

Math 5555

Abstract Algebra II

Book 2

Induction takes class functions on H to class functions on G
 characters representations of H ——— characters on G representations of G

Let χ be a class function on $H \leq G$ i.e. $\chi: H \rightarrow \mathbb{C}$, $\chi(xhx^{-1}) = \chi(h)$ for all $x, h \in H$.
 To get a class function on G , start with the trivial extension

$$\hat{\chi}: G \rightarrow \mathbb{C}$$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H; \\ 0 & \text{if } g \notin H. \end{cases}$$

To make this into a class function, use an averaging over conjugates as we did before. This leads to $\chi^G: G \rightarrow \mathbb{C}$:

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1}) \quad (\chi^G \text{ is induced from } \chi, \quad \chi^G = \text{Ind}_H^G \chi)$$

$$\text{If } u \in G \text{ then } \chi^G(ugu^{-1}) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xug u^{-1} x^{-1}) = \frac{1}{|H|} \sum_{w \in G} \hat{\chi}(w g w^{-1})$$

$$\begin{aligned} w^{-1} &= u^{-1} x^{-1} \\ w &= x u \\ w u &= x \end{aligned}$$

So χ^G is a class function on G . $= \chi^G(g)$

Note: Let T be a set of right coset representatives for H in G .

So every element $g \in G$ is uniquely expressible as $g = ht$, $h \in H, t \in T$

$$|G| = |H| |T| \quad (\text{Lagrange's Theorem}), \quad |T| = \frac{|G|}{|H|} = [G:H].$$

$\chi: H \rightarrow \mathbb{C}$ is a class function on H ; T is a right transversal for H in G (set of right coset representatives)

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1})$$

$$x = ht, \quad \begin{matrix} h \in H \\ t \in T \end{matrix}$$

$$T = \{t_1, \dots, t_l\}$$

$$G = Ht_1 \sqcup Ht_2 \sqcup \dots \sqcup Ht_l = \bigsqcup_{t \in T} Ht$$

$$= \frac{1}{|H|} \sum_{t \in T} \sum_{h \in H} \hat{\chi}(htgt^{-1}h^{-1})$$

$$htgt^{-1}h^{-1} \in H$$

$$= \frac{1}{|H|} \sum_{t \in T} \sum_{h \in H} \hat{\chi}(tgt^{-1})$$

$$\iff tgt^{-1} \in H$$

$|H|$ terms equal to $\hat{\chi}(tgt^{-1})$

$$= \frac{1}{|H|} \sum_{t \in T} |H| \hat{\chi}(tgt^{-1}) = \sum_{t \in T} \hat{\chi}(tgt^{-1}) = \sum_{i=1}^l \hat{\chi}(t_i g t_i^{-1})$$

Special case: $\chi = \chi_1 =$ trivial (principal) character of H , $\chi(h) = 1$.

χ^G isn't the principal character of G unless $G=H$.

G permutes the right cosets of H by right multiplication giving a permutation representation $g \in G$ permutes $Ht_i \mapsto Ht_j = Ht_i g$.

$\chi^G = \left(\mathbb{1}_H \right)_G$ is the perm.

$$G \longrightarrow S_l$$

$$l = [G:H] = |T|.$$

character of G acting on cosets of H .

Ex. Construct the character table of $G = S_4$ making use of the character table of $S_3 = H \leq G$.

H:

| | | | |
|----------|-----|------|-------|
| $K_H(g)$ | 6 | 2 | 3 |
| g | (1) | (12) | (123) |
| χ_1 | 1 | 1 | 1 |
| χ_2 | 1 | -1 | 1 |
| χ_3 | 2 | 0 | -1 |

check: $[\chi_3, \chi_3] = 1$

$G = S_4$:

| | | | | | |
|------------|-----|------|----------|-------|--------|
| $(C_G(g))$ | 24 | 4 | 8 | 3 | 4 |
| g | (1) | (12) | (12)(34) | (123) | (1234) |
| ψ_1 | 1 | 1 | 1 | 1 | 1 |
| ψ_2 | 1 | -1 | 1 | 1 | -1 |
| ψ_3 | 2 | 0 | 2 | -1 | 0 |
| ψ_4 | 3 | | | | |
| ψ_5 | 3 | | | | |
| ψ | 3 | 1 | 3 | 0 | 1 |

S_4 has $k=5$ conjugacy classes

irreducible representations/characters of degree $n_1, n_2, \dots, n_5 \geq 1$, $n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = 16 = 2^4$

$[\psi_1, \psi_2] = 0$

$\psi(1) = \psi_1(1) + \psi_2(1) = 1 + 1 = 2$

$\psi = \sum_{i=1}^5 a_i \psi_i$

$[\psi, \psi]_G = \frac{1^3}{24} + \frac{1}{4} + \frac{9}{8} + \frac{0}{3} + \frac{1}{4} = \frac{3+2+9+0+2}{8} = \frac{16}{8} = 2$

$= \sum_{i=1}^5 a_i^2 = 2 \Rightarrow a_1^2 + a_2^2 = 2$

$\psi = \psi_1 + \psi_3$

(degree 1) (degree 2)

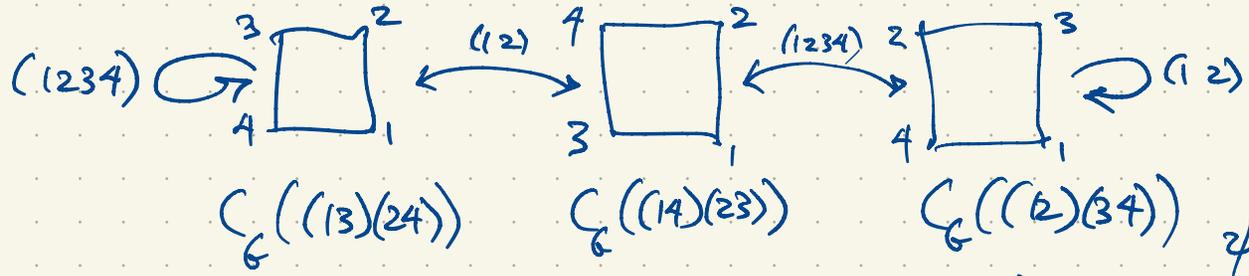
$a_i = [\psi, \psi_i] = \frac{3}{24} + \frac{1}{4} + \frac{3}{8} + \frac{0}{3} + \frac{1}{4} = \frac{1+2+3+0+2}{8} = 1$

S_4 has normal subgroups

$1, S_4, A_4, K = \langle (12)(34), (13)(24) \rangle = \{ (1), (12)(34), (13)(24), (14)(23) \}$

S_4 permutes the conjugacy class of $(12)(34)$ in all $3! = 6$ possible ways

there is a permutation action $S_4 \rightarrow \text{Sym} \{ (12)(34), (13)(24), (14)(23) \} \cong S_3$ with kernel K of order 4.



This gives a permutation representation of S_4 of degree 3. Its character is

$\psi((12)) = 1$, $\psi((1234)) = 1$, $\psi((123)) = 0$

$\psi(k) = 3$ for $k \in K$

$\psi((1)) = 3$, $\psi((12)(34)) = 3$

$$H: \begin{array}{c|ccc} K_H(g) & 6 & 2 & 3 \\ \hline g & (1) & (12) & (123) \\ \chi_1 & 1 & 1 & 1 \\ \chi_2 & 1 & -1 & 1 \\ \chi_3 & 2 & 0 & -1 \end{array}$$

$$G = S_4: \begin{array}{c|ccccc} (C_G) & 24 & 4 & 8 & 3 & 4 \\ \hline g & (1) & (12) & (12)(34) & (123) & (1234) \\ \chi_1 & 1 & 1 & 1 & 1 & 1 \\ \chi_2 & 1 & -1 & 1 & 1 & -1 \\ \chi_3 & 2 & 0 & 2 & -1 & 0 \\ \chi_4 & 3 & & & & \\ \chi_5 & 3 & & & & \\ \chi_6 & 4 & & & & \end{array}$$

$$T = \langle (1234) \rangle \\ = \{ (1), (1234), (13)(24), (1432) \}$$

$\chi^G = \chi_2^G$ is a character on G

$$\chi^G(g) = \sum_{t \in T} \hat{\chi}(tgt^{-1})$$

where $\hat{\chi}: G \rightarrow \mathbb{C}$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

$$\chi^G(1) = 1 + 1 + 1 + 1 = 4$$