \in G$. We want to find a complementary subspace $W \leq \mathbb{C}^n$ which is also invariant. Start with any complementary subspace U' where $\mathbb{C}^n = U \oplus U'$, $\dim U = k$, $\dim U' = n-k$; every $v \in \mathbb{C}^n$ is uniquely expressible as $v = u + u'$, $u \in U$, $u' \in U'$.

Unfortunately U' is not invariant in general. Let $P: \mathbb{C}^n \rightarrow U$ be the projection onto U along U' i.e. $P(v) = P(u + u') = u$, so $U = PV$, $U' = \ker P$. Consider the new map $\tilde{P}: V \rightarrow V$ defined by $\tilde{P} = \frac{1}{|G|} \sum_{g \in G} \pi(g) P \pi(g)$. $\tilde{P}V \subseteq U$. Since $\tilde{P}v = \frac{1}{|G|} \sum_g \pi(g^{-1}) P \underbrace{\pi(g)v}_{\substack{\in V \\ \in U}} \in U$.

For all $u \in U$, $\tilde{P}u = u$ i.e. $\tilde{P}|_U = \text{id}|_U$. Why?

$$\tilde{P}u = \frac{1}{|G|} \sum_g \pi(g^{-1}) P \pi(g) u = \frac{1}{|G|} \sum_g \pi(g^{-1}) \pi(g) u = \frac{1}{|G|} |G| u = u.$$

$= \pi(g)u$ since $\pi(g)u \in U$.

Next: show $\tilde{P}$ commutes with all $\pi(g)$, $g \in G$. (P doesn't satisfy this in general!)

$$\text{Then } \tilde{P}\pi(g) = \frac{1}{|G|} \sum_{h \in G} \pi(h^{-1}) P \pi(h) \pi(g) = \frac{1}{|G|} \sum_{h \in G} \pi(h^{-1}) P \pi(hg)$$

$$h \longleftrightarrow \begin{aligned} x &= hg \\ x^{-1} &= g^{-1}h^{-1} \\ g x^{-1} &= h^{-1} \end{aligned}$$

$$= \frac{1}{|G|} \sum_{x \in G} \underbrace{\pi(gx^{-1})}_{\pi(g)\pi(x)^{-1}} P \pi(x) = \pi(g) \cdot \frac{1}{|G|} \sum_{x \in G} \pi(x^{-1}) P \pi(x) = \pi(g) \tilde{P}.$$

Next: show $\tilde{P}^2 = \tilde{P}$. If $v \in V$ then $\tilde{P}v \in U$ so $\tilde{P}^2v = \tilde{P}\tilde{P}v = \tilde{P}v$. So $\tilde{P}$ is idempotent so

$\tilde{P}$ is a projection on $\tilde{P}V = U$ along $W = \ker \tilde{P}$. Note: if $k = \dim U = \text{rank } \tilde{P} = \text{tr } \tilde{P}$, $\dim W = \dim \ker \tilde{P} = n - k$. W is also invariant under $\pi(g)$: if $v \in W$ then $\tilde{P}v = 0$

so $\pi(g)\tilde{P}v = \pi(g)0 = 0$ so $\pi(g)v \in W$

$\tilde{P}\pi(g)v$



The irreducible representations of S_3 are

$$\pi_1(g) = [1] \in GL_1(\mathbb{C}) \quad \text{trivial representation}$$

$$\pi_2(g) = [\text{sgn}(g)] \in GL_1(\mathbb{C}) \quad \text{sign representation:}$$

$$= [\pm 1] \text{ according as } g \text{ is an even or odd permutation.}$$

$$\text{sgn}((1)) = 1$$

$$\text{sgn}((123)) = 1$$

$$\text{sgn}((132)) = 1$$

$$\text{sgn}((12)) = \text{sgn}((13)) = \text{sgn}((23)) = -1$$

$$\text{sgn}(gh) = \text{sgn}(g) \text{sgn}(h)$$

$$\pi_3: S_3 \rightarrow GL_2(\mathbb{C})$$

$$(1) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(12) \mapsto \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$(123) \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$(132) \mapsto \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$(13) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(23) \mapsto \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\pi_3((13)) = \pi_3((123)(12)) = \pi_3((123)) \pi_3((12))$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\pi_3((23)) = \pi_3((123)(13)) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

An irreducible character is a map $G \rightarrow \mathbb{C}$
where π is an irreducible representation $g \mapsto \chi(g)$.

$0, \langle [1] \rangle, \langle [-1] \rangle, \mathbb{C}^2$ are the only subspaces invariant under $\pi_3((13))$; but the 1-dimensional invariant subspaces are not invariant under $\pi_3((123))$.

The character table of S_3 is

	(1)	(12)	(23)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$$\pi_3((123)) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \text{ has char. poly. } x^2 + x + 1$$

$$\pi_3((13)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ has char. poly. } x^2 - 1$$

↑ eigenspaces $\langle [1] \rangle, \langle [-1] \rangle$
(eigenvalues 1, -1 respectively)

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Given any finite group G , a class function on G is a function $f: G \rightarrow \mathbb{C}$ which only depends on conjugacy class, i.e. $f(g') = f(g)$ whenever $g, g' \in G$ are conjugate i.e. $g' = ugu$ for some $u \in G$.

All characters of G (irreducible or otherwise) are class functions.

$V = \{\text{class functions on } G\}$ is a complex vector space

$$f, f' \in V \Rightarrow f + f' \in V, \quad (f + f')(x) = f(x) + f'(x)$$

V is in fact a complex inner product space:

$$[f, f']_G = [f, f'] = \frac{1}{|G|} \sum_{x \in G} f(x) \overline{f'(x)} \quad [f', f] = \overline{[f, f']}$$

sesquilinear
 $\frac{1}{2}$ -linear

$\dim V = \text{number of conjugacy classes}$.

The irreducible characters of G always give an orthonormal basis for $V = \{\text{class functions on } G\}$.

Recall the permutation representation with permutation character

$$\pi: S_3 \rightarrow GL_3(\mathbb{C})$$

$$\pi: e_i \mapsto e_{\pi(i)}$$

$$(12) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(123) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$(1) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\chi(1) = 3$$

$$\chi((12)) = 1$$

$$\chi((123)) = 0$$

$$[\chi, \chi_1] = \frac{1}{6}(1 \cdot 3 + 1 + 1 + 1 + 0) = 1$$

$$[\chi, \chi_2] = \frac{1}{6}(3 - 1 - 1 + 0 + 0) = 0$$

$$[\chi, \chi_3] = \frac{1}{6}(6 + 0 + 0 + 0 + 0 + 0) = 1$$

$$\Rightarrow \pi \simeq \pi_1 \oplus \pi_3$$

Character table of S_3

	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1
χ	3	1	0

$$[\chi_1, \chi_1] = \frac{1}{6} \sum_{g \in S_3} \chi_1(g) \overline{\chi_1(g)} = \frac{1}{6}(1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1) = 1$$

$$[\chi_1, \chi_2] = \frac{1}{6} \sum_g \chi_1(g) \overline{\chi_2(g)} = \frac{1}{6}(1 \cdot 1 - 1 \cdot 1 - 1 \cdot 1 - 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1) = 0$$

$$[\chi_1, \chi_3] = \frac{1}{6}(1 \cdot 2 + 0 + 0 + 0 - 1 \cdot 1) = 0$$

$$[\chi_2, \chi_2] = \frac{1}{6}(1 + 1 + 1 + 1 + 1 + 1) = 1$$

$$[\chi_2, \chi_3] = \frac{1}{6}(2 + 0 + 0 + 0 - 1 \cdot 1) = 0$$

$$[\chi_3, \chi_3] = \frac{1}{6}(4 + 0 + 0 + 0 + 1 \cdot 1) = 1$$

The long way to check the decomposition $\pi \simeq \pi_1 \oplus \pi_3$ is found in the course notes: with respect to new basis

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

basis for
first invariant
subspace

basis for
second invariant
subspace

$\pi(g)$ has matrix $\left[\begin{array}{c|cc} \pi_1(g) & 0 & 0 \\ \hline 0 & \pi_3(g) \end{array} \right]$ with respect to v_1, v_2, v_3

(see p. 3 of course notes).

Character tables also have orthogonality of columns.

$ C(g) $	6	2	3
g	(1)	(12)	(23)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$$C_G(g) = \{x \in G: xg = gx\}$$

The conjugacy class of $g \in G$ is the index of the centralizer

$$[G : C_G(g)] = \frac{|G|}{|C_G(g)|}$$

The column orthogonality is expressed as: given $g, h \in G$,
(k = number of irreducible characters = no. of conjugacy classes)

$$\sum_{i=1}^k \chi_i(g) \overline{\chi_i(h)} = \begin{cases} 0 & \text{if } g, h \\ & \text{not conjugate} \\ |C_G(g)| & \text{if } g, h \\ & \text{are conjugate} \end{cases}$$

Row orthogonality says: Let $g_1, \dots, g_k \in G$ be reps of conj. classes