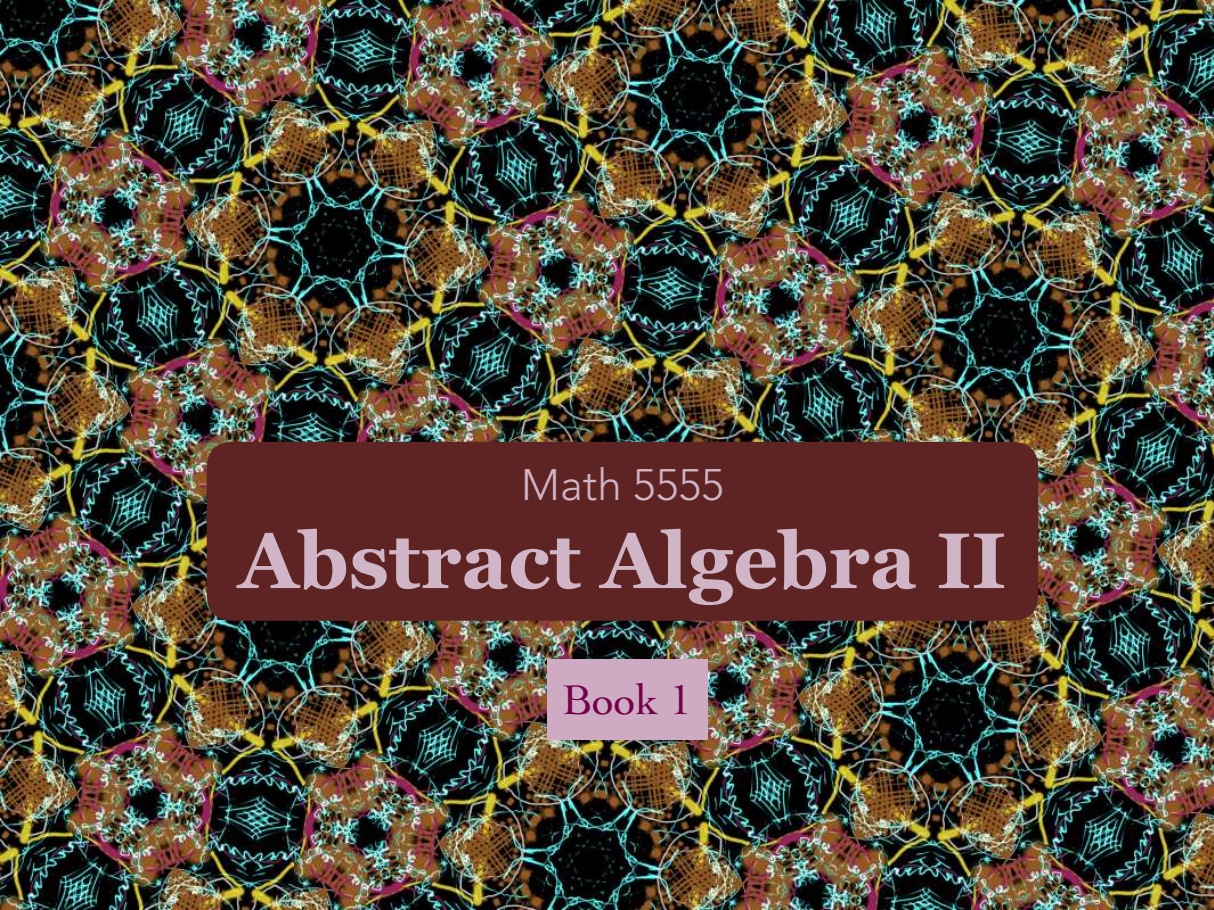




Math 5555

Abstract Algebra II

Book 1

In group theory we have

• permutation representations: homomorphisms $\pi: G \rightarrow S_n$ permutation representation of degree n
(if π is 1-to-1 then π is a faithful representation: then $\pi(G) \leq S_n$)

• linear representations: homomorphism $\pi: G \rightarrow GL_n(F)$ F : field
linear representation of degree n over F
If $F = \mathbb{C}$ (or $\mathbb{R}$ or ...) then π is an ordinary representation.

If $\text{char } F = p$ (prime) then π is a modular representation.

G : unless otherwise specified, G finite group. (Until later...)

Usually $F = \mathbb{C}$ (or $\mathbb{R}$) and $|G| < \infty$.

If $\pi_i: G \rightarrow GL_{n_i}(\mathbb{C})$ ($i=1,2$) then $\pi_1 \oplus \pi_2: G \rightarrow GL_{n_1+n_2}(\mathbb{C})$, $g \mapsto \left[\begin{array}{c|c} \pi_1(g) & 0 \\ \hline 0 & \pi_2(g) \end{array} \right]$ is a representation of degree $n_1 + n_2$

$\pi: G \rightarrow GL_n(\mathbb{C})$ is decomposable if there is a decomposition $\mathbb{C}^n = U \oplus V$ such that $U, V \neq 0$
($\dim U = n_1$, $\dim V = n_2$, n_1, n_2 positive integers, $n_1 + n_2 = n$)
 U, V invariant under all matrices $\pi(g)$, $g \in G$.

$G = S_3$ acting naturally on $\mathbb{C}^3$ by permuting the standard basis vectors $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

via $\sigma: e_i \mapsto e_{\sigma(i)}$

i.e.

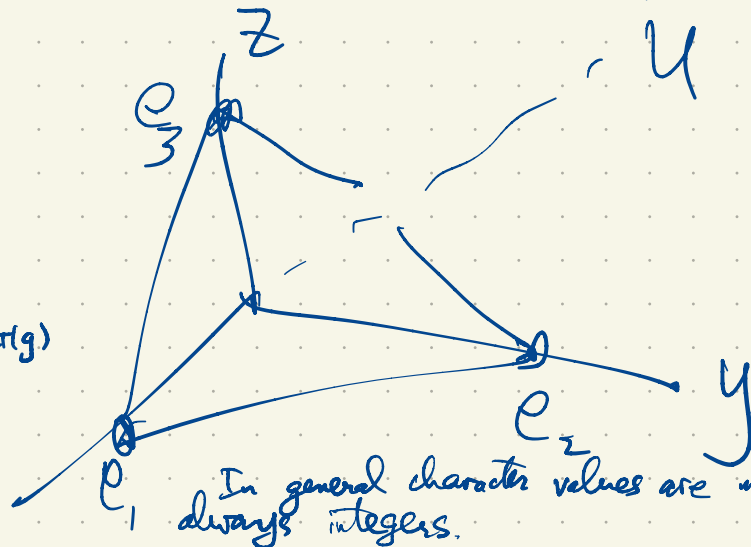
$$(12) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(123) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$(1) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is a faithful representation of degree 3.

It is decomposable: $\mathbb{C}^3 = U \oplus V$, $U = \langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rangle$, $V = U^\perp = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \rangle$
indecomposable.



The representation $\pi: S_3 \rightarrow GL_3(\mathbb{C})$ has (affords) character

$$\chi(g) = \text{tr } \pi(g)$$

$$\chi(1) = 3 = \text{degree of } \pi = \deg \chi$$

$$\chi((12)) = \chi((13)) = \chi((23)) = 1$$

$$\chi((123)) = \chi((132)) = 0$$

In general character values are not always integers.

For a permutation representation

$$\pi: G \rightarrow S_n \subset GL_n(\mathbb{C})$$

the associated character $\chi(g) = \text{tr } \pi(g)$
(called the permutation character)

is $\chi(g) = \text{no. of fixed points of } g$
 $= |\{i: g(i) = i\}|$, $1 \leq i \leq n$.

Given a representation $\pi: G \rightarrow GL_n(\mathbb{C})$, the character of π is

$$\chi(g) = \text{tr } \pi(g) \in \mathbb{C}.$$

Character values $\chi(g) \in \mathbb{C}$ are always algebraic integers ($g \in G$, $|G| < \infty$) and character values of S_n are ordinary integers.

$\chi(g)$ depends only on the conjugacy class of g .

If $g, h \in G$ then $g \sim h'gh$ (conjugate in G) so

$$\pi(h'gh) = \pi(h')\pi(g)\pi(h) \sim \pi(g) \text{ (similar in } GL_n(\mathbb{C}) \text{ i.e. conjugate)}$$

$\uparrow$
inverses in $GL_n(\mathbb{C})$

$$\text{so } \text{tr } \pi(g) = \text{tr } (\pi(h')\pi(g)\pi(h)) = \text{tr } \pi(h'gh)$$

$$\text{tr } (AB) = \text{tr } (BA)$$

$$\text{tr } (B^{-1}AB) = \text{tr } (AB \cdot B^{-1}) = \text{tr } A$$

$$\chi(h'gh) = \chi(g).$$

If $\pi: G \rightarrow GL_n(\mathbb{C})$ is any representation i.e. homomorphism, and $B \in GL_n(\mathbb{C})$, then

$\tilde{\pi}(g) = B^{-1}\pi(g)B \in GL_n(\mathbb{C})$ is also a representation.

$\pi, \tilde{\pi}$ are equivalent (via a change of basis).

$$\begin{aligned} \tilde{\pi}(gh) &= B^{-1}\pi(gh)B = B^{-1}\pi(g)\pi(h)B = B^{-1}\pi(g)B \cdot B^{-1}\pi(h)B \\ &= \tilde{\pi}(g)\tilde{\pi}(h). \end{aligned}$$

They have the same character: the character of $\tilde{\pi}$ is

$$\tilde{\chi}(g) = \text{tr } \tilde{\pi}(g) = \text{tr } (B^{-1}\pi(g)B) = \text{tr } \pi(g) = \chi(g).$$

It's not obvious but the converse is true: χ determines π up to equivalence. Two representations have the same character iff they are equivalent.

Let $\pi: G \rightarrow GL_n(\mathbb{C})$ be a representation (i.e. homomorphism).

π is reducible if there exists nontrivial ^{proper} subspace $U \subset \mathbb{C}^n$ ($\dim U \in \{1, 2, \dots, n-1\}$) such that U is invariant under $\pi(g)$ for all $g \in G$ i.e. $\pi(g)U \subseteq U$ for all $g \in G$.

$$\text{i.e. } \pi(g) = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \begin{matrix} \left\{ \begin{smallmatrix} k \\ n-k \end{smallmatrix} \right\} \\ \text{for all } g \in G. \end{matrix}$$

$$k = \dim U.$$

when $|G| < \infty$ and $\pi: G \rightarrow GL_n(\mathbb{C})$
 $\text{char } F = 0$, π reducible $\Leftrightarrow \pi$ decomposable
 π irreducible $\Leftrightarrow \pi$ indecomposable

In general, π decomposable $\Rightarrow \pi$ reducible
 $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \Rightarrow \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$
 π indecomposable $\Leftarrow \pi$ irreducible.

The representation $\pi: \mathbb{R} \rightarrow GL_2(\mathbb{R})$, $\pi(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, $\pi(a+b) = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \pi(a)\pi(b)$
 additive group

is reducible: $\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$ is an invariant subspace. There is no complementary invariant subspace (in particular the complementary subspace $\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle$ is not invariant).

Maschke's Theorem Let G be a finite group and $F = \mathbb{C}$ (or more generally F is any field of characteristic not dividing $|G|$). Then any representation $\pi: G \rightarrow GL(F)$ is reducible iff it's decomposable (i.e. π is irreducible iff π is indecomposable; i.e. whenever every invariant subspace $U \leq V$ has a complementary subspace U' which is also invariant).

$U \leq V$ has a complementary subspace $U' \iff V = U \oplus U'$. If $\dim V = n$ and $\dim U = k$ then U' is a complement to U iff $\dim U' = n-k$ and $U + U' = V$
 iff $\dim U' = n-k$ and $U \cap U' = \{0\}$.

In this case every $v \in V$ is uniquely represented as $v = u + u'$, $u \in U$, $u' \in U'$.

In this, the projection $V \rightarrow U$ along U' is the map $P: v \mapsto Pv = u$. $P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \text{size } k \\ \text{size } n-k \end{matrix}$
 P has image $PV = U$; $\ker P = U'$. Note: $I-P$ is the projection from V onto U' along U .

A linear transformation $P: V \rightarrow V$ is a projection iff $P^2 = P$. In this case P is a root of $x^2 - x$ so the eigenvalues of P are in $\{0, 1\}$. Take $U = 1$ -eigenspace of $P = \ker(I-P)$, $U' = \ker P$.

Proof of Maschke's Theorem: (in the nontrivial direction)

Suppose $\pi: G \rightarrow GL(\mathbb{C})$ is a representation having an invariant subspace $U \leq \mathbb{C}^n$ i.e. $\pi(g)U \subseteq U$ for all $g \in G$. We want to find a complementary subspace $W \leq \mathbb{C}^n$ which is also invariant. Start with any complementary subspace U' where $\mathbb{C}^n = U \oplus U'$, $\dim U = k$, $\dim U' = n-k$; every $v \in \mathbb{C}^n$ is uniquely expressible as $v = u + u'$, $u \in U$, $u' \in U'$.

Unfortunately U' is not invariant in general. Let $P: \mathbb{C}^n \rightarrow U$ be the projection onto U along U' i.e. $P(v) = P(u + u') = u$, so $U = PV$, $U' = \ker P$. Consider the new map $\tilde{P}: V \rightarrow V$ defined by $\tilde{P} = \frac{1}{|G|} \sum_{g \in G} \pi(g) P \pi(g)$. $\tilde{P}V \subseteq U$. Since $\tilde{P}v = \frac{1}{|G|} \sum_g \pi(g^{-1}) P \underbrace{\pi(g)v}_{\substack{\in V \\ \in U}} \in U$.

For all $u \in U$, $\tilde{P}u = u$ i.e. $\tilde{P}|_U = \text{id}|_U$. Why?

$$\tilde{P}u = \frac{1}{|G|} \sum_g \pi(g^{-1}) P \pi(g) u = \frac{1}{|G|} \sum_g \pi(g^{-1}) \pi(g) u = \frac{1}{|G|} |G| u = u.$$

$= \pi(g)u$ since $\pi(g)u \in U$.

Next: show $\tilde{P}$ commutes with all $\pi(g)$, $g \in G$. (P doesn't satisfy this in general!)

$$\text{Then } \tilde{P}\pi(g) = \frac{1}{|G|} \sum_{h \in G} \pi(h^{-1}) P \pi(h) \pi(g) = \frac{1}{|G|} \sum_{h \in G} \pi(h^{-1}) P \pi(hg)$$

$$h \longleftrightarrow \begin{aligned} x &= hg \\ x^{-1} &= g^{-1}h^{-1} \\ g x^{-1} &= h^{-1} \end{aligned}$$

$$= \frac{1}{|G|} \sum_{x \in G} \underbrace{\pi(gx^{-1})}_{\pi(g)\pi(x)^{-1}} P \pi(x) = \pi(g) \cdot \frac{1}{|G|} \sum_{x \in G} \pi(x^{-1}) P \pi(x) = \pi(g) \tilde{P}.$$

Next: show $\tilde{P}^2 = \tilde{P}$. If $v \in V$ then $\tilde{P}v \in U$ so $\tilde{P}^2v = \tilde{P}\tilde{P}v = \tilde{P}v$. So $\tilde{P}$ is idempotent so

$\tilde{P}$ is a projection on $\tilde{P}V = U$ along $W = \ker \tilde{P}$. Note: if $k = \dim U = \text{rank } \tilde{P} = \text{tr } \tilde{P}$, $\dim W = \dim \ker \tilde{P} = n - k$. W is also invariant under $\pi(g)$: if $v \in W$ then $\tilde{P}v = 0$

so $\pi(g)\tilde{P}v = \pi(g)0 = 0$ so $\pi(g)v \in W$

$\tilde{P}\pi(g)v$



The irreducible representations of S_3 are

$$\pi_1(g) = [1] \in GL_1(\mathbb{C}) \quad \text{trivial representation}$$

$$\pi_2(g) = [\text{sgn}(g)] \in GL_1(\mathbb{C}) \quad \text{sign representation:}$$

$$= [\pm 1] \text{ according as } g \text{ is an even or odd permutation.}$$

$$\text{sgn}((1)) = 1$$

$$\text{sgn}((123)) = 1$$

$$\text{sgn}((132)) = 1$$

$$\text{sgn}((12)) = \text{sgn}((13)) = \text{sgn}((23)) = -1$$

$$\text{sgn}(gh) = \text{sgn}(g) \text{sgn}(h)$$

$$\pi_3: S_3 \rightarrow GL_2(\mathbb{C})$$

$$(1) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(12) \mapsto \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$(123) \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$(132) \mapsto \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$(13) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(23) \mapsto \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\pi_3((13)) = \pi_3((123)(12)) = \pi_3((123)) \pi_3((12))$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\pi_3((23)) = \pi_3((123)(13)) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

An irreducible character is a map $G \rightarrow \mathbb{C}$

where π is an irreducible representation $g \mapsto \chi \pi(g)$.

$0, \langle [1] \rangle, \langle [-1] \rangle, \mathbb{C}^2$ are the only subspaces invariant under $\pi_3((13))$; but the 1-dimensional invariant subspaces are not invariant under $\pi_3((123))$.

The character table of S_3 is

	(1)	(12)	(23)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$$\pi_3((123)) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \text{ has char. poly. } x^2 + x + 1$$

$$\pi_3((13)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ has char. poly. } x^2 - 1$$

↑ eigenspaces $\langle [1] \rangle, \langle [-1] \rangle$
(eigenvalues 1, -1 respectively)

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Given any finite group G , a class function on G is a function $f: G \rightarrow \mathbb{C}$ which only depends on conjugacy class, i.e. $f(g') = f(g)$ whenever $g, g' \in G$ are conjugate i.e. $g' = ugu$ for some $u \in G$.

All characters of G (irreducible or otherwise) are class functions.

$V = \{\text{class functions on } G\}$ is a complex vector space

$$f, f' \in V \Rightarrow f + f' \in V, \quad (f + f')(x) = f(x) + f'(x)$$

V is in fact a complex inner product space:

$$[f, f']_G = [f, f'] = \frac{1}{|G|} \sum_{x \in G} f(x) \overline{f'(x)} \quad [f', f] = \overline{[f, f']}$$

sesquilinear
 $\frac{1}{2}$ -linear

$\dim V = \text{number of conjugacy classes}$.

The irreducible characters of G always give an orthonormal basis for $V = \{\text{class functions on } G\}$.

Recall the permutation representation with permutation character

$$\pi: S_3 \rightarrow GL_3(\mathbb{C})$$

$$\pi: e_i \mapsto e_{\pi(i)}$$

$$(12) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(123) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$(1) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\chi(1) = 3$$

$$\chi((12)) = 1$$

$$\chi((123)) = 0$$

$$[\chi, \chi_1] = \frac{1}{6}(1 \cdot 3 + 1 + 1 + 1 + 0) = 1$$

$$[\chi, \chi_2] = \frac{1}{6}(3 - 1 - 1 + 0 + 0) = 0$$

$$[\chi, \chi_3] = \frac{1}{6}(6 + 0 + 0 + 0 + 0) = 1$$

$$\Rightarrow \pi \simeq \pi_1 \oplus \pi_3$$

Character table of S_3

	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1
χ	3	1	0

$$[\chi_1, \chi_1] = \frac{1}{6} \sum_{g \in S_3} \chi_1(g) \overline{\chi_1(g)} = \frac{1}{6}(1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1) = 1$$

$$[\chi_1, \chi_2] = \frac{1}{6} \sum_g \chi_1(g) \overline{\chi_2(g)} = \frac{1}{6}(1 \cdot 1 - 1 \cdot 1 - 1 \cdot 1 - 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1) = 0$$

$$[\chi_1, \chi_3] = \frac{1}{6}(1 \cdot 2 + 0 + 0 + 0 - 1 \cdot 1) = 0$$

$$[\chi_2, \chi_2] = \frac{1}{6}(1 + 1 + 1 + 1 + 1 + 1) = 1$$

$$[\chi_2, \chi_3] = \frac{1}{6}(2 + 0 + 0 + 0 - 1 \cdot 1) = 0$$

$$[\chi_3, \chi_3] = \frac{1}{6}(4 + 0 + 0 + 0 + 1 \cdot 1) = 1$$

The long way to check the decomposition $\pi \simeq \pi_1 \oplus \pi_3$ is found in the course notes: with respect to new basis

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

basis for
first invariant
subspace

basis for
second invariant
subspace

$$\pi(g) \text{ has matrix } \left[\begin{array}{c|cc} \pi_1(g) & 0 & 0 \\ \hline 0 & \pi_3(g) \end{array} \right] \text{ with respect to } v_1, v_2, v_3$$

(see p. 3 of course notes).

Character tables also have orthogonality of columns.

$ C_g $	6	2	3
g	(1)	(12)	(23)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$$C_G(g) = \{x \in G: xg = gx\}$$

the conjugacy class of $g \in G$ is
the index of the centralizer

$$[G : C_G(g)] = \frac{|G|}{|C_G(g)|}$$

The column orthogonality is expressed as: given $g, h \in G$,
(k = number of irreducible characters = no. of conjugacy classes)

$$\sum_{i=1}^k \chi_i(g) \overline{\chi_i(h)} = \begin{cases} 0 & \text{if } g, h \text{ not conjugate} \\ |C_G(g)| & \text{if } g, h \text{ are conjugate} \end{cases}$$

Row orthogonality says: Let $g_1, \dots, g_k \in G$ be reps of conj. classes

$$[f, f']_G = \frac{1}{|G|} \sum_{a \in G} f(a) \overline{f'(a)} = \frac{1}{|G|} \sum_{j=1}^k \frac{|G|}{|C_G(g_j)|} f(g_j) \overline{f'(g_j)} = \sum_{j=1}^k \frac{1}{|C_G(g_j)|} f(g_j) \overline{f'(g_j)}$$

$$[\chi_i, \chi_j] = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Let's construct the character table of A_5 , $|A_5| = 60$, the smallest nonabelian simple group. $G = A_5$ has $k=5$ conjugacy classes $= 2^2 \cdot 3 \cdot 5$

$g_1 = ()$	identity	1A	size 1	$ C_G(g_1) = 60$
$g_2 = (12)(34)$		2A	size 15	$ C_G(g_2) = 4$
$g_3 = (123)$		3A	size 20	$ C_G(g_3) = 3$
$g_4 = (12345)$		5A	size 12	$ C_G(g_4) = 5$
$g_5 = (12354)$		5B	size 12	$ C_G(g_5) = 5$
			<u>60</u>	

(12345) and (12354) are conjugate in S_5 since $(45)^{-1}(12345)(45) = (12354)$

$$(1234)^{-1}(12345)(1234) = (23415) = (15234)$$

but (12345) and (12354) are not conjugate in A_5 .

$$(1235)^{-1}(12345)(1235) = (23541) = (12354)$$

Character table of A_5 :

$ C_G(g) $	60	4	3	5	5
g	$()$	$(12)(34)$	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	3				
χ_3	3				
χ_4	4				
χ_5	5				
χ	5	1	2	0	0

principal character

$$\chi_i(g) = \text{tr}(\rho_i(g)) = \text{tr}([1]) = 1$$

standard permutation character

$$\sum_{j=1}^k |\chi_j(1)|^2 = |G| \quad \text{special case of}$$

$$\sum_{j=1}^k |\chi_j(g)|^2 = |C_G(g)|$$

Write $60 = n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2$ where n_1, n_2, n_3, n_4, n_5 are positive integers.

$\chi = a_1 \chi_1 + a_2 \chi_2 + \dots + a_5 \chi_5$, $a_i \in \{0, 1, 2, \dots\}$ is the number of copies of χ_i in χ .

$$[\chi, \chi_i] = a_i$$

$$[\chi, \chi] = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = \frac{25}{60} + \frac{1}{4} + \frac{4}{3} + \frac{0}{5} + \frac{0}{5} = \frac{25+15+80}{60} = \frac{120}{60} = 2.$$

$\Rightarrow a_1, a_2, \dots, a_5 = 1, 1, 0, 0, 0$ in some order.

$\Rightarrow \chi = \chi_i + \chi_j$ for some $i \neq j$

$$[\chi, \chi_1] = \frac{1}{60} \cdot 5 \cdot 1 + \frac{1}{4} \cdot 1 \cdot 1 + \frac{1}{3} \cdot 2 \cdot 1 + \frac{1}{5} \cdot 0 \cdot 1 + \frac{1}{5} \cdot 0 \cdot 1$$

$$= \frac{1}{12} + \frac{1}{4} + \frac{2}{3} + 0 + 0$$

$$= \frac{1+3+8+0+0}{12} = \frac{12}{12} = 1$$

$$\chi = \chi_1 + \chi_j, \quad j \in \{2, 3, 4, 5\}.$$

$$\chi(1) = \chi_1(1) + \chi_j(1)$$

$$5 = 1 + \underbrace{\chi_j(1)}_4 \Rightarrow j=4$$

$$\chi = \chi_1 + \chi_4$$

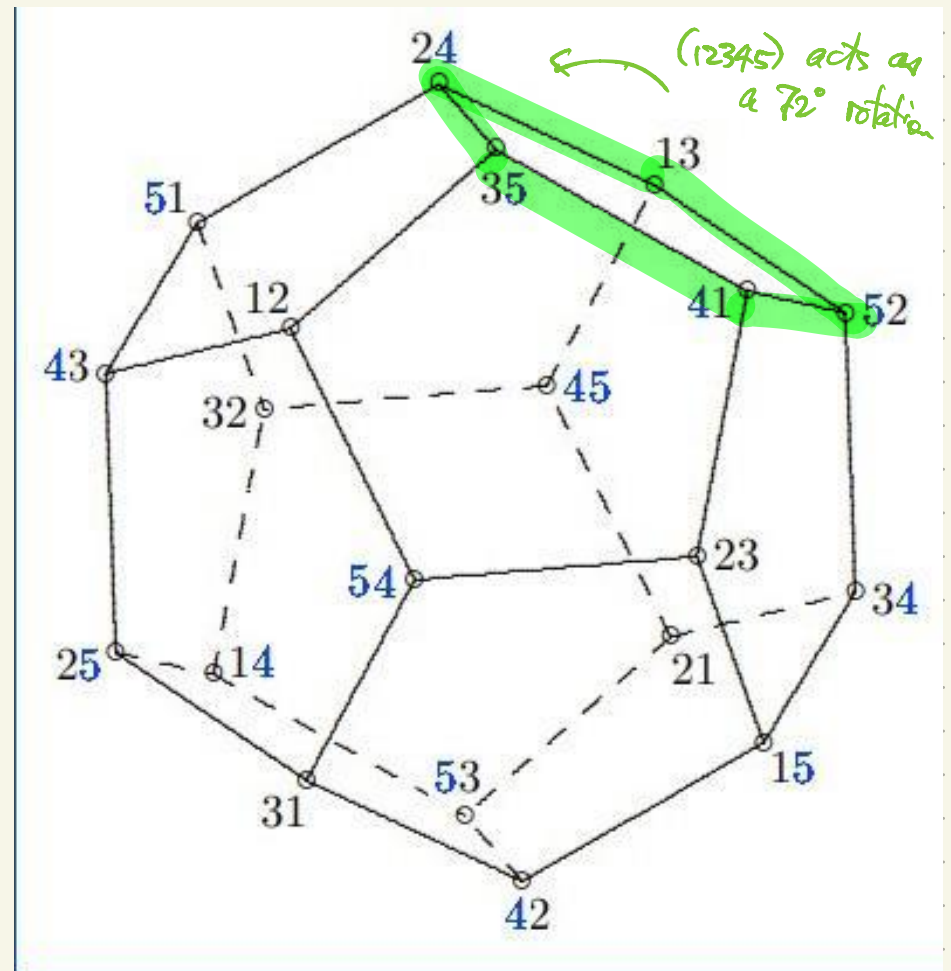
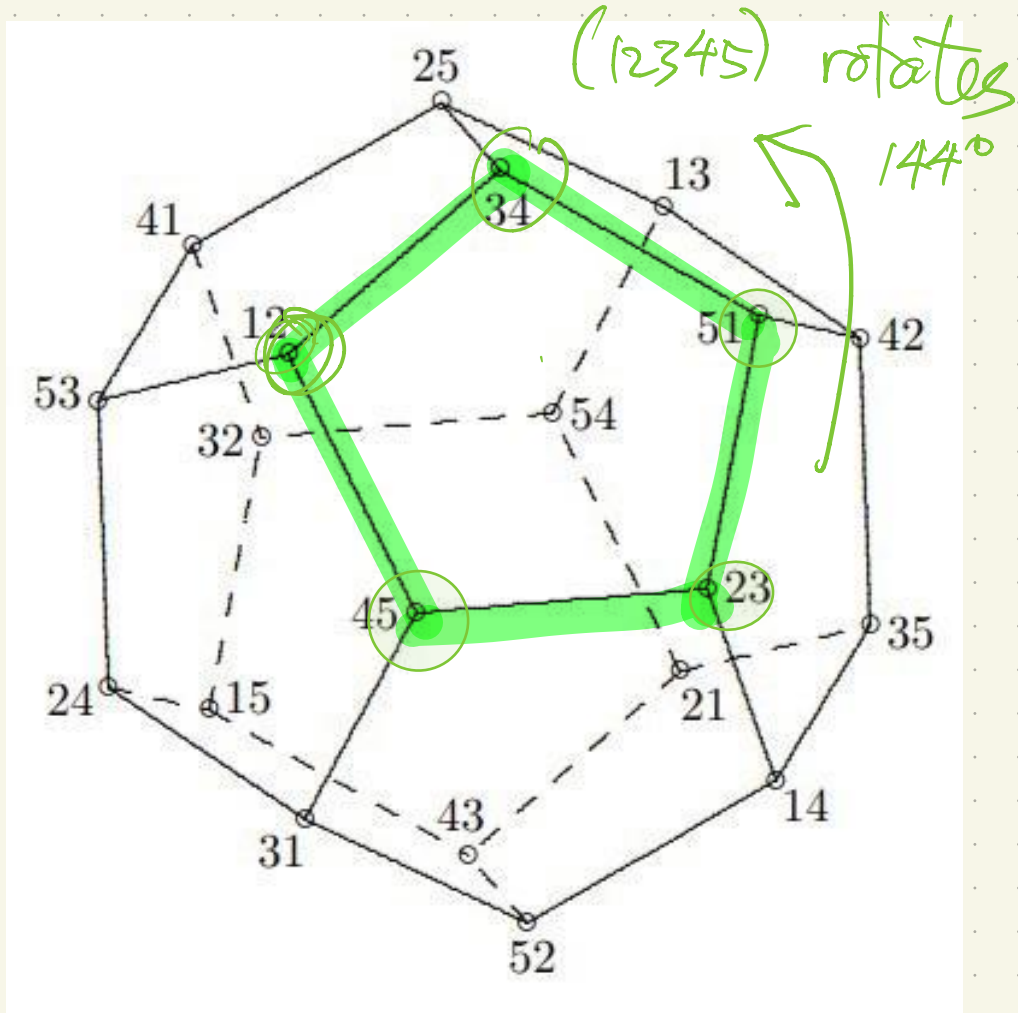
(C_g)	60	4	3	5	5
g	(1)	(12)(34)	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_3	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0
χ	5	1	2	0	0

$$1 + 3 \frac{1+\sqrt{5}}{2} + 3 \frac{1-\sqrt{5}}{2} - 4 + 5 \square = 0$$

$$[\chi_5, \chi_5] = \frac{5^2}{60} + \frac{1^2}{4} + \frac{(-1)^2}{3} + \frac{0^2}{5} + \frac{0^2}{5} = \frac{25}{60} + \frac{1}{4} + \frac{1}{3} + 0 + 0 = \frac{5+3+4+0+0}{12}$$

$$= \frac{12}{12} = 1.$$

Another way to get χ_5 is to use the transitive permutation action of $G = A_5$ of degree 6.



Dodecahedron with vertices (i,j) ,
 $i \neq j$ in $\{1,2,3,4,5\}$
 A_5 acts on the dodecahedron as a
 rotational symmetry group.

$$\begin{aligned}\cos 0 &= 1 \\ \cos \pi &= -1 \\ \cos \frac{2\pi}{5} &= -\frac{1}{2} \\ \cos \frac{2\pi}{5} &= \cos 72^\circ = \frac{-1+\sqrt{5}}{4} \\ \cos \frac{4\pi}{5} &= \cos 144^\circ = \frac{-1-\sqrt{5}}{4}\end{aligned}$$

Every rotation matrix $A \in SO_3(\mathbb{R})$ is similar to
 $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix}$ where θ is the angle of the rotation.



$$\text{tr } A = 1 + 2 \cos \theta.$$

θ	$\text{tr } A$
0	3
$\pi = 180^\circ$	-1 (half-turn)
$\frac{2\pi}{3} = 120^\circ$	0
$\frac{2\pi}{5} = 72^\circ$	$\frac{1+\sqrt{5}}{2}$ golden ratio ≈ 1.618
$\frac{4\pi}{5} = 144^\circ$	$\frac{1-\sqrt{5}}{2} \approx -0.618$

$G = A_5 \cong \text{PSL}_2(5)$ acting as fractional linear transformations on $\mathbb{F}_5 \cup \{\infty\} = \{0, 1, 2, 3, 4, \infty\}$

$$\text{PSL}_2(5) = \left\{ \pm \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad-bc=1, a, b, c, d \in \mathbb{F}_5 \right\} / \{\pm I\}$$

$$\pm \begin{bmatrix} a & b \\ c & d \end{bmatrix} : x \mapsto \frac{ax+b}{cx+d}$$

identity fixes all six points,
element of order 2:

$$g_2 = \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : x \mapsto -\frac{1}{x} = -\frac{1}{x}$$

$$\chi(g_2) = 6$$

$$g_1 = \pm I$$

element of order 3: i.e. $(0, \infty)(1, 4)(2)(3) = (0, \infty)(1, 4)$, $\chi(g_3) = 2$.

$$g_3 = \pm \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} : x \mapsto \frac{-1}{x+1} = -\frac{1}{x+1} \quad \text{i.e. } (0, 4, \infty)(1, 2, 3), \quad \chi(g_3) = 0$$

element of order 5: $g_4 = \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} : x \mapsto x+1$ i.e. $(0, 1, 2, 3, 4)(\infty)$
 $= (0, 1, 2, 3, 4)$, $\chi(g_4) = 1$

Another conjugacy class of elements of order 5:

$$g_5 = g_4^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} : x \mapsto x+2 \quad \text{i.e. } (0, 2, 4, 1, 3) \quad \chi(g_5) = 1$$

$$\begin{aligned} [\chi, \chi] &= \frac{6^2}{60} + \frac{2^2}{4} + \frac{0^2}{3} + \frac{1^2}{5} + \frac{1^2}{5} \\ &= \frac{3 + 5 + 0 + 1 + 1}{5} = \frac{10}{5} = 2 \end{aligned}$$

$ C_G(g) $	60	4	3	5	5
g	(1)	(12)(34)	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_3	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0
χ	6	2	0	1	1

$$\chi = a_1 \chi_1 + a_2 \chi_2 + \dots + a_5 \chi_5$$

$$[\chi, \chi] = \sum a_i^2 = 2$$

$\Rightarrow a_1, \dots, a_5: 1, 1, 0, 0, 0$ in some order

$$[\chi, \chi_1] = \frac{6}{60} + \frac{2}{4} + \frac{0}{3} + \frac{1}{5} + \frac{1}{5} = \frac{1+5+0+2+2}{10} = \frac{10}{10} = 1$$

$$\frac{\chi(1)}{6} = \frac{\chi_1(1)}{1} + \frac{\chi_5(1)}{5} \quad (j \neq 1)$$

$$\chi = \chi_1 + \chi_5$$

Let G be a finite group with k conj. classes and $|G| = n$ then its char. table looks like

$ C_G(g) $	n	n_1	n_2	$\dots$	n_k
g	g_1	g_2	$\dots$	g_k	
χ_1	1	1	$\dots$	1	
χ_2	n_1				
$\vdots$					
χ_k	n_k				
χ	n	0	$\dots$	0	

$$n_i = \deg \chi_i = \chi_i(1)$$

Let χ be the perm. character of the regular representation of G : each $g \in G$ permutes G by $g: x \mapsto gx$. The perm. character of this representation is $\begin{cases} \chi(1) = n \\ \chi(g) = 0 \text{ for } g \neq 1 \end{cases}$

$$\chi = (?)\chi_1 + (?)\chi_2 + \dots + (?)\chi_k$$

$\uparrow$ $\uparrow$ $\uparrow$
 $[\chi, \chi_1]$ $[\chi, \chi_2]$ $[\chi, \chi_k]$

$$[\chi, \chi_i] = \frac{n n_i}{n} + 0 + \dots + 0 = n_i$$

$$\chi = n_1 \chi_1 + n_2 \chi_2 + \dots + n_k \chi_k, \quad n_i = \chi_i(1) \geq 1$$

Every irreducible character occurs with multiplicity $n_i \geq 1$ in the regular representation.

Evaluate at $1 \in G$:

$$\chi(1) = n = n_1 \chi_1(1) + n_2 \chi_2(1) + \dots + n_k \chi_k(1) = n_1^2 + n_2^2 + \dots + n_k^2$$

Character values are algebraic integers. If χ is a character of G with $\chi(g) = \text{tr } \pi(g)$ then $\pi(g)^m = \pi(g^m) = \pi(1) = I$ where m is the exponent of G (lcm of orders of elements of G). Eigenvalues of $\pi(g)$ are n^{th} roots of 1. $\uparrow$ identity matrix of size $\chi(1) = \deg \pi = n$

$\pi(g)$ has upper triangular form $\pi(g) \sim \begin{bmatrix} \lambda_1 & * \\ 0 & \ddots & \lambda_n \end{bmatrix}$, $\lambda_j \in \mathbb{C}$.

$\pi(g)^m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\lambda_j^m = 1$ for all $j=1, \dots, n$. Roots of $x^n - 1$ (roots of unity; in particular algebraic integers).

$\chi(g) = \lambda_1 + \dots + \lambda_n$ is an algebraic integer.

Why do we care?

An early success of character theory is Burnside's Theorem: If p, q are primes then every group of order $p^a q^b$ (a, b positive integers) is solvable.

$$|A_5| = 2^3 \cdot 3 \cdot 5$$

The proof of Burnside's Theorem uses the fact that character values are algebraic integers. (see handout)

Moreover the degree of each irreducible representation of G divides $|G|$.

eg. A_5 has irreducible representations of degree $1, 3, 3, 4, 5 \mid 60$.

For any ^{finite} abelian group G , all irreducible representations have degree 1.

If $|G| = n$ then G has n conjugacy classes (each of size 1) so G has n irreducible characters χ_i ($1 \leq i \leq n$) each of degree $n_i = \chi_i(1) = 1$. $\sum_{i=1}^n n_i^2 = n$.

If G is cyclic of order n , $G = \langle g \rangle = \{1, g, g^2, \dots, g^{n-1}\}$ then the char. table:

$ G(g^i) $	n	n	n	n	n
g^i	1	g	g^2	g^3	g^{n-1}
χ_1	1	1	1	1	1
χ_2	1	ζ	ζ^2	ζ^3	ζ^{n-1}
$\vdots$	$\vdots$	ζ^2	ζ^4	ζ^6	$\zeta^{2(n-1)}$
χ_n	1	ζ^{n-1}	$\zeta^{2(n-1)}$	$\vdots$	$\vdots$

$\zeta = e^{2\pi i/n}$ or any primitive n^{th} root of unity.

For $G = C_n \times C_m$ (direct product of cyclic groups of order n, m)
 the character table is a tensor product (Kronecker product) of char. tables
 of C_n and C_m .

If G is Klein 4-group $\{(\pm 1, \pm 1)\} = C_2 \times C_2$ under multiplication then

	4	4	4	4
	(1, 1)	(1, -1)	(-1, 1)	(-1, -1)
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

$$= \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\text{Char table of } C_2} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

If G is an abel. group then the irreducible characters of G are homomorphisms.
 (since $\chi(g) = \text{tr } \pi(g) = \pi(g)$ in this case). If $\chi, \chi' \in \text{Irr}(G) = \{\text{all irred. characters of } G\}$
 then $\chi\chi' \in \text{Irr } G$ and the irred. characters form a group $\hat{G} = \text{Hom}(G, \mathbb{C}^\times) \cong G$
 but not canonically.

Consider a nonabelian group of order 8, either $G = \text{dihedral of order 8} = D_4$

D_4 has 2 elements of order 4
 5 " " " " " 2
 1 " " " " " 1

or $G = Q = \{\pm 1, \pm i, \pm j, \pm k\}$
 Q has 6 elements of order 4
 1 " " " " 2
 1 " " " " 1

Both have $|Z(G)| = 2$

$G/Z(G) \cong \text{Klein 4-group}$

G has 5 conjugacy classes.

G (nonabel. of order 8) has char. table

$(C_2 \times C_2)$	8	8	4	4	4
g_i	1	-1	g_3	g_4	g_5
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

$$Z(G) = \langle -1 \rangle$$

g_3, g_4, g_5 are $\pm i, \pm j, \pm k$ or $\{R, R^3\}, \{H, V\}, \{D, D^3\}$

$$\pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Character table for $G/Z(G) = \text{Klein 4-group}$

	4	4	4	4
	1	g_2	g_3	g_4
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

$K \trianglelefteq G$ (normal subgroup)

If $\pi: G/K \longrightarrow GL_n(\mathbb{C})$ is a representation then composing with the canonical map gives a representation of G . If π is irreducible then lifting to G gives an irreducible representation of G .

$$G \longrightarrow G/K \longrightarrow GL_n(\mathbb{C})$$

$$g \longmapsto gK$$

S_3 character table

	1	1	2	2	3
1	1	1	1	1	1
2	1	-1	1	-1	0
3	2	0	-1	1	0

$$S_3/A_3 \cong C_2 \text{ cyclic of order 2}$$

	1	-1
χ_1	1	1
χ_2	1	-1

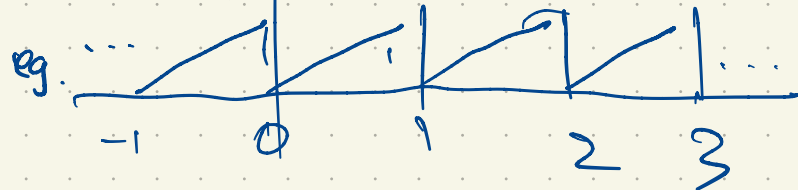
S_4 has a normal subgroup $K = \langle (12)(34), (13)(24) \rangle = \{1, (12)(34), (13)(24), (14)(23)\}$
 $S_4/K \cong S_3$. Three irred. representations of S_3 give three irred. reps of S_4 of degree 1, 1, 2.

$$G = S^1 = \{z \in \mathbb{C} : |z| = 1\} = \{e^{2\pi i \theta} : \theta \in \mathbb{R}\} \cong \mathbb{R}/\mathbb{Z} \text{ (additively)}$$

$$\hat{G} = \text{Hom}(S^1, \mathbb{C}^\times) = \{\phi_n : n \in \mathbb{Z}\}, \quad \phi_n(z) = z^n. \quad \phi_n(zz') = (zz')^n = z^n (z')^n = \phi_n(z) \phi_n(z')$$

S^1 acts on $L^2(S^1)$ = square-integrable functions $S^1 \rightarrow \mathbb{C}$.
 $= \{f : f: S^1 \rightarrow \mathbb{C}, \int_{S^1} |f|^2 < \infty\}$

f is defined on $S^1 \cong \mathbb{R}/\mathbb{Z}$ iff f is a periodic function $f: \mathbb{R} \rightarrow \mathbb{C}$ with $f(z+1) = f(z)$.



$$f(e^{2\pi i \theta}) = \theta - \lfloor \theta \rfloor = \text{fractional part of } \theta$$

$$\int_0^1 \theta d\theta = \frac{1}{2} \theta^2 \Big|_0^1 = \frac{1}{2}$$

$$\int_0^1 |f(\theta)|^2 d\theta = \int_0^1 \theta^2 d\theta = \frac{1}{3} \theta^3 \Big|_0^1 = \frac{1}{3}$$

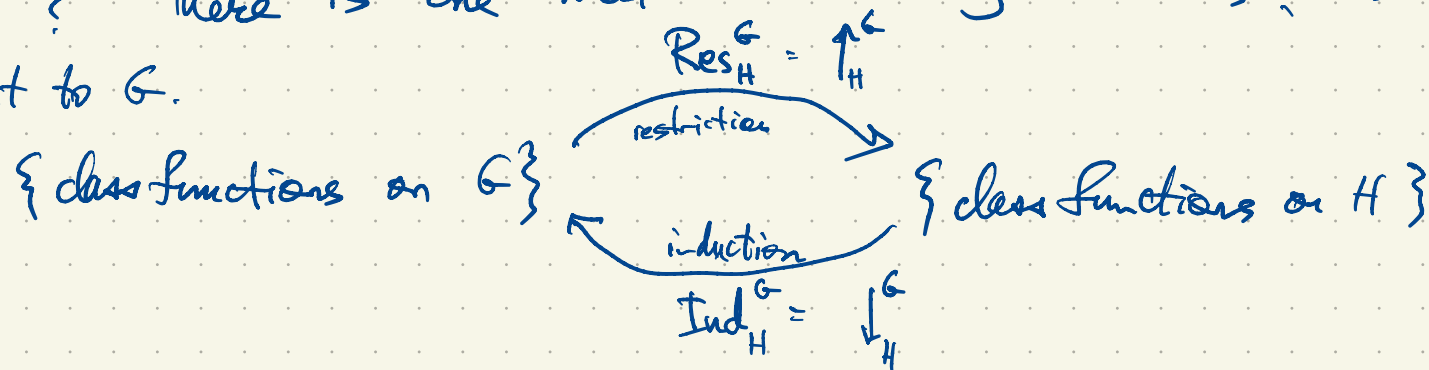
Functions $\sin(2\pi k\theta)$, $\cos(2\pi k\theta)$ ($k \in \mathbb{Z}$) span (a dense subspace of) $L^2(S^1)$.

S^1 acts on $L^2(S^1)$ by translation:

Alternatively: $\{e^{2\pi i k \theta} : k \in \mathbb{Z}\}$.

Suppose $H \leq G$. Let χ be a class function on G . By restricting we get a class function $\chi|_H$ on H . If $x, h \in H$ then $\chi|_H(h^{-1}xh) = \chi|_H(x)$.

Given a class function ψ on H , how does one construct a class function on G ? There is one most natural way to do this, called induction from H to G .



$$\chi \longmapsto \text{Res}_H^G \chi = \chi|_H$$

$$\text{Ind}_H^G \psi = \psi^G \longleftarrow \psi$$

Adjunction: Ind_H^G is left adjoint to Res_H^G ; Res_H^G is right adjoint to Ind_H^G .

Frobenius Reciprocity: Let χ be a class function on G and ψ a class function on H .

$$\text{Then } [\chi, \psi^G]_G = [\chi|_H, \psi]_H.$$

Eg. $G = A_5$
 $H \cong S_3 = \langle (123), (12)(45) \rangle$

$ C_G(g) $	60 1A (1)	4 2A (12)(34)	3 3A (123)	5 5A (12345)	5 5B (12354)
g					
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_3	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

$ C_H(*) $	6 1A (1)	2 2A (12)	3 3A (123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1
χ_H	5	1	-1

Let $\chi = \chi_5$ class function on G . Then $\chi|_H$ is a class function on H . Since $\text{Irr } H = \{\chi_1, \chi_2, \chi_3\}$ is a basis for all class functions on H , we can express $\chi_H = a_1\chi_1 + a_2\chi_2 + a_3\chi_3$.
 Solve 3 linear equations in 3 unknowns:

At 1A: $5 = a_1 + a_2 + 2a_3$

At 2A: $1 = a_1 - a_2$

At 3A: $-1 = a_1 + a_2 - a_3$

$\Rightarrow a_3 = 2, a_2 = 0, a_1 = 1$

$\Rightarrow \chi_H = \chi_1 + 2\chi_3$

Alternatively:

$[\chi_5, \chi_1] = \frac{5}{6} + \frac{1}{2} - \frac{1}{3} = \frac{6}{6} = 1$

$[\chi_5, \chi_2] = \frac{5}{6} - \frac{1}{2} - \frac{1}{3} = \frac{0}{6} = 0$

$[\chi_5, \chi_3] = \frac{10}{6} + \frac{0}{2} + \frac{1}{3} = \frac{16}{6} = 2$

Character table of A_5

$ C(g) $	60 1A (1)	4 2A (12)(34)	3 3A (123)	5 5A (12345)	5 5B (12354)
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_3	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

$$A_5 = \text{PSL}_2(5), |A_5| = 60$$

$$|\text{GL}_2(5)| = 480$$

$$|\text{SL}_2(5)| = 120$$

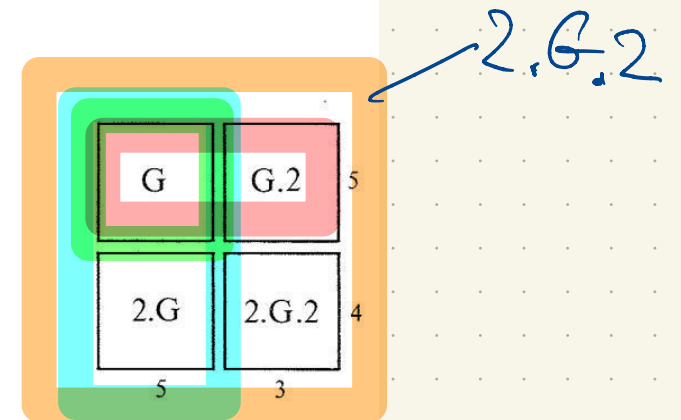
$$|\text{PGL}_2(5)| = |S_5| = 120$$

$$G = A_5$$

$$2.G = \text{SL}_2(5)$$

Atlas:

	60	4	3	5	5		6	2	3		
p power	A	A	A	A	A		A	A	AB		
p' part	A	A	A	A	A		A	A	AB		
ind	1A	2A	3A	5A	B*	fus ind	2B	4A	6A		
χ_1	+	1	1	1	1	:	++	1	1	1	
χ_2	+	3	-1	0	-b5	*	+	0	0	0	
χ_3	+	3	-1	0	*	-b5					
χ_4	+	4	0	1	-1	-1	:	++	2	0	-1
χ_5	+	5	1	-1	0	0	:	++	1	-1	1
ind	1	4	3	5	5	fus ind	2	8	6		
	2		6	10	10			8	6		
χ_6	-	2	0	-1	b5	*	-	0	0	0	
χ_7	-	2	0	-1	*	b5					
χ_8	-	4	0	1	-1	-1	:	oo	0	0	i3
χ_9	-	6	0	0	1	1	:	oo	0	i2	0



$$G.2 = S_5 \cong \text{PGL}_2(5)$$

A group 2.G.2 of order 240 having a quotient group $G.2 \cong S_5$
 (homomorphic image)
 and a subgroup $2.G \cong SL_2(5)$

$$2.G.2 = \langle SL_2(5), \begin{bmatrix} 0 & \sqrt{2} \\ -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \rangle \subset SL_2(25) \quad \sqrt{2} \in \mathbb{F}_{25} \supset \mathbb{F}_5$$

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} : \begin{matrix} a, b, c, d \in \mathbb{F}_5 \\ ad - bc = 1 \end{matrix} \quad \mathbb{F}_5[\sqrt{2}]$$

$$M = F_1$$

Sporadic Fischer-Griess "Monster" or "Friendly Giant" group $M \cong FG \cong F_1$

194 conjugacy classes

Order = 808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000
 $= 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

		e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e
		80801742479451287588645	83095629624528523	139511839126	376561712757	48000	000	45440	39360	94880	192	192	856	992	328	8	8	2	34	23
		9904961710757005754368000000000	823551610880000000	336328171520000	1985163878400	BB	BB	A	CA	EA	E	E	A	BD	AD	E	E	A	BD	AD
		p power	A	A	A	BB	BB	A	AA	BA	A	A	A	BA	AA	A	A	A	BA	AA
		p' part	A	A	A	10D	10E	11A	12A	12B	16B	16C	17A	18A	18B	16B	16C	17A	18A	18B
1nd		1A	2A	2B	3A	10D	10E	11A	12A	12B	16B	16C	17A	18A	18B	16B	16C	17A	18A	18B
X ₁	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X ₂	+	196883	4371	275	782	20	0	16	14	5	-1	7	6	-3	6	-1	7	6	-3	6
X ₃	+	21296876	91884	-2324	7889	41	1	29	17	-10	0	8	7	3	3	0	8	7	3	3
X ₄	+	842609326	1139374	12974	55912	79	-1	53	40	-5	2	10	8	1	10	2	10	8	1	10
X ₅	+	18538750076	8507516	123004	249458	104	4	66	50	14	0	8	6	5	14	0	8	6	5	14
X ₆	+	19360062527	9362495	-58305	297482	155	-5	91	74	20	-1	15	8	2	2	-1	15	8	2	2
X ₇	+	293553734298	53981850	98970	1055310	175	-5	99	14	-3	-2	6	7	-6	3	-2	6	7	-6	3

X₁₉₄

$$B = F_{2+}$$

Sporadic "Baby monster" group $B \cong F_{2+}$

184 conjugacy classes

Order = 4,154,781,481,226,426,191,177,580,544,000,000
 $= 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$

	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e
	4154781481226	30612991873	3548835956	26489012	11689182387370501	19	9	13981	746	256	256	68	296	648	426191177580544000000	5099415756800	61213696000	826931200	992793600	9926400	
	p power	A	A	A	A	A	A	A	A	AD	BD	A	CB	GB	K	M	A	BD	BD		
	p' part	A	A	A	A	A	A	A	A	AD	BD	A	AB	BB	A	A	A	BA	BA		
1nd	1A	2A	2B	2C	2D	3A	10E	10F	11A	12A	12B	16G	16H	17A	18A	18B					
x ₁	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
x ₂	+	4371	-493	275	-53	19	78	-1	4	4	-12	-3	1	1	2	2	2	1	1	2	2
x ₃	+	96255	4863	2047	103	255	351	5	5	5	25	7	1	1	1	3	3	1	1	1	3
x ₄	+	1139374	-26962	8878	782	-338	1000	-3	7	5	-36	-9	-2	2	0	2	2	-2	2	0	2
x ₅	+	9458750	118846	37950	3486	2110	2729	5	0	4	53	-1	2	-2	1	1	1	2	-2	1	1
x ₆	+	9550635	119339	35627	-781	-2005	3003	-10	10	6	65	2	-3	1	1	-1	-1	-3	1	1	-1
x ₇	+	63532485	-468027	134597	-4811	1989	8073	-1	9	5	-101	7	1	1	0	0	0	-1	9	5	-101

X₁
X₂
X₃
X₄
X₅
X₆
X₇
X₁₈₄

Theorem B (= Baby Monster) is NOT isomorphic to a subgroup of M (= Monster.)
 (although B is involved in M : if $\tau \in M$ is of type 2A then
 $C_M(\tau) \cong 2.B$ nonsplit extension: B is a homomorphic image $C_M(\tau)/\langle \tau \rangle \cong B$.)

Suppose M has a subgroup B iso. to Baby Monster.
 Let $\chi_2 \in \text{Irr}(M)$, $\chi_2(1A) = 196883$. Restricting to B gives a character

$\chi_2|_B$ of B of degree 196883. (Not irreducible of course)

$$\Rightarrow \chi_2|_B = k_1 \psi_1 + k_2 \psi_2 + k_3 \psi_3 \quad \text{where } k_1, k_2, k_3 \geq 0$$

$$k_i = [\chi_2|_B, \psi_i]_B$$

Let $g_i \in B$ be an element of order $i \in \{1, 11, 17\}$.

These have conj. class 1A, 11A, 17A in Baby Monster, also in Monster.

$$\chi_2(g_i) = k_1 \psi_1(g_i) + k_2 \psi_2(g_i) + k_3 \psi_3(g_i)$$

$$i=1: \quad 196883 = k_1 + 4371 k_2 + 96255 k_3$$

$$i=11: \quad 16 = k_1 + 4 k_2 + 5 k_3$$

$$i=17: \quad 6 = k_1 + 2 k_2 + k_3$$

$\rightarrow$ non-integer values k_1, k_2, k_3
 Contradiction.

Induction takes class functions on H to class functions on G
 characters representations of H ——— characters on G representations of G

Let χ be a class function on $H \leq G$ i.e. $\chi: H \rightarrow \mathbb{C}$, $\chi(xhx^{-1}) = \chi(h)$ for all $x, h \in H$.
 To get a class function on G , start with the trivial extension

$$\hat{\chi}: G \rightarrow \mathbb{C}$$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H; \\ 0 & \text{if } g \notin H. \end{cases}$$

To make this into a class function, use an averaging over conjugates as we did before. This leads to $\chi^G: G \rightarrow \mathbb{C}$:

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1}) \quad (\chi^G \text{ is induced from } \chi, \quad \chi^G = \text{Ind}_H^G \chi)$$

$$\text{If } u \in G \text{ then } \chi^G(ugu^{-1}) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xugu^{-1}x^{-1}) = \frac{1}{|H|} \sum_{w \in G} \hat{\chi}(wuw^{-1})$$

$$\begin{aligned} w^{-1} &= u^{-1}x^{-1} \\ w &= xu \\ wu &= x \end{aligned}$$

So χ^G is a class function on G .
 $= \chi^G(g)$

Note: Let T be a set of right coset representatives for H in G .

So every element $g \in G$ is uniquely expressible as $g = ht$, $h \in H, t \in T$.

$$|G| = |H||T| \quad (\text{Lagrange's Theorem}), \quad |T| = \frac{|G|}{|H|} = [G:H].$$

$\chi: H \rightarrow \mathbb{C}$ is a class function on H ; T is a right transversal for H in G (set of right coset representatives)

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1})$$

$$= \frac{1}{|H|} \sum_{t \in T} \sum_{h \in H} \hat{\chi}(htgt^{-1}h^{-1})$$

$$= \frac{1}{|H|} \sum_{t \in T} \sum_{h \in H} \hat{\chi}(tgt^{-1})$$

$$x = ht, \quad h \in H, \quad t \in T$$

$$htgt^{-1}h^{-1} \in H$$

$$\iff tgt^{-1} \in H$$

$$T = \{t_1, \dots, t_l\}$$

$$G = Ht_1 \sqcup Ht_2 \sqcup \dots \sqcup Ht_l$$

$$= \bigsqcup_{t \in T} Ht$$

$$= \frac{1}{|H|} \sum_{t \in T} |H| \hat{\chi}(tgt^{-1}) = \sum_{t \in T} \hat{\chi}(tgt^{-1}) = \sum_{i=1}^l \hat{\chi}(t_i g t_i^{-1})$$

|H| terms equal to $\hat{\chi}(tgt^{-1})$

Special case: $\chi = \chi_1$ = trivial (principal) character of H , $\chi(h) = 1$.

χ^G isn't the principal character of G unless $G=H$.

G permutes the right cosets of H by right multiplication giving a permutation representation $g \in G$ permutes $Ht_i \mapsto Ht_j, g = Ht_j$.

$\chi^G = \left(\underset{H}{1} \right)^G$ is the perm.

$$G \longrightarrow S_l$$

$$l = [G:H] = |T|.$$

character of G acting on cosets of H .