Math 5555

Abstract Algebra II

Book 1

In group theory we have

- permutation representations: homomorphisms $\pi : G \to S_n$ permutation representation of degree n
  (if π is 1-to-1 then π is a faithful representation: then $\underset{\cong G}{\pi(G)} \leq S_n$)
- linear representations: homomorphism $\pi : G \to GL_n(F)$ F: field
  linear representation of degree n over F

  If $F = \mathbb{C}$ (or $\mathbb{R}$ or ...) then π is an ordinary representation.

  If char $F = p$ (prime) then π is a modular representation.

  G: unless otherwise specified, G finite group. (until later...)

Usually $F = \mathbb{C}$ (or $\mathbb{R}$) and $|G| < \infty$.

If $\pi_i : G \to GL_{n_i}(\mathbb{C})$ $(i = 1,2)$ then $\pi_1 \oplus \pi_2 : G \to GL_{n_1+n_2}(\mathbb{C})$, $g \mapsto \begin{bmatrix} \pi_1(g) & O \\ O & \pi_2(g) \end{bmatrix}$ is a representation of degree $n_1 + n_2$

$\pi : G \to GL_n(\mathbb{C})$ is decomposable if there is a decomposition $\mathbb{C}^n = U \oplus V$ such that $U, V \neq 0$ (dim $U = n_1$, dim $V = n_2$, n_1, n_2 positive integers, $n_1 + n_2 = n$)

U, V invariant under all matrices $\pi(g)$, $g \in G$.

$G = S_3$ acting naturally on $\mathbb{C}^3$ by permuting the standard basis vectors $e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

via $\sigma : e_i \mapsto e_{\sigma(i)}$

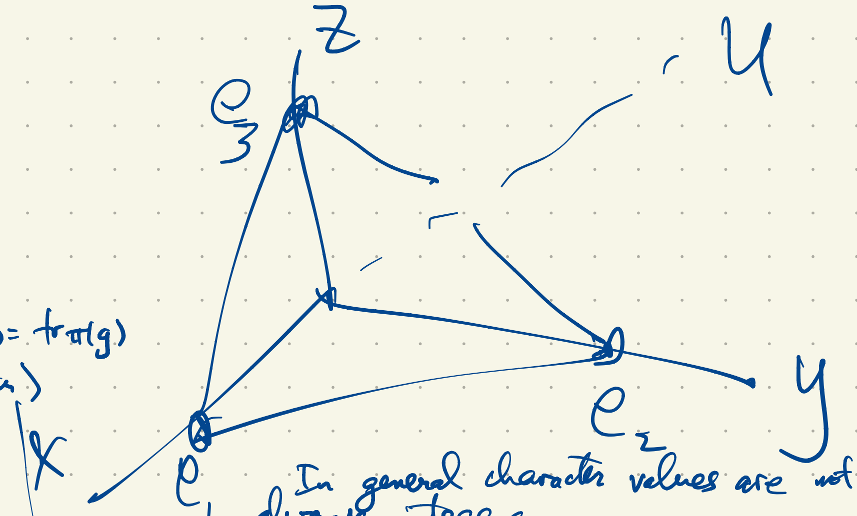
i.e. $(12) \mapsto \begin{bmatrix} 0&1&0\\1&0&0\\0&0&1 \end{bmatrix}$

$(123) \mapsto \begin{bmatrix} 0&0&1\\1&0&0\\0&1&0 \end{bmatrix}$

$() \mapsto \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}$

This is a faithful representation of degree 3.

It is decomposable: $\mathbb{C}^3 = U \oplus V$, $U = \left\langle \begin{bmatrix}1\\1\\1\end{bmatrix} \right\rangle$, $V = U^\perp = \underbrace{\left\langle \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix} \right\rangle}_{\text{indecomposable.}}$



For a permutation representation

$\pi : G \to S_n \subset GL_n(\mathbb{C})$

the associated character $\chi(g) = \operatorname{tr} \pi(g)$

(called the permutation character)

is $\chi(g) =$ no. of fixed points of g

$= |\{i : g(i) = i\}|$, $1 \leq i \leq n$.

In general character values are not always integers.

The representation $\pi : S_3 \to GL_3(\mathbb{C})$ has (affords) character

$\chi(g) = \operatorname{tr} \pi(g)$

$\chi(1) = 3 =$ degree of $\pi = \deg \chi$

$\chi((12)) = \chi((13)) = \chi((23)) = 1$

$\chi((123)) = \chi((132)) = 0$

Given a representation $\pi : G \to GL_n(\mathbb{C})$, the character of π is

$$\chi(g) = \operatorname{tr}\pi(g) \in \mathbb{C}.$$

Character values $\chi(g) \in \mathbb{C}$ are always algebraic integers ($g \in G$, $|G| < \infty$) and character values of S_n are ordinary integers.

$\chi(g)$ depends only on the conjugacy class of g.

If $g, h \in G$ then $g \sim h^{-1}gh$ (conjugate in G) so

$$\pi(h^{-1}gh) = \pi(h)^{-1}\pi(g)\pi(h) \sim \pi(g) \quad \text{(similar in } GL_n(\mathbb{C}) \text{ i.e. conjugate)}$$

inverses in $GL_n(\mathbb{C})$

so $\operatorname{tr}\pi(g) = \operatorname{tr}(\pi(h^{-1})\pi(g)\pi(h)) = \operatorname{tr}\pi(h^{-1}gh)$

$\operatorname{tr}(AB) = \operatorname{tr}(BA)$

$\operatorname{tr}(B^{-1}AB) = \operatorname{tr}(AB \cdot B^{-1}) = \operatorname{tr} A$

$\chi(h^{-1}gh) = \chi(g)$.

If $\pi : G \to GL_n(\mathbb{C})$ is any representation i.e. homomorphism, and $B \in GL_n(\mathbb{C})$, then

$\tilde{\pi}(g) = B^{-1}\pi(g)B \in GL_n(\mathbb{C})$ is also a representation

$\pi, \tilde{\pi}$ are equivalent (via a change of basis).

$$\tilde{\pi}(gh) = B^{-1}\pi(gh)B = B^{-1}\pi(g)\pi(h)B = B^{-1}\pi(g)B \cdot B^{-1}\pi(h)B = \tilde{\pi}(g)\tilde{\pi}(h).$$

They have the same character: the character of $\tilde{\pi}$ is

$$\tilde{\chi}(g) = \operatorname{tr}\tilde{\pi}(g) = \operatorname{tr}(B^{-1}\pi(g)B) = \operatorname{tr}\pi(g) = \chi(g).$$

It's not obvious but the converse is true: χ determines π up to equivalence.
Two representations have the same character iff they are equivalent.

Let $\pi : G \to GL_n(\mathbb{C})$ be a representation (i.e. homomorphism).

π is reducible if there exists nontrivial proper subspace $U < \mathbb{C}^n$ ($\dim U \in \{1, 2, \dots, n-1\}$) such that U is invariant under $\pi(g)$ for all $g \in G$ i.e. $\pi(g)U \subseteq U$ for all $g \in G$.

i.e.
$$\pi(g) = \begin{array}{c} \left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right] \begin{array}{l} \}k \\ \}n-k \end{array} \\ \begin{array}{cc} k & n-k \end{array} \end{array} \qquad k = \dim U$$
for all $g \in G$.

when $|G| < \infty$ and $\pi : G \to GL_n(F)$, $\operatorname{char} F = 0$,
π reducible $\Longleftrightarrow$ π decomposable
π irreducible $\Longleftrightarrow$ π indecomposable

In general, π decomposable $\Longrightarrow$ π reducible

$$\left[\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right] \Longrightarrow \left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right]$$

π indecomposable $\Longleftarrow$ π irreducible.

The representation $\pi : \underset{\substack{\text{additive} \\ \text{group}}}{\mathbb{R}} \longrightarrow GL_2(\mathbb{R})$, $\pi(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, $\pi(a+b) = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \pi(a)\pi(b)$

is reducible: $\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$ is an invariant subspace. There is no complementary invariant subspace (in particular the complementary subspace $\left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$ is not invariant).