Math 5555

Abstract Algebra II

Book 2

Induction takes class functions on H to class functions on G
characters ⋯ ⋯ characters on G
representations of H ⋯ representations of G.

Let χ be a class function on $H \le G$ i.e. $\chi : H \to \mathbb{C}$, $\chi(xhx^{-1}) = \chi(h)$ for all $x, h \in H$.

To get a class function on G, start with the trivial extension

$$\hat{\chi} : G \longrightarrow \mathbb{C}$$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H; \\ 0 & \text{if } g \notin H. \end{cases}$$

To make this into a class function, use an averaging over conjugates as we did before. This leads to $\chi^G : G \longrightarrow \mathbb{C}$:

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1}) \qquad (\chi^G \text{ is \underline{induced} from } \chi, \quad \chi^G = \mathrm{Ind}_H^G \chi)$$

$w^{-1} = u^{-1}x^{-1}$
$w = xu$
$wu = x$

If $u \in G$ then $\chi^G(ugu^{-1}) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xugu^{-1}x^{-1}) = \frac{1}{|H|} \sum_{w \in G} \hat{\chi}(wgw^{-1})$

$$= \chi^G(g).$$

So χ^G is a class function on G.

Note: Let T be a set of right coset representatives for H in G.
So every element $g \in G$ is uniquely expressible as $g = ht$, $h \in H$, $t \in T$

$$|G| = |H||T| \quad \text{(Lagrange's Theorem)}, \quad |T| = \frac{|G|}{|H|} = [G : H].$$

$\chi : H \to \mathbb{C}$ is a class function on H; T is a right transversal for H in G (set of right coset representatives)

$T = \{t_1, \dots, t_\ell\}$

$G = Ht_1 \sqcup Ht_2 \sqcup \cdots \sqcup Ht_\ell = \bigsqcup_{t \in T} Ht.$

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1})$$

$x = ht$, $h \in H$, $t \in T$

$$= \frac{1}{|H|} \sum_{t \in T} \sum_{h \in H} \hat{\chi}(htgt^{-1}h^{-1})$$

$htgt^{-1}h^{-1} \in H \iff tgt^{-1} \in H$

$$= \frac{1}{|H|} \sum_{t \in T} \underbrace{\sum_{h \in H} \hat{\chi}(tgt^{-1})}_{|H| \text{ terms equal to } \hat{\chi}(tgt^{-1})}$$

$$= \frac{1}{|H|} \sum_{t \in T} |H| \hat{\chi}(tgt^{-1}) = \sum_{t \in T} \hat{\chi}(tgt^{-1}) = \sum_{i=1}^{\ell} \hat{\chi}(t_i g t_i^{-1})$$

Special case: $\chi = \chi_1$ = trivial (principal) character of H, $\chi(h) = 1$.

χ^G isn't the principal character of G unless $G = H$.

G permutes the right cosets of H by right multiplication giving a permutation representation $g \in G$ permutes $Ht_i \longmapsto Ht_i g = Ht_j$

$G \longrightarrow S_\ell$ $\quad \ell = [G:H] = |T|$.

$\chi^G = (1_H)^G$ is the perm. character of G acting on cosets of H.

Eg. Construct the character table of $G = S_4$ making use of the character table of $S_3 = H \leq G$.

H:

$\lvert C_H(g)\rvert$	6	2	3
g	$()$	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$G = S_4$:

$\lvert C_G(g)\rvert$	24	4	8	3	4
g	$()$	(12)	$(12)(34)$	(123)	(1234)
ψ_1	1	1	1	1	1
ψ_2	1	-1	1	1	-1
ψ_3	2	0	2	-1	0
ψ_4	3				
ψ_5	3				
ψ	3	1	3	0	1

Check: $[\psi_3, \psi_3] = 1$

$[\psi_1, \psi_2] = 0$

$$\underbrace{\psi(1)}_{3} = \underbrace{\psi_1(1)}_{1} + \underbrace{\psi_3(1)}_{2}$$

$$\psi = \sum_{i=1}^{5} a_i \psi_i$$

$$[\psi, \psi]_G = \frac{\cancel{9}^{3}}{\cancel{24}_{8}} + \frac{1}{4} + \frac{9}{8} + \frac{0}{3} + \frac{1}{4}$$

$$= \frac{3+2+9+0+2}{8} = \frac{16}{8}$$

$$= 2$$

$$= \sum_{i=1}^{5} a_i^2 = 1^2 + 1^2 + 0^2 + 0^2 + 0^2$$

$$\psi = \psi_1 + \psi_3$$

(degree 1) (degree 2)

$$a_1 = [\psi, \psi_1] = \frac{3}{24} + \frac{1}{4} + \frac{3}{8} + \frac{0}{3} + \frac{1}{4} = \frac{1+2+3+0+2}{8} = 1$$

S_4 has $k = 5$ conjugacy classes

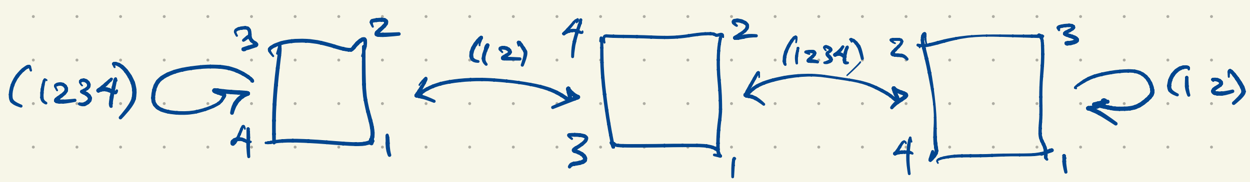
... irreducible representations/characters of degree $n_1, n_2, \ldots, n_5 \geq 1$, $n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = |G| = 24$

S_4 has normal subgroups

1, S_4, A_4, $K = \langle (12)(34), (13)(24) \rangle = \{ (), (12)(34), (13)(24), (14)(23) \}$.

S_4 permutes the conjugacy class of $(12)(34)$ in all $3! = 6$ possible ways

There is a permutation action $S_4 \longrightarrow \mathrm{Sym}\{(12)(34), (13)(24), (14)(23)\} \cong S_3$ with kernel K of order 4.



$C_G((13)(24))$ $\quad$ $C_G((14)(23))$ $\quad$ $C_G((12)(34))$

This gives a permutation representation of S_4 of degree 3.

Its permutation character is

$\psi((1)) = 3$

$\psi(k) = 3$ for $k \in K$

$\psi((12)) = 1$

$\psi((1234)) = 1$ $\quad$ $\psi((123)) = 0$ $\quad$ $\psi((12)(34)) = 3$

$H = S_3$:

$\lvert C_H(g)\rvert$	6	2	3
g	$()$	(12)	(123)
χ_1	1	1	1
$\chi = \chi_2$	1	-1	1
χ_3	2	0	-1

$G = S_4$:

$\lvert C_G(g)\rvert$	24	4	8	3	4
g	$()$	(12)	$(12)(34)$	(123)	(1234)
ψ_1	1	1	1	1	1
ψ_2	1	-1	1	1	-1
ψ_3	2	0	2	-1	0
ψ_4	3	-1	-1	0	1
ψ_5	3	1	-1	0	-1
χ^G	4	-2	0	1	0

$$[\chi^G, \chi^G]_G = \frac{16}{24} + \frac{4}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 2$$

$\Rightarrow \chi^G$ has irreducible constituents of multiplicity 1, 1, 0, 0, 0

$$[\chi^G, \psi_1] = \frac{4}{24} - \frac{2}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 0$$

$$[\chi^G, \psi_2] = \frac{4}{24} + \frac{2}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 1$$

$T = \langle (1234) \rangle = \{(), (1234), (13)(24), (1432)\}$ right transversal for H in G

$\chi^G = \chi_2^G$ is a character on G

$$\chi^G(g) = \sum_{t \in T} \hat{\chi}(tgt^{-1})$$

where $\hat{\chi} : G \longrightarrow \mathbb{C}$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

$$\chi^G = \psi_2 + \psi_4$$

$$\psi_4 = \chi^G - \psi_2$$

$$[\psi_4, \psi_4]_G = \frac{9}{24} + \frac{1}{4} + \frac{1}{8} + \frac{0}{3} + \frac{1}{4} = \frac{3+2+1+0+2}{8} = 1$$

$$\chi^G(()) = 1 + 1 + 1 + 1 = 4$$

$$\chi^G((12)) = \hat{\chi}((12)) + \hat{\chi}((23)) + \underbrace{\cancel{\hat{\chi}((34))}}_{0} + \underbrace{\cancel{\hat{\chi}((14))}}_{0} = -1 - 1 + 0 + 0 = -2$$

$$\chi^G((12)(34)) = \hat{\chi}((12)(34)) + \hat{\chi}((23)(41)) + \hat{\chi}((34)(12)) + \hat{\chi}((41)(23)) = 0 + 0 + 0 + 0 = 0$$

$$\chi^G((123)) = \hat{\chi}((123)) + \cancel{\hat{\chi}((234))} + \cancel{\hat{\chi}((341))} + \cancel{\hat{\chi}((412))} = 1 + 0 + 0 + 0 = 1$$

Frobenius Reciprocity Let χ be a class function on $H \leq G$ and let ψ be a class function on G. Then

$$\chi^G(g) = \frac{1}{|H|}\sum_{x\in G}\hat{\chi}(xgx^{-1}) \qquad \hat{\chi}(g) = \begin{cases}\chi(g) & \text{if } g\in H\\ 0 & \text{if } g\notin H\end{cases}$$

$$[\psi_H, \chi]_H = [\psi, \chi^G]_G$$

$\psi_H = \psi|_H$

Proof $$[\chi^G, \psi]_G = \frac{1}{|G|}\sum_{g\in G}\chi^G(g)\overline{\psi(g)}$$

$$= \frac{1}{|G|}\sum_{g\in G}\sum_{x\in G}\frac{1}{|H|}\hat{\chi}(x^{-1}gx)\overline{\psi(g)}$$

$x\in G = HT$
$x = ht,\ h\in H,\ t\in T$

$$= \frac{1}{|G||H|}\sum_{g\in G}\sum_{x\in G}\hat{\chi}(xgx^{-1})\overline{\psi(g)}$$

$htgt^{-1}h^{-1}\in H \iff tgt^{-1}\in H$

$$= \frac{1}{|G||H|}\sum_{g\in G}\sum_{h\in H}\sum_{t\in T}\hat{\chi}(htgt^{-1}h^{-1})\overline{\psi(g)}$$

$$= \frac{1}{|G||H|}\sum_{g\in G}\sum_{h\in H}\sum_{t\in T}\underbrace{\hat{\chi}(tgt^{-1})\overline{\psi(g)}}_{|H| \text{ identical terms } \cancel{\text{for}}\ h\in H}$$

$$= \frac{1}{|G|}\sum_{g\in G}\sum_{t\in T}\hat{\chi}(tgt^{-1})\overline{\psi(g)}$$