Math 5555

Abstract Algebra II

Book 2

Induction takes class functions on H to class functions on G
characters — characters on G
representations of H — representations of G

Let χ be a class function on $H \leq G$ i.e. $\chi: H \longrightarrow \mathbb{C}$, $\chi(xhx^{-1}) = \chi(h)$ for all $x, h \in H$.

To get a class function on G, start with the trivial extension

$$\hat{\chi}: G \longrightarrow \mathbb{C}$$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H; \\ 0 & \text{if } g \notin H. \end{cases}$$

To make this into a class function, use an averaging over conjugates as we did before. This leads to $\chi^G: G \longrightarrow \mathbb{C}$:

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1}) \qquad (\ \chi^G \text{ is } \underline{\text{induced}} \text{ from } \chi, \qquad \chi^G = \text{Ind}_H^G \chi \ )$$

If $u \in G$ then $\chi^G(ugu^{-1}) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xugu^{-1}x^{-1}) = \frac{1}{|H|} \sum_{w \in G} \hat{\chi}(wgw^{-1})$

$$w^{-1} = u^{-1}x^{-1}$$
$$w = xu$$
$$wu = x$$

$$= \dot{\chi}^G(g).$$

So χ^G is a class function on G.

Note: Let T be a set of right coset representatives for H in G. So every element $g \in G$ is uniquely expressible as $g = ht$, $h \in H, t \in T$.
$$|G| = |H||T| \qquad \text{(Lagrange's Theorem)}, \qquad |T| = \frac{|G|}{|H|} = [G:H].$$

$\chi : H \longrightarrow \mathbb{C}$ is a class function on H; T is a right transversal for H in G (set of right coset representatives)

$$T = \{t_1, \cdots, t_\ell\}$$

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1})$$

$$G = Ht_1 \sqcup Ht_2 \sqcup \cdots \sqcup Ht_\ell$$

$x = ht, \quad \begin{matrix} h \in H \\ t \in T \end{matrix}$

$$= \coprod_{t \in T} Ht.$$

$$= \frac{1}{|H|} \sum_{t \in T} \sum_{h \in H} \hat{\chi}(htgt^{-1}h^{-1})$$

$htgt^{-1}h^{-1} \in H$

$$= \frac{1}{|H|} \sum_{t \in T} \sum_{h \in H} \hat{\chi}(tgt^{-1})$$

$\Longleftrightarrow tgt^{-1} \in H$

$\underbrace{}$

$|H|$ terms equal to $\hat{\chi}(tgt^{-1})$

$$= \frac{1}{|H|} \sum_{t \in T} |H| \hat{\chi}(tgt^{-1}) = \sum_{t \in T} \hat{\chi}(tgt^{-1}) = \sum_{i=1}^{\ell} \hat{\chi}(t_i g t_i^{-1})$$

Special case: $\chi = \chi_1 =$ trivial (principal) character of H, $\chi(h) = 1$.

χ^G isn't the principal character of G unless $G = H$.

G permutes the right cosets of H by right multiplication giving a permutation representation $g \in G$ permutes $Ht_i \longmapsto Ht_i g = Ht_j$

$$G \longrightarrow S_\ell \qquad \ell = [G : H] = |T|.$$

$\chi^G = (1_H)^G$ is the perm. character of G acting on cosets of H.

Eg. Construct the character table of $G = S_4$ making use of the character table of $S_3 = H \leq G$.

$$[\psi, \psi_2] = 0$$
$$\psi(1) = \psi_1(1) + \psi_3(1)$$
$$\underbrace{3}_{} \quad \underbrace{1}_{} \quad \underbrace{2}_{}$$

H:

$|K_H(g)|$	6	2	3
g	()	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Check: $[\psi_3, \psi_3] = 1$

$G = S_4$:

$|C_G(g)|$	24	4	8	3	4
g	()	(12)	(12)(34)	(123)	(1234)
ψ_1	1	1	1	1	1
ψ_2	1	-1	1	1	-1
ψ_3	2	0	2	-1	0
ψ_4	3				
ψ_5	3				
ψ	3	1	3	0	1

$$\psi = \sum_{i=1}^{5} a_i \psi_i$$

$$[\psi, \psi]_G = \frac{9^3}{24} + \frac{1}{4} + \frac{9}{8} + \frac{0}{3} + \frac{1}{4}$$
$$= \frac{3 + 2 + 9 + 0 + 2}{8} = \frac{16}{8}$$
$$= 2$$
$$= \sum_{i=1}^{5} a_i^2 = 1^2 + 1^2 + 0 + 0 + 0$$
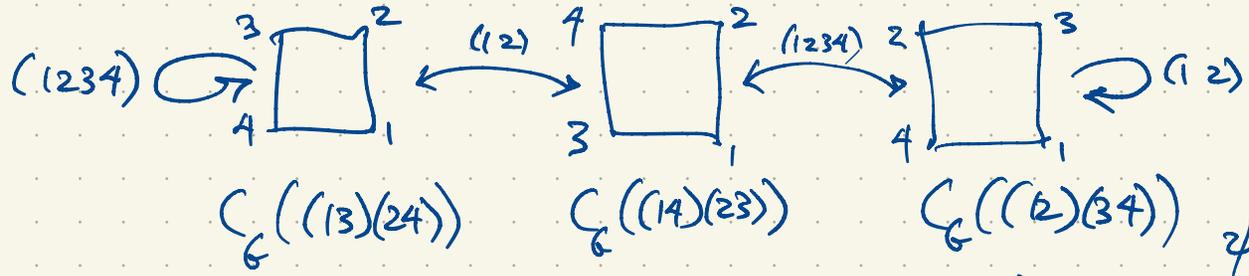$$\psi = \psi_1 + \psi_3$$
$$\quad \text{(degree 1)} \quad \text{(degree 2)}$$

$a_1 = [\psi, \psi_1] = \frac{3}{24} + \frac{1}{4} + \frac{3}{8} + \frac{0}{3} + \frac{1}{4} = \frac{1 + 2 + 3 + 0 + 2}{8} = 1$

S_4 has $k = 5$ conjugacy classes

... irreducible representations / characters of degree $n_1, n_2, \ldots, n_5 \geq 1$, $\quad n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = |G| = 24$

S_4 has normal subgroups

$1, S_4, A_4, \quad K = \langle (12)(34), (13)(24) \rangle = \{ (), (12)(34), (13)(24), (14)(23) \}$.

S_4 permutes the conjugacy class of (12)(34) in all $3! = 6$ possible ways there is a permutation action $S_4 \longrightarrow Sym\{ (12)(34), (13)(24), (14)(23) \} \cong S_3$ with kernel K of order 4.

This gives a permutation representation of S_4 of degree 3. Its permutation character is

$$\psi(1) = 3$$
$$\psi(k) = 3 \text{ for } k \in K$$

$C_G((13)(24))$ $\quad$ $C_G((14)(23))$ $\quad$ $C_G((12)(34))$

$\psi((12)) = 1$
$\psi((1234)) = 1 \quad \psi((123)) = 0 \quad \psi((12)(34)) = 3$

$H = S_3:$

$|K_H(g)|$	6	2	3
g	$()$	(12)	(123)
χ_1	1	1	1
$\chi = \chi_2$	1	-1	1
χ_3	2	0	-1

$G = S_4:$

$|C_G(g)|$	24	4	8	3	4
g	$()$	(12)	$(12)(34)$	(123)	(1234)
ψ_1	1	1	1	1	1
ψ_2	1	-1	1	1	-1
ψ_3	2	0	2	-1	0
ψ_4	3	-1	-1	0	1
ψ_5	3	1	-1	0	-1
χ^G	4	-2	0	1	0

$$[\chi^G, \chi^G]_G = \frac{16}{24} + \frac{4}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4}$$
$$= 2$$

$\Rightarrow \chi^G$ has irreducible constituents of multiplicity 1, 1, 0, 0, 0

$$[\chi^G, \psi_1] = \frac{4}{24} - \frac{2}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4}$$
$$= 0$$

$$[\chi^G, \psi_2] = \frac{4}{24} + \frac{2}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 1$$

$T = \langle (1234) \rangle$
$= \{(), (1234), (13)(24), (1432)\}$
right transversal for H in G

$\chi^G = \chi_2^G$ is a character on G

$$\chi^G(g) = \sum_{t \in T} \hat{\chi}(tgt^{-1})$$

where $\hat{\chi}: G \longrightarrow C$
$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

$\chi^G = \psi_2 + \psi_4$
$\psi_4 = \chi^G - \psi_2$

$$[\psi_4, \psi_4]_G = \frac{9}{24} + \frac{1}{4} + \frac{1}{8} + \frac{0}{3} + \frac{1}{4} = \frac{3+2+1+0+2}{8} = 1$$

$\chi^G(1) = 1 + 1 + 1 + 1 = 4$

$\chi^G((12)) = \hat{\chi}((12)) + \hat{\chi}((23)) + \hat{\chi}((34)) + \hat{\chi}((14))$
$$= -1 - 1 + 0 + 0 = -2$$

$\chi^G((12)(34)) = \hat{\chi}((12)(34))$
$+ \hat{\chi}((23)(41))$
$+ \hat{\chi}((34)(12))$
$+ \hat{\chi}((41)(23))$
$= 0 + 0 + 0 + 0 = 0$

$\chi^G((123)) = \hat{\chi}((123)) + \hat{\chi}((234))$
$+ \hat{\chi}((341)) + \hat{\chi}((412))$
$$= 1 + 0 + 0 + 0 = 1$$

Frobenius Reciprocity Let χ be a class function on $H \leq G$ and let ψ be a class function on G. Then

$$[\psi_H, \chi]_H = [\psi, \chi^G]_G$$

$$\uparrow \psi_H = \psi|_H$$

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xg x^{-1}) \qquad \hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

Proof

$$[\chi^G, \psi]_G = \frac{1}{|G|} \sum_{g \in G} \chi^G(g) \overline{\psi(g)}$$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{x \in G} \frac{1}{|H|} \hat{\chi}(x^{-1}g x) \overline{\psi(g)}$$

$$x \in G = HT$$
$$x = ht, \quad h \in H, \ t \in T$$
$$|G| = |H||T|$$

$$= \frac{1}{|G||H|} \sum_{g \in G} \sum_{x \in G} \hat{\chi}(x g x^{-1}) \overline{\psi(g)}$$

$$htgt^{-1}h^{-1} \in H \iff tgt^{-1} \in H$$

$$= \frac{1}{|G||H|} \sum_{g \in G} \sum_{h \in H} \sum_{t \in T} \hat{\chi}(htgt^{-1}h^{-1}) \overline{\psi(g)}$$

$$= \frac{1}{|G||H|} \sum_{g \in G} \sum_{h \in H} \sum_{t \in T} \underbrace{\hat{\chi}(tgt^{-1})}_{\text{}} \overline{\psi(g)}$$

$|H|$ identical terms for $h \in H$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{t \in T} \hat{\chi}(tgt^{-1}) \overline{\psi(g)}$$

$$= \frac{1}{|G|} \sum_{t \in T} \left(\sum_{g \in G} \hat{\chi}(tgt^{-1}) \overline{\psi(g)} \right)$$

For each $t \in T$ reparameterize the inner sum, $u = tgt^{-1}$, $g = t^{-1}ut$

$$= \frac{1}{|G|} \sum_{t \in T} \left(\sum_{u \in G} \hat{\chi}(u) \overline{\psi(t^{-1}ut)} \right)$$

$$= \frac{1}{|G|} \sum_{t \in T} \sum_{h \in H} \chi(h) \overline{\psi(t^{-1}ht)} = \frac{1}{|G|} \sum_{t \in T} \sum_{h \in H} \chi(h) \overline{\psi(h)}$$

$$= \frac{1}{|G|} |T| \sum_{h \in H} \chi(h) \overline{\psi(h)}$$

$$= \frac{1}{|H|} \sum_{h \in H} \chi(h) \overline{\psi(h)}_H$$

$$= [\chi, \psi_H]_H \quad \square$$

If ψ is a class function on G then so is $\bar{\psi}$ where $\bar{\psi}(g) = \overline{\psi(g)} = \psi(\bar{g}^{-1})$.

Indeed if ψ is a character, $\psi(g) = \operatorname{tr}\pi(g)$, $\pi: G \to GL_n(\mathbb{C})$
homomorphism

$\pi(g) \underset{\text{similar to}}{\sim} \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ (actually $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ in the case of finite groups)

$\lambda_1, \dots, \lambda_n$ are m^{th} roots of unity, where $m = $ exponent of $G = \operatorname{lcm}(|g| : g \in G)$

$\bar{\lambda}_i = \lambda_i^{-1}$ since $\lambda_i^m = 1$, $\quad \bar{\lambda}_i \lambda_i = |\lambda_i|^2 = 1$

$|\lambda_i|^m = 1$

$|\lambda_i| = 1$

$\psi(g) = \sum \lambda_i$

$\psi(\bar{g}^{-1}) = \sum \lambda_i^{-1} = \sum \bar{\lambda}_i = \overline{\psi(g)}$

For every finite group G, the irreducible representations of G can be chosen to be $\underline{\text{unitary}}$.

$U_n(\mathbb{C}) = \{A \in \mathbb{C}^{n \times n} : AA^* = A^*A = I\}, \quad A^* = \bar{A}^T = \overline{A^T}$.

eg. for S_3, we have an irreducible representation of degree 2.

$\pi_3 : S_3 \longrightarrow GL_2(\mathbb{C})$ $\qquad \chi(g)$

$() \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\qquad 2$

$(12) \longmapsto \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$ $\qquad 0$

$(123) \longmapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ $\qquad -1$

$(132) \longmapsto \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ $\qquad -1$

$(13) \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\qquad 0$

$(23) \longmapsto \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$ $\qquad 0$

This representation uses integer matrix entries.

left-to-right composition

$(13) = (123)(12)(132)$

Character values:

$$\chi(g) = \mathrm{tr}\,\pi(g)$$

A change of basis from $\binom{1}{0}, \binom{0}{1}$ takes one representation to the other.

A different choice of basis for $\mathbb{C}^2$ yields an equivalent representation using unitary matrices: $\qquad \chi(g)$

$() \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad 2$

$(123) \longmapsto \begin{bmatrix} \omega & 0 \\ 0 & \bar\omega \end{bmatrix}$ $\qquad -1$ $\qquad \omega = e^{2\pi i/3}$

$(132) \longmapsto \begin{bmatrix} \bar\omega & 0 \\ 0 & \omega \end{bmatrix}$ $\qquad -1$ $\qquad \bar\omega = \omega^2 = \omega^{-1}$

root of $x^2 + x + 1$

$\omega^2 + \omega + 1 = 0$

$\bar\omega + \omega = \omega^2 + \omega = -1$

$(12) \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\qquad 0$

$(13) \longmapsto \begin{bmatrix} \omega & 0 \\ 0 & \bar\omega \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \bar\omega & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ \bar\omega & 0 \end{bmatrix}\begin{bmatrix} \bar\omega & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} 0 & \bar\omega \\ \omega & 0 \end{bmatrix}$ $\qquad 0$

$(23) \longmapsto \begin{bmatrix} 0 & \omega \\ \bar\omega & 0 \end{bmatrix}$ $\qquad 0$

To prove the claim (that every finite group has its irred. repns. equivalent to equivalent to unitary representations) we use a fact from linear algebra: any inner product on $\mathbb{C}^n$ is equivalent (under change of basis) to standard inner product $B(x,y) = \sum x_i \bar{y}_i$.

~~Any~~ inner product has the form $(x,y) \longmapsto x M y^*$ where $x = (x_1, \cdots, x_n)$

$M^* = M$ $n \times n$ matrix $y^* = \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{pmatrix}$