Math 5555

Abstract Algebra II

Book 1

In group theory we have

- permutation representations: homomorphisms $\pi : G \to S_n$ permutation representation of degree n
  (if π is 1-to-1 then π is a faithful representation: then $\underset{\cong G}{\pi(G)} \le S_n$)
- linear representations: homomorphism $\pi : G \to GL_n(F)$ F: field
  linear representation of degree n over F
  If $F = \mathbb{C}$ (or $\mathbb{R}$ or ...) then π is an ordinary representation.
  If char $F = p$ (prime) then π is a modular representation.

G: unless otherwise specified, G finite group. (until later...)

Usually $F = \mathbb{C}$ (or $\mathbb{R}$) and $|G| < \infty$.

If $\pi_i : G \to GL_{n_i}(\mathbb{C})$ $(i = 1,2)$ then $\pi_1 \oplus \pi_2 : G \to GL_{n_1+n_2}(\mathbb{C})$, $g \mapsto \begin{array}{c} \left[\begin{array}{c|c} \pi_1(g) & O \\ \hline O & \pi_2(g) \end{array}\right] \begin{array}{l} \}\, n_1 \\ \}\, n_2 \end{array} \\ \underbrace{\quad}_{n_1}\ \underbrace{\quad}_{n_2} \end{array}$ is a representation of degree $n_1 + n_2$

$\pi : G \to GL_n(\mathbb{C})$ is <u>decomposable</u> if there is a decomposition $\mathbb{C}^n = U \oplus V$ such that $U, V \neq 0$
(dim $U = n_1$, dim $V = n_2$, n_1, n_2 positive integers, $n_1 + n_2 = n$)
U, V invariant under all matrices $\pi(g)$, $g \in G$.

$G = S_3$ acting naturally on $\mathbb{C}^3$ by permuting the standard basis vectors $e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

via $\sigma : e_i \mapsto e_{\sigma(i)}$

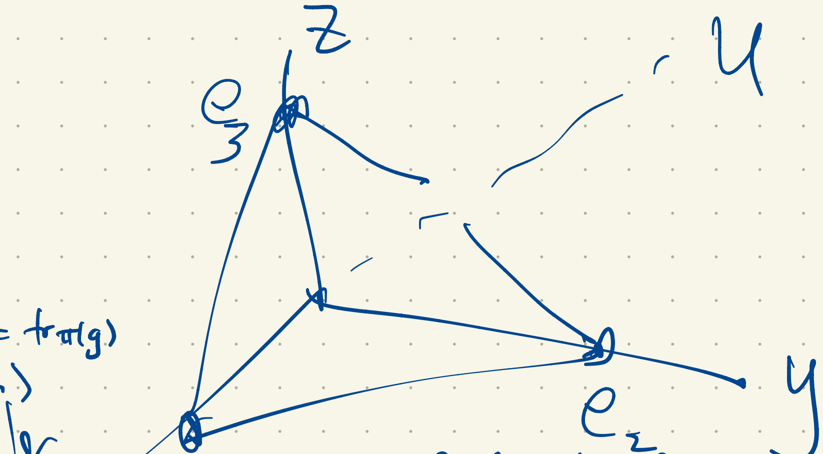
i.e. $(12) \mapsto \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$

$(123) \mapsto \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$

$() \mapsto \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

This is a faithful representation of degree 3.

It is decomposable: $\mathbb{C}^3 = U \oplus V$, $U = \left\langle \begin{bmatrix}1\\1\\1\end{bmatrix} \right\rangle$, $V = U^\perp = \underbrace{\left\langle \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix} \right\rangle}_{\text{indecomposable.}}$



For a permutation representation
$\pi : G \to S_n \subset GL_n(\mathbb{C})$
the associated character $\chi(g) = \mathrm{tr}\,\pi(g)$
(called the permutation character)
is $\chi(g) =$ no. of fixed points of g
$= |\{i : g(i) = i\}|$, $1 \le i \le n$.

In general character values are not always integers.

The representation $\pi : S_3 \to GL_3(\mathbb{C})$
has (affords) character
$\chi(g) = \mathrm{tr}\,\pi(g)$

$\chi(1) = 3 =$ degree of π = deg χ.

$\chi((12)) = \chi((13)) = \chi((23)) = 1$

$\chi((123)) = \chi((132)) = 0$

Given a representation $\pi : G \to GL_n(\mathbb{C})$, the character of π is

$$\chi(g) = \operatorname{tr} \pi(g) \in \mathbb{C}.$$

Character values $\chi(g) \in \mathbb{C}$ are always algebraic integers ($g \in G$, $|G| < \infty$)
and character values of S_n are ordinary integers.

$\chi(g)$ depends only on the conjugacy class of g.

If $g, h \in G$ then $g \sim h^{-1}gh$ (conjugate in G) so

$$\pi(h^{-1}gh) = \pi(h^{-1})\pi(g)\pi(h) \sim \pi(g) \quad \text{(similar in } GL_n(\mathbb{C}) \text{ i.e. conjugate)}$$

inverses in $GL_n(\mathbb{C})$

so $\operatorname{tr} \pi(g) = \operatorname{tr}(\pi(h^{-1})\pi(g)\pi(h)) = \operatorname{tr} \pi(h^{-1}gh)$

$\operatorname{tr}(AB) = \operatorname{tr}(BA)$

$\operatorname{tr}(B^{-1}AB) = \operatorname{tr}(AB \cdot B^{-1}) = \operatorname{tr} A$

$\chi(h^{-1}gh) = \chi(g)$.

If $\pi : G \to GL_n(\mathbb{C})$ is any representation i.e. homomorphism, and $B \in GL_n(\mathbb{C})$, then
$\tilde{\pi}(g) = B^{-1}\pi(g)B \in GL_n(\mathbb{C})$ is also a representation

$$\tilde{\pi}(gh) = B^{-1}\pi(gh)B = B^{-1}\pi(g)\pi(h)B = B^{-1}\pi(g)B \cdot B^{-1}\pi(h)B = \tilde{\pi}(g)\tilde{\pi}(h).$$

$\pi, \tilde{\pi}$ are equivalent (via a change of basis).

They have the same character: the character of $\tilde{\pi}$ is

$$\tilde{\chi}(g) = \operatorname{tr} \tilde{\pi}(g) = \operatorname{tr}(B^{-1}\pi(g)B) = \operatorname{tr} \pi(g) = \chi(g).$$

It's not obvious but the converse is true: χ determines π up to equivalence.
Two representations have the same character iff they are equivalent.

Let $\pi : G \to GL_n(\mathbb{C})$ be a representation (i.e. homomorphism).
π is reducible if there exists nontrivial proper subspace $U < \mathbb{C}^n$ ($\dim U \in \{1, 2, \dots, n-1\}$) such that
U is invariant under $\pi(g)$ for all $g \in G$ i.e. $\pi(g)U \subseteq U$ for all $g \in G$.

i.e.
$$\pi(g) = \begin{array}{cc} \left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right] & \begin{array}{l} \}\, k \\ \}\, n-k \end{array} \\ \begin{array}{cc} k & n-k \end{array} & \end{array} \qquad k = \dim U.$$

for all $g \in G$.

When $|G| < \infty$ and $\pi : G \to GL_n(F)$
char $F = 0$, π reducible $\Longleftrightarrow$ π decomposable
π irreducible $\Longleftrightarrow$ π indecomposable

In general, π decomposable $\Rightarrow$ π reducible

$$\left[\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right] \Rightarrow \left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right]$$

π indecomposable $\Leftarrow$ π irreducible.

The representation $\underset{\substack{\text{additive}\\ \text{group}}}{\pi : \mathbb{R}} \longrightarrow GL_2(\mathbb{R})$, $\quad \pi(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, $\quad \pi(a+b) = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \pi(a)\pi(b)$

is reducible: $\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$ is an invariant subspace. There is no complementary invariant subspace (in particular the complementary subspace $\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle$ is not invariant).

<u>Maschke's Theorem</u> Let G be a finite group and $F = \mathbb{C}$ (or more generally F is any field of characteristic not dividing $|G|$) Then any representation $\pi : G \to GL_n(F)$ is reducible iff it's decomposable (ie. π is irreducible iff π is indecomposable; ie. whenever every invariant subspace $U \le F^n$ has a complementary subspace U' which is also invariant).

$U \le V$ has a complementary subspace U' $\Leftrightarrow$ $V = U \oplus U'$. If $\dim V = n$ and $\dim U = k$ then U' is a complement to U iff $\dim U' = n-k$ and $U + U' = V$

iff $\dim U' = n-k$ and $U \cap U' = \{0\}$.

In this case every $v \in V$ is uniquely represented as $v = u + u'$, $u \in U$, $u' \in U'$.

In this, the projection $V \to U$ along U' is the map $P : v \mapsto Pv = u$. $\quad P = \begin{array}{c} \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right] \begin{array}{l} \}k \\ \}n-k \end{array} \\ \underbrace{\ }_{k}\ \underbrace{\ }_{n-k} \end{array}$

P has image $PV = U$; $\ker P = U'$. Note: $I - P$ is the projection from V onto U' along U.

A linear transformation $P : V \to V$ is a <u>projection</u> iff $P^2 = P$. In this case P is a root of $x^2 - x$ so the eigenvalues of P are in $\{0, 1\}$. Take U = 1-eigenspace of P = $\ker(I - P)$, $U' = \ker P$.

Proof of Maschke's Theorem: (in the nontrivial direction)

Suppose $\pi: G \to GL_n(\mathbb{C})$ is a representation having an invariant subspace $U \le \mathbb{C}^n$ i.e. $\pi(g)U \subseteq U$ for all $g \in G$. We want to find a complementary subspace $W \le \mathbb{C}^n$ which is also invariant. Start with any complementary subspace U' where $\mathbb{C}^n = U \oplus U'$, $\dim U = k$, $\dim U' = n-k$; every $v \in \mathbb{C}^n$ is uniquely expressible as $v = u + u'$, $u \in U$, $u' \in U'$. Unfortunately U' is not invariant in general. Let $P: \mathbb{C}^n \to U$ be the projection onto U along U' i.e. $P(v) = P(u+u') = u$, so $U = PV$, $U' = \ker P$. Consider the new map $\tilde{P}: V \to V$ defined by

$$\tilde{P} = \frac{1}{|G|} \sum_{g \in G} \underbrace{\pi^{-1}(g) P \pi(g)}_{n \times n \text{ matrix}}.$$

$\tilde{P} V \subseteq U$ since $\tilde{P} v = \frac{1}{|G|} \sum_g \underbrace{\pi(g^{-1}) \underbrace{P \underbrace{\pi(g) v}_{\in V}}_{\in U}}_{\in U} \in U$.

For all $u \in U$, $\tilde{P} u = u$ i.e. $\tilde{P}|_U = id|_U$. Why?

$$\tilde{P} u = \frac{1}{|G|} \sum_g \pi(g^{-1}) \underbrace{P \pi(g) u}_{= \pi(g) u \text{ since } \pi(g) u \in U} = \frac{1}{|G|} \sum_g \pi(g)^{-1} \pi(g) u = \frac{1}{|G|} |G| u = u.$$

Next: show $\tilde{P}$ commutes with all $\pi(g)$, $g \in G$. (P doesn't satisfy this in general!)

Then

$$\tilde{P} \pi(g) = \frac{1}{|G|} \sum_{h \in G} \pi(h^{-1}) P \pi(h) \pi(g) = \frac{1}{|G|} \sum_{h \in G} \pi(h^{-1}) P \pi(hg)$$

$$= \frac{1}{|G|} \sum_{x \in G} \underbrace{\pi(g x^{-1})}_{\pi(g)\pi(x)^{-1}} P \pi(x) = \pi(g) \cdot \frac{1}{|G|} \sum_{x \in G} \pi(x)^{-1} P \pi(x) = \pi(g) \tilde{P}.$$

$h \longleftrightarrow x = hg$

$x^{-1} = g^{-1} h^{-1}$

$g x^{-1} = h^{-1}$

Finish this proof on Wed.