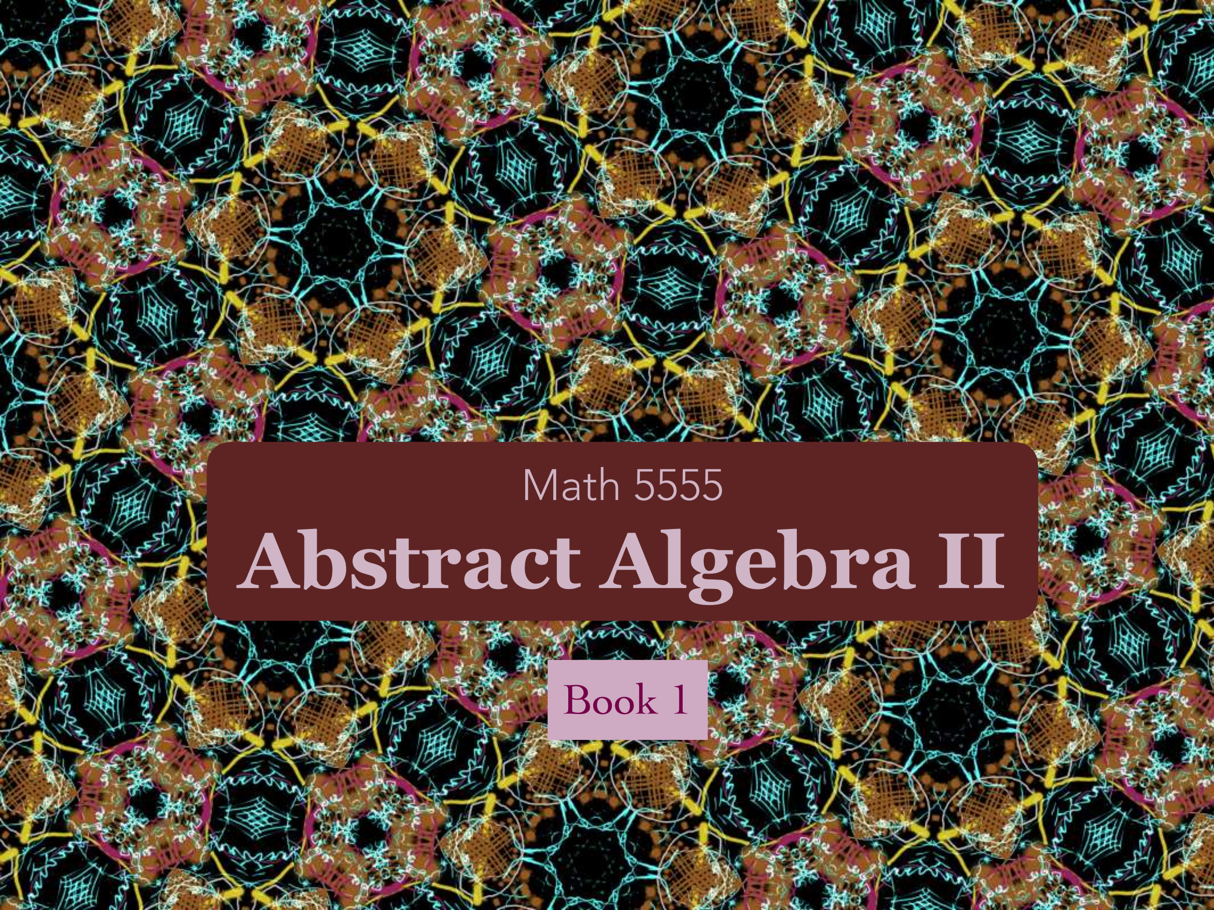


Math 5555

Abstract Algebra II

Book 1

In group theory we have

- permutation representations: homomorphisms $\pi : G \to S_n$ permutation representation of degree n
  (if π is 1-to-1 then π is a faithful representation: then $\underset{G}{\overset{\cong}{\pi(G)}} \leq S_n$)
- linear representations: homomorphism $\pi : G \to GL_n(F)$ F: field
  linear representation of degree n over F

  If $F = \mathbb{C}$ (or $\mathbb{R}$ or ...) then π is an ordinary representation.
  If char $F = p$ (prime) then π is a modular representation.

G : unless otherwise specified, G finite group. (until later...)

Usually $F = \mathbb{C}$ (or $\mathbb{R}$) and $|G| < \infty$.

If $\pi_i : G \to GL_{n_i}(\mathbb{C})$ $(i = 1,2)$ then $\pi_1 \oplus \pi_2 : G \to GL_{n_1+n_2}(\mathbb{C})$, $g \mapsto \begin{array}{cc} \left[\begin{array}{c|c} \pi_1(g) & O \\ \hline O & \pi_2(g) \end{array}\right] & \begin{array}{l} \}\, n_1 \\ \}\, n_2 \end{array} \\ \begin{array}{cc} \underbrace{\qquad}_{n_1} & \underbrace{\qquad}_{n_2} \end{array} & \end{array}$ is a representation of degree $n_1 + n_2$

$\pi : G \to GL_n(\mathbb{C})$ is decomposable if there is a decomposition $\mathbb{C}^n = U \oplus V$ such that $U, V \neq 0$
(dim $U = n_1$, dim $V = n_2$, n_1, n_2 positive integers, $n_1 + n_2 = n$)
U, V invariant under all matrices $\pi(g)$, $g \in G$.

$G = S_3$ acting naturally on $\mathbb{C}^3$ by permuting the standard basis vectors $e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

via $\sigma : e_i \mapsto e_{\sigma(i)}$

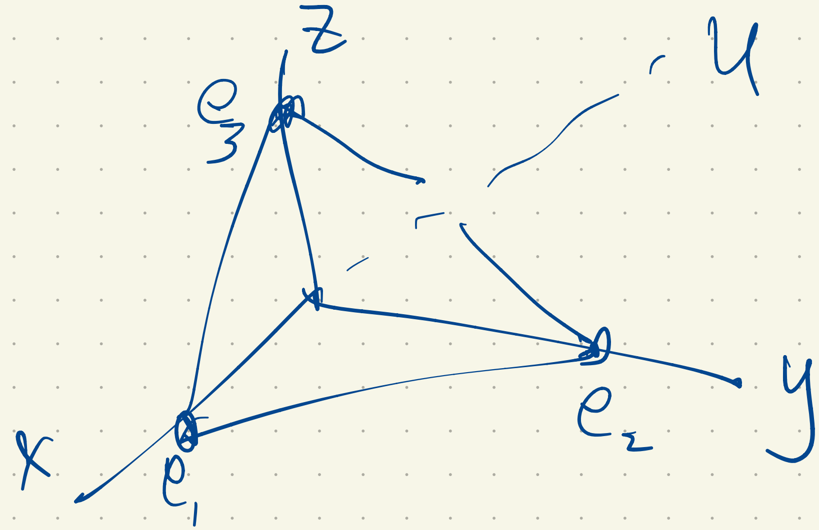
i.e. $(12) \mapsto \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$

$(123) \mapsto \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$

$() \mapsto \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

This is a faithful representation of degree 3.

It is decomposable: $\mathbb{C}^3 = U \oplus V$, $U = \left\langle \begin{bmatrix}1\\1\\1\end{bmatrix} \right\rangle$, $V = U^{\perp} = \underbrace{\left\langle \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix} \right\rangle}_{\text{indecomposable.}}$



Given a representation $\pi : G \longrightarrow GL_n(\mathbb{C})$, the character of π is

$$\chi(g) = \operatorname{tr} \pi(g) \in \mathbb{C}.$$

For our group G above ...