Math 5555

Abstract Algebra II

Book 1

In group theory we have

- permutation representations: homomorphisms $\pi: G \to S_n$ permutation representation of degree n

  (if π is 1-to-1 then π is a faithful representation: then $\underset{\cong G}{\pi(G)} \le S_n$)

- linear representations: homomorphism $\pi: G \to GL_n(F)$ F: field

  linear representation of degree n over F

  If $F = \mathbb{C}$ (or $\mathbb{R}$ or ...) then π is an ordinary representation.

  If char $F = p$ (prime) then π is a modular representation.

  G: unless otherwise specified, G finite group. (until later ...)

Usually $F = \mathbb{C}$ (or $\mathbb{R}$) and $|G| < \infty$.

If $\pi_i : G \to GL_{n_i}(\mathbb{C})$ $(i = 1,2)$ then $\pi_1 \oplus \pi_2 : G \to GL_{n_1+n_2}(\mathbb{C})$, $g \mapsto \begin{bmatrix} \pi_1(g) & 0 \\ 0 & \pi_2(g) \end{bmatrix}$ is a representation of degree $n_1 + n_2$

$\pi : G \to GL_n(\mathbb{C})$ is decomposable if there is a decomposition $\mathbb{C}^n = U \oplus V$ such that $U, V \neq 0$ (dim $U = n_1$, dim $V = n_2$, n_1, n_2 positive integers, $n_1 + n_2 = n$)

U, V invariant under all matrices $\pi(g)$, $g \in G$.

$G = S_3$ acting naturally on $\mathbb{C}^3$ by permuting the standard basis vectors $e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

via $\sigma : e_i \mapsto e_{\sigma(i)}$

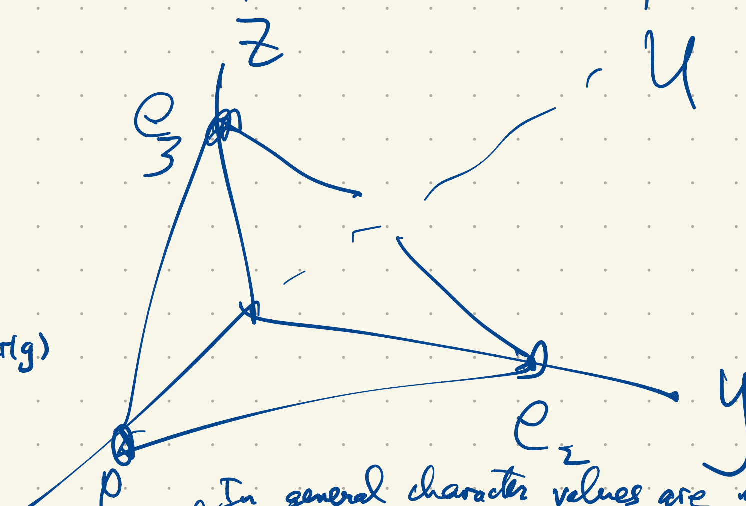
i.e. $(12) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$(123) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$() \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

This is a faithful representation of degree 3.

It is decomposable: $\mathbb{C}^3 = U \oplus V$, $U = \left\langle \begin{bmatrix}1\\1\\1\end{bmatrix} \right\rangle$, $V = U^\perp = \underbrace{\left\langle \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix} \right\rangle}_{\text{indecomposable.}}$



For a permutation representation

$\pi : G \to S_n \subset GL_n(\mathbb{C})$

the associated character $\chi(g) = \operatorname{tr} \pi(g)$

(called the permutation character)

is $\chi(g) = $ no. of fixed points of g

$= |\{i : g(i) = i\}|$, $1 \le i \le n$.

In general character values are not always integers.

The representation $\pi : S_3 \to GL_3(\mathbb{C})$ has (affords) character

$\chi(g) = \operatorname{tr} \pi(g)$

$\chi(1) = 3 = $ degree of π = deg χ.

$\chi((12)) = \chi((13)) = \chi((23)) = 1$

$\chi((123)) = \chi((132)) = 0$

Given a representation $\pi : G \to GL_n(\mathbb{C})$, the character of π is

$$\chi(g) = \operatorname{tr} \pi(g) \in \mathbb{C}.$$

Character values $\chi(g) \in \mathbb{C}$ are always algebraic integers ($g \in G$, $|G| < \infty$)
and character values of S_n are ordinary integers.
$\chi(g)$ depends only on the conjugacy class of g.

If $g, h \in G$ then $g \sim h^{-1}gh$ (conjugate in G) so

$$\pi(h^{-1}gh) = \pi(h)^{-1}\pi(g)\pi(h) \sim \pi(g) \quad \text{(similar in } GL_n(\mathbb{C}) \text{ i.e. conjugate)}$$

inverses in $GL_n(\mathbb{C})$

so $\operatorname{tr} \pi(g) = \operatorname{tr}(\pi(h^{-1})\pi(g)\pi(h)) = \operatorname{tr} \pi(h^{-1}gh)$

$\operatorname{tr}(AB) = \operatorname{tr}(BA)$

$\operatorname{tr}(B^{-1}AB) = \operatorname{tr}(AB \cdot B^{-1}) = \operatorname{tr} A$

$\chi(h^{-1}gh) = \chi(g)$.

If $\pi : G \to GL_n(\mathbb{C})$ is any representation i.e. homomorphism, and $B \in GL_n(\mathbb{C})$, then
$\tilde{\pi}(g) = B^{-1}\pi(g)B \in GL_n(\mathbb{C})$ is also a representation
$\pi, \tilde{\pi}$ are equivalent (via a change of basis).

$$\tilde{\pi}(gh) = B^{-1}\pi(gh)B = B^{-1}\pi(g)\pi(h)B = B^{-1}\pi(g)B \cdot B^{-1}\pi(h)B = \tilde{\pi}(g)\tilde{\pi}(h).$$

They have the same character: the character of $\tilde{\pi}$ is

$$\tilde{\chi}(g) = \operatorname{tr} \tilde{\pi}(g) = \operatorname{tr}(B^{-1}\pi(g)B) = \operatorname{tr} \pi(g) = \chi(g).$$

It's not obvious but the converse is true: χ determines π up to equivalence.
Two representations have the same character iff they are equivalent.

Let $\pi : G \to GL_n(\mathbb{C})$ be a representation (i.e. homomorphism).
π is reducible if there exists nontrivial proper subspace $U < \mathbb{C}^n$ ($\dim U \in \{1, 2, \dots, n-1\}$) such that
U is invariant under $\pi(g)$ for all $g \in G$ i.e. $\pi(g)U \subseteq U$ for all $g \in G$.

i.e. $$\pi(g) = \begin{array}{cc} \left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right] & \begin{array}{l} \}k \\ \}n-k \end{array} \\ \begin{array}{cc} k & n-k \end{array} & \end{array} \qquad k = \dim U.$$

for all $g \in G$.

when $|G| < \infty$ and $\pi : G \to GL_n(F)$
char $F = 0$, π reducible $\Longleftrightarrow$ π decomposable
π irreducible $\Longleftrightarrow$ π indecomposable

In general, π decomposable $\Rightarrow$ π reducible

$$\left[\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right] \Rightarrow \left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right]$$

π indecomposable $\Leftarrow$ π irreducible.

The representation $\underset{\substack{\text{additive}\\ \text{group}}}{\pi : \mathbb{R}} \longrightarrow GL_2(\mathbb{R})$, $\pi(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, $\pi(a+b) = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \pi(a)\pi(b)$

is reducible: $\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$ is an invariant subspace. There is no complementary invariant subspace (in particular the complementary subspace $\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle$ is not invariant).

<u>Maschke's Theorem</u> Let G be a finite group and $F = \mathbb{C}$ (or more generally F is any field of characteristic not dividing $|G|$) Then any representation $\pi : G \to GL_n(F)$ is reducible iff it's decomposable (ie. π is irreducible iff π is indecomposable; ie. whenever every invariant subspace $U \le F^n$ has a complementary subspace U' which is also invariant).

$U \le V$ has a complementary subspace U' $\Leftrightarrow$ $V = U \oplus U'$. If $\dim V = n$ and $\dim U = k$ then U' is a complement to U iff $\dim U' = n-k$ and $U + U' = V$
iff $\dim U' = n-k$ and $U \cap U' = \{0\}$.

In this case every $v \in V$ is uniquely represented as $v = u + u'$, $u \in U$, $u' \in U'$.

In this, the projection $V \to U$ along U' is the map $P : v \mapsto Pv = u$. $P = \begin{array}{c} \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right] \begin{array}{l} \}k \\ \}n-k \end{array} \\ \underbrace{}_{k} \; \underbrace{}_{n-k} \end{array}$

P has image $PV = U$; $\ker P = U'$. Note: $I - P$ is the projection from V onto U' along U.

A linear transformation $P : V \to V$ is a projection iff $P^2 = P$. In this case P is a root of $x^2 - x$ so the eigenvalues of P are in $\{0, 1\}$. Take U = 1-eigenspace of P = $\ker(I - P)$, $U' = \ker P$.

Proof of Maschke's Theorem: (in the nontrivial direction)

Suppose $\pi : G \to GL_n(\mathbb{C})$ is a representation having an invariant subspace $U \le \mathbb{C}^n$ i.e. $\pi(g)U \subseteq U$ for all $g \in G$. We want to find a complementary subspace $W \le \mathbb{C}^n$ which is also invariant. Start with any complementary subspace U' where $\mathbb{C}^n = U \oplus U'$, $\dim U = k$, $\dim U' = n-k$; every $v \in \mathbb{C}^n$ is uniquely expressible as $v = u + u'$, $u \in U$, $u' \in U'$. Unfortunately U' is not invariant in general. Let $P : \mathbb{C}^n \to U$ be the projection onto U along U' i.e. $P(v) = P(u+u') = u$, so $U = PV$, $U' = \ker P$. Consider the new map $\tilde{P} : V \to V$ defined by

$$\tilde{P} = \frac{1}{|G|}\sum_{g\in G} \underbrace{\pi(g^{-1})P\pi(g)}_{n\times n \text{ matrix}}.$$

$\tilde{P}V \subseteq U$ since $\tilde{P}v = \frac{1}{|G|}\sum_g \underbrace{\pi(g^{-1})\underbrace{P\underbrace{\pi(g)v}_{\in V}}_{\in U}}_{\in U} \in U$.

For all $u \in U$, $\tilde{P}u = u$ i.e. $\tilde{P}|_U = id|_U$. Why?

$$\tilde{P}u = \frac{1}{|G|}\sum_g \pi(g^{-1})\underbrace{P\pi(g)u}_{=\pi(g)u \text{ since } \pi(g)u \in U} = \frac{1}{|G|}\sum_g \cancel{\pi(g^{-1})\pi(g)}u = \cancel{\frac{1}{|G|}}\cancel{|G|}u = u.$$

Next: show $\tilde{P}$ commutes with all $\pi(g)$, $g \in G$ (P doesn't satisfy this in general!)

Then
$$\tilde{P}\pi(g) = \frac{1}{|G|}\sum_{h\in G}\pi(h^{-1})P\pi(h)\pi(g) = \frac{1}{|G|}\sum_{h\in G}\pi(h^{-1})P\pi(hg)$$

$h \longleftrightarrow x = hg$
$x^{-1} = g^{-1}h^{-1}$
$gx^{-1} = h^{-1}$

$$= \frac{1}{|G|}\sum_{x\in G}\underbrace{\pi(gx^{-1})}_{\pi(g)\pi(x)^{-1}}P\pi(x) = \pi(g)\cdot\frac{1}{|G|}\sum_{x\in G}\pi(x)^{-1}P\pi(x) = \pi(g)\tilde{P}.$$

Next: show $\tilde{P}^2 = \tilde{P}$. If $v \in V$ then $\tilde{P}v \in U$ so $\tilde{P}^2v = \tilde{P}\underbrace{\tilde{P}v}_{\hat{u}} = \tilde{P}v$. So $\tilde{P}$ is idempotent so $\tilde{P}$ is a projection on $\tilde{P}V = U$ along $W = \ker\tilde{P}$. Note: if $k = \dim U = \operatorname{rank}\tilde{P} = \operatorname{tr}\tilde{P}$, $\dim W = \dim\ker\tilde{P} = n-k$. W is also invariant under $\pi(g)$: if $v \in W$ then $\tilde{P}v = 0$ so $\underbrace{\pi(g)\tilde{P}v}_{=\tilde{P}\pi(g)v} = \pi(g)0 = 0$ so $\pi(g)v \in W$.

$\square$

The irreducible representations of S_3 are

$\pi_1(g) = [1] \in GL_1(\mathbb{C})$ trivial representation

$\pi_2(g) = [\mathrm{sgn}(g)] \in GL_1(\mathbb{C})$ sign representation:

$= [\pm 1]$ according as g is an even or odd permutation.

$\mathrm{sgn}(()) = 1$ $\quad$ $\mathrm{sgn}((12)) = \mathrm{sgn}((13)) = \mathrm{sgn}((23)) = -1$

$\mathrm{sgn}((123)) = 1$

$\mathrm{sgn}((132)) = 1$

$\mathrm{sgn}(gh) = \mathrm{sgn}(g)\,\mathrm{sgn}(h)$

$\pi_3 : S_3 \to GL_2(\mathbb{C})$

$$() \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(12) \longmapsto \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$(123) \longmapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$(132) \longmapsto \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$(13) \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(23) \longmapsto \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\pi_3((13)) = \pi_3((123)(12)) = \pi_3((123))\,\pi_3((12)) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\pi_3((23)) = \pi_3((123)(13)) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

An irreducible character is a map $G \to \mathbb{C}$, $g \longmapsto \mathrm{tr}\,\pi(g)$

where π is an irreducible representation.

0, $\left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$, $\left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle$, $\mathbb{C}^2$ are the only subspaces invariant under $\pi_3((13))$; but the 1-dimensional invariant subspaces are not invariant under $\pi_3((123))$.

$\pi_3((123)) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ has char. poly. $x^2 + x + 1$

$\pi_3((13)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has char. poly. $x^2 - 1$

eigenspaces $\left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$, $\left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle$ (eigenvalue $1, -1$ respectively)

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

The character table of S_3 is

	$()$	(12)	(23)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Given any finite group G, a <u>class function</u> on G is a function $f: G \to \mathbb{C}$ which only depends on conjugacy class, i.e. $f(g') = f(g)$ whenever $g, g' \in G$ are conjugate i.e. $g' = u^{-1} g u$ for some $u \in G$.

All characters of G (irreducible or otherwise) are class functions.

$V = \{\text{class functions on } G\}$ is a complex vector space

$f, f' \in V \Rightarrow f + f' \in V, \quad (f + f')(x) = f(x) + f'(x)$

V is in fact a complex inner product space:

$[f, f']_G = [f, f'] = \frac{1}{|G|} \sum_{x \in G} f(x) \overline{f'(x)} \qquad [f', f] = \overline{[f, f']}$

sesquilinear = $1\frac{1}{2}$-linear

$\dim V$ = number of conjugacy classes.

The irreducible characters of G always give an orthonormal basis for $V = \{\text{class functions on } G\}$.

Recall the permutation representation with permutation character

$\pi : S_3 \longrightarrow GL_3(\mathbb{C})$

$\pi : e_i \longmapsto e_{\sigma(i)}$

$(12) \longmapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$(123) \longmapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$() \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\chi(()) = 3$

$\chi((12)) = 1$

$\chi((123)) = 0$

$$\left.\begin{aligned} [\chi, \chi_1] &= \tfrac{1}{6}(1 \cdot 3 + 1 + 1 + 1 + 0) = 1 \\ [\chi, \chi_2] &= \tfrac{1}{6}(3 - 1 - 1 - 1 + 0 + 0) = 0 \\ [\chi, \chi_3] &= \tfrac{1}{6}(6 + 0 + 0 + 0 + 0 + 0) = 1 \end{aligned}\right\} \Rightarrow \pi \cong \pi_1 \oplus \pi_3$$

Character table of S_3

	$()$	(12)	(123)
χ_1	1	1	1
χ_2	1	−1	1
χ_3	2	0	−1
χ	3	1	0

$[\chi_1, \chi_1] = \frac{1}{6} \sum_{g \in S_3} \chi_1(g) \overline{\chi_1(g)}$
$= \frac{1}{6}(1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1)$
$= 1$

$[\chi_1, \chi_2] = \frac{1}{6} \sum_g \chi_1(g) \overline{\chi_2(g)}$
$= \frac{1}{6}(1 \cdot 1 - 1 \cdot 1 - 1 \cdot 1 - 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1)$
$= 0$

$[\chi_1, \chi_3] = \frac{1}{6}(1 \cdot 2 + 0 + 0 + 0 - 1 - 1) = 0$

$[\chi_2, \chi_2] = \frac{1}{6}(1 + 1 + 1 + 1 + 1 + 1) = 1$

$[\chi_2, \chi_3] = \frac{1}{6}(2 + 0 + 0 + 0 - 1 - 1)$
$= 0$

$[\chi_3, \chi_3] = \frac{1}{6}(4 + 0 + 0 + 0 + 1 + 1)$
$= 1$

The long way to check the decomposition $\pi \cong \pi_1 \oplus \pi_3$ is found in the course notes: with respect to new basis $v_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $v_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$, $v_3 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$

v_1: basis for first invariant subspace

v_2, v_3: basis for second invariant subspace

$\pi(g)$ has matrix $\left[\begin{array}{c|cc} \pi_1(g) & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\pi_3(g)$}} \\ 0 & & \end{array}\right]$ with respect to v_1, v_2, v_3

(see p. 3 of course notes).

Character tables also have orthogonality of columns.

$\lvert C_G(g)\rvert$	6	2	3
g	$()$	(12)	(23)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$C_G(g) = \{x \in G : xg = gx\}$

The conjugacy class of $g \in G$ is the index of the centralizer

$$[G : C_G(g)] = \frac{|G|}{|C_G(g)|}$$

The column orthogonality is expressed as: given $g, h \in G$,

$$\sum_{i=1}^{k} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} 0 & \text{if } g, h \text{ not conjugate} \\ |C_G(g)| & \text{if } g, h \text{ are conjugate} \end{cases}$$

(k = number of irreducible characters = no. of conjugacy classes)

Row orthogonality says: Let $g_1, \dots, g_k \in G$ be reps of conj. classes

$$[f, f']_G = \frac{1}{|G|}\sum_{a \in G} f(a)\overline{f'(a)} = \cancel{\frac{1}{|G|}}\sum_{j=1}^{k} \frac{\cancel{|G|}}{|C_G(g_j)|} f(g_j)\overline{f'(g_j)} = \sum_{j=1}^{k} \frac{1}{|C_G(g_j)|} f(g_j)\overline{f'(g_j)}$$

$$[\chi_i, \chi_j] = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Let's construct the character table of A_5, $|A_5| = 60 = 2^2 \cdot 3 \cdot 5$, the smallest nonabelian simple group. $G = A_5$ has $k=5$ conjugacy classes

$g_1 = ()$ identity	1A	size 1	,	$\lvert C_G(g_1)\rvert = 60$
$g_2 = (12)(34)$	2A	size 15		$\lvert C_G(g_2)\rvert = 4$
$g_3 = (123)$	3A	size 20		$\lvert C_G(g_3)\rvert = 3$
$g_4 = (12345)$	5A	size 12		$\lvert C_G(g_4)\rvert = 5$
$g_5 = (12354)$	5B	size $\underline{12}$ 60		$\lvert C_G(g_5)\rvert = 5$

(12345) and (12354) are conjugate in S_5 since $(45)^{-1}(12345)(45) = (12354)$

$(1234)^{-1}(12345)(1234) = (234\,1\,5) = (15234)$

$(1235)^{-1}(12345)(1235) = (23541) = (12354)$

but (12345) and (12354) are not conjugate in A_5.

Character table of A_5:

$\lvert C_G(g)\rvert$	60	4	3	5	5
g	$()$	$(12)(34)$	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	3				
χ_3	3				
χ_4	4				
χ_5	5				
χ	5	1	2	0	0

principal character $\chi_1(g) = \mathrm{tr}(\pi_1(g)) = \mathrm{tr}([1]) = 1$

standard permutation character

$$\sum_{j=1}^{k} |\chi_j(1)|^2 = |G|$$ special case of $$\sum_{j=1}^{k} |\chi_j(g)|^2 = |C_G(g)|.$$

Write $60 = n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2$ where n_1, n_2, n_3, n_4, n_5 are positive integers.

$\chi = a_1\chi_1 + a_2\chi_2 + \cdots + a_5\chi_5$. $a_i \in \{0,1,2,\cdots\}$ is the number of copies of χ_i in χ.

$[\chi, \chi_i] = a_i$

$$[\chi,\chi] = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = \frac{25}{60} + \frac{1}{4} + \frac{4}{3} + \frac{0}{5} + \frac{0}{5} = \frac{25+15+80}{60} = \frac{120}{60} = 2.$$

$\Longrightarrow$ $a_1, a_2, \ldots, a_5 = 1,1,0,0,0$ in some order.

$\Longrightarrow$ $\chi = \chi_i + \chi_j$ for some $i \neq j$

$\lvert C_G(g)\rvert$	60	4	3	5	5
g	$()$	$(12)(34)$	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\frac{1+\sqrt5}{2}$	$\frac{1-\sqrt5}{2}$
χ_3	3	-1	0	$\frac{1-\sqrt5}{2}$	$\frac{1+\sqrt5}{2}$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0
χ	5	1	2	0	0

$$[\chi,\chi_1] = \frac{1}{60}\cdot 5\cdot 1 + \frac{1}{4}\cdot 1\cdot 1 + \frac{1}{3}\cdot 2\cdot 1 + \frac{1}{5}\cdot 0\cdot 1 + \frac{1}{5}\cdot 0\cdot 1$$
$$= \frac{1}{12} + \frac{1}{4} + \frac{2}{3} + 0 + 0$$
$$= \frac{1+3+8+0+0}{12} = \frac{12}{12} = 1$$

$\chi = \chi_1 + \chi_j$, $j \in \{2,3,4,5\}$.

$$\chi(()) = \chi_1(()) + \chi_j(())$$
$$5 = 1 + \underbrace{\chi_j(())}_{4} \Longrightarrow j = 4$$

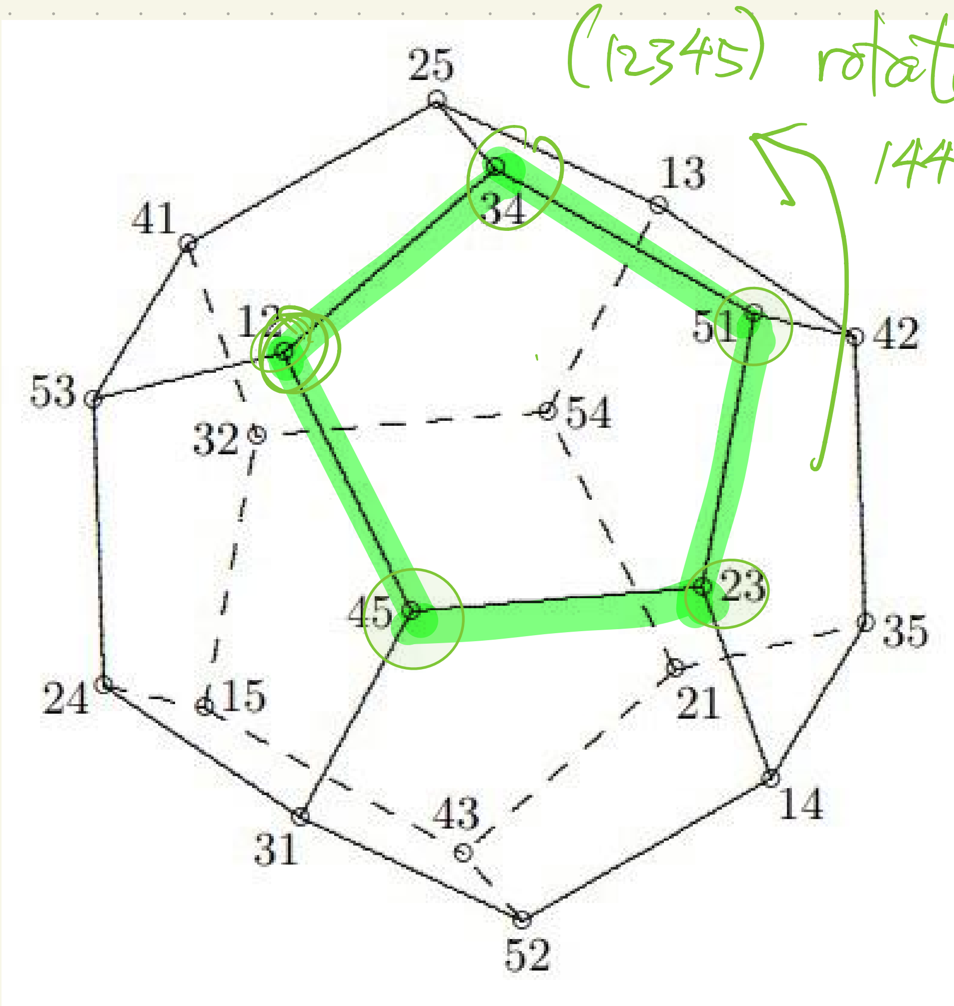
$\chi = \chi_1 + \chi_4$

$$1 + 3\frac{1+\sqrt5}{2} + 3\frac{1-\sqrt5}{2} - 4 + 5\square = 0$$

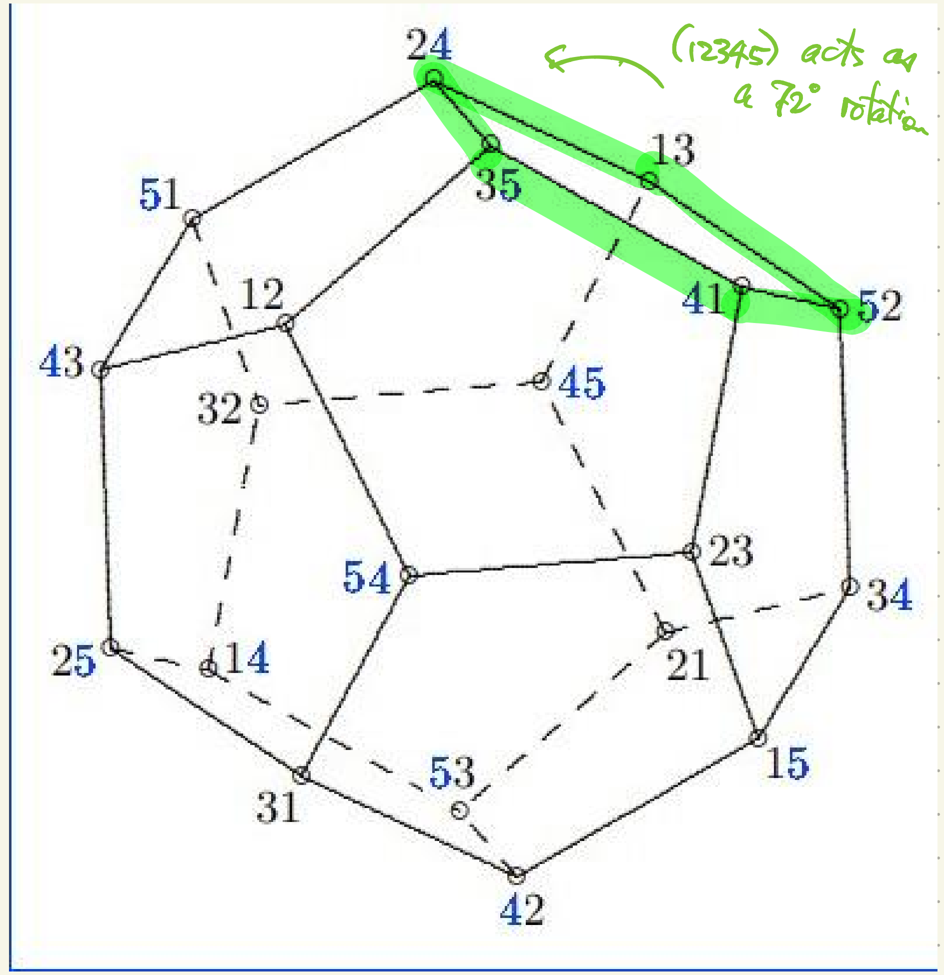
$$[\chi_5,\chi_5] = \frac{5^2}{60} + \frac{1^2}{4} + \frac{(-1)^2}{3} + \frac{0^2}{5} + \frac{0^2}{5} = \frac{25}{60} + \frac{1}{4} + \frac{1}{3} + 0 + 0 = \frac{5+3+4+0+0}{12}$$
$$= \frac{12}{12} = 1.$$

Another way to get χ_5 is to use the transitive permutation action of $G = A_5$ of of degree 6.

(12345) rotates 144°



(12345) acts as a 72° rotation



Dodecahedron with vertices (i,j), $i \neq j$ in $\{1,2,3,4,5\}$

A_5 acts on the dodecahedron as a rotational symmetry group.

Every rotation matrix $A \in SO_3(\mathbb{R})$ is similar to

$$A = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{array}\right]$$

where θ is the angle of the rotation.

$$\operatorname{tr} A = 1 + 2\cos\theta.$$

θ	$\operatorname{tr} A$	
0	3	
$\pi = 180°$	-1	(half turn)
$\frac{2\pi}{3} = 120°$	0	
$\frac{2\pi}{5} = 72°$	$\frac{1+\sqrt{5}}{2}$	golden ratio ≈ 1.618
$\frac{4\pi}{5} = 144°$	$\frac{1-\sqrt{5}}{2}$	≈ -0.618

$\cos 0 = 1$

$\cos \pi = -1$

$\cos \frac{2\pi}{3} = -\frac{1}{2}$

$\cos \frac{2\pi}{5} = \cos 72° = \frac{-1+\sqrt{5}}{4}$

$\cos \frac{4\pi}{5} = \cos 144° = \frac{-1-\sqrt{5}}{4}$

$G = A_5 \cong PSL_2(5)$ acting as fractional linear transformations on $\mathbb{F}_5 \cup \{\infty\} = \{0,1,2,3,4,\infty\}$

$PSL_2(5) = \left\{ \pm\begin{bmatrix} a & b \\ c & d \end{bmatrix} : \quad ad-bc=1, \quad a,b,c,d \in \mathbb{F}_5 \right\} \Big/ \{\pm I\}$

$\pm\begin{bmatrix} a & b \\ c & d \end{bmatrix} : x \longmapsto \dfrac{ax+b}{cx+d}$

identity fixes all six points, $\quad \chi(\underset{g_1 = \pm I}{1}) = 6.$

element of order 2:

$g_2 = \pm\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} : \quad x \longmapsto \dfrac{1}{-x} = -\dfrac{1}{x}$

i.e. $(0,\infty)(1,4)(2)(3) = (0,\infty)(1,4), \quad \chi(g_2) = 2.$

element of order 3:

$g_3 = \pm\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} : \quad x \longmapsto \dfrac{-1}{x+1} = -\dfrac{1}{x+1} \quad$ i.e. $(0\,4\,\infty)(1\,2\,3), \quad \chi(g_3) = 0$

element of order 5: $g_4 = \pm\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} : \quad x \longmapsto x+1 \quad$ i.e. $(0\,1\,2\,3\,4)(\infty) = (0\,1\,2\,3\,4), \quad \chi(g_4) = 1$

Another conjugacy class of elements of order 5:

$g_5 = g_4^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} : \quad x \longmapsto x+2 \quad$ i.e. $(0\,2\,4\,1\,3) \quad \chi(g_5) = 1$

$\lvert C_G(g)\rvert$	60	4	3	5	5
g	$()$	$(12)(34)$	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\frac{1+\sqrt5}{2}$	$\frac{1-\sqrt5}{2}$
χ_3	3	-1	0	$\frac{1-\sqrt5}{2}$	$\frac{1+\sqrt5}{2}$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0
χ	6	2	0	1	1

$[\chi,\chi] = \dfrac{6^2}{60} + \dfrac{2^2}{4} + \dfrac{0^2}{3} + \dfrac{1^2}{5} + \dfrac{1^2}{5}$

$= \dfrac{3+5+0+1+1}{5} = \dfrac{10}{5} = 2$

$\chi = a_1\chi_1 + a_2\chi_2 + \cdots + a_5\chi_5$

$[\chi,\chi] = \sum a_i^2 = 2$

$\Rightarrow a_1,\dots,a_5$: $1,1,0,0,0$ in some order

$[\chi,\chi_1] = \dfrac{6}{60} + \dfrac{2}{4} + \dfrac{0}{3} + \dfrac{1}{5} + \dfrac{1}{5} = \dfrac{1+5+0+2+2}{10} = \dfrac{10}{10} = 1$

$\underbrace{\chi(1)}_{6} = \underbrace{\chi_1(1)}_{1} + \underbrace{\chi_j(1)}_{5} \quad (j \neq 1)$

$\chi = \chi_1 + \chi_5$

Let G be a finite group with k conj. classes and $|G| = n$ then its char. table looks like

$$\begin{array}{c|cccc}
|C_G(g)| & n & * & & * \\
g & g_1 & g_2 & \cdots & g_k \\
\hline
\chi_1 & n_1 = 1 & 1 & \cdots & 1 \\
\chi_2 & n_2 & & & \\
\vdots & & & & \\
\chi_k & n_k & & & \\
\hline
\chi & n & 0 & \cdots & 0
\end{array}$$

$n_i = \deg \chi_i = \chi_i(1)$

Let χ be the perm. character of the regular representation of G: each $g \in G$ permutes G by $g : x \longmapsto gx$. The perm. character of this representation is $\begin{cases} \chi(1) = n \\ \chi(g) = 0 & \text{for } g \neq 1. \end{cases}$

$$\chi = \underset{[\chi,\chi_1]}{(?)}\chi_1 + \underset{[\chi,\chi_2]}{(?)}\chi_2 + \cdots + \underset{[\chi,\chi_k]}{(?)}\chi_k$$

$$[\chi, \chi_i] = \frac{n n_i}{n} + 0 + \cdots + 0 = n_i$$

$$\chi = n_1\chi_1 + n_2\chi_2 + \cdots + n_k\chi_k, \qquad n_i = \chi_i(1) \geq 1$$

Every irreducible character occurs with multiplicity $n_i \geq 1$ in the regular representation.

Evaluate at $1 \in G$:

$$\chi(1) = n = n_1\chi_1(1) + n_2\chi_2(1) + \cdots + n_k\chi_k(1) = n_1^2 + n_2^2 + \cdots + n_k^2.$$

Character values are algebraic integers. If χ is a character of G with $\chi(g) = \operatorname{tr}\pi(g)$ then $\pi(g)^m = \pi(g^m) = \pi(1) = I$ where m is the exponent of G (lcm of orders of elements of G). I = identity matrix of size $\chi(1) = \deg \pi = n$

Eigenvalues of $\pi(g)$ are n^{th} roots of 1.

$\pi(g)$ has upper triangular form $\pi(g) \sim \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, $\lambda_j \in \mathbb{C}$.

$\pi(g)^m = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$, $\lambda_j^m = 1$ for all $j = 1, \dots, n$. Roots of $x^n - 1$ (roots of unity; in particular algebraic integers).

$\chi(g) = \lambda_1 + \cdots + \lambda_n$ is an algebraic integer.

Why do we care?

An early success of character theory is Burnside's Theorem: If p, q are primes then every group of order $p^a q^b$ (a, b positive integers) is solvable.

$|A_5| = 2^2 \cdot 3 \cdot 5$

The proof of Burnside's Theorem uses the fact that character values are algebraic integers. (see handout)

Moreover the degree of each irreducible representation of G divides $|G|$.

eg. A_5 has irreducible representations of degree $1, 3, 3, 4, 5 \mid 60$.

For any finite abelian group G, all irreducible representations have degree 1.

If $|G| = n$ then G has n conjugacy classes (each of size 1) so G has n irreducible characters χ_i $(1 \le i \le n)$ each of degree $n_i = \chi_i(1) = 1$. $\sum_{i=1}^{n} n_i^2 = n$.

If G is cyclic of order n, $G = \langle g \rangle = \{1, g, g^2, \dots, g^{n-1}\}$ then the char. table:

$$\begin{array}{c|cccccc}
|C_G(g^i)| & n & n & n & n & & n \\
g^i & 1 & g & g^2 & g^3 & \cdots & g^{n-1} \\
\hline
\chi_1 & 1 & 1 & 1 & 1 & \cdots & 1 \\
\chi_2 & 1 & \zeta & \zeta^2 & \zeta^3 & \cdots & \zeta^{n-1} \\
 & 1 & \zeta^2 & \zeta^4 & \zeta^6 & \cdots & \zeta^{2(n-1)} \\
\vdots & \vdots & & & & & \\
\chi_n & 1 & \zeta^{n-1} & \zeta^{2(n-1)} & \cdots & & \cdots
\end{array}$$

$\zeta = e^{2\pi i/n}$ or any primitive n^{th} root of unity.

For $G = C_n \times C_m$ (direct product of cyclic groups of order n, m)
the character table is a tensor product (Kronecker product) of char. tables of C_n and C_m.

If G is Klein 4-group $\{(\pm 1, \pm 1)\} = C_2 \times C_2$ under multiplication then

	4	4	4	4
	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

$$= \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\text{char table of } C_2} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

If G is an abel. group then the irreducible characters of G are homomorphisms.
(since $\chi(g) = \mathrm{tr}\,\pi(g) = \pi(g)$ in this case). If $\chi, \chi' \in \mathrm{Irr}(G) = \{$all irred. characters of $G\}$ then $\chi\chi' \in \mathrm{Irr}\,G$ and the irred. characters form a group $\hat{G} = \mathrm{Hom}(G, \mathbb{C}^\times) \cong G$ but not canonically)

Consider a nonabelian group of order 8, either $G =$ dihedral of order 8 $= D_4$
or $G = Q = \{\pm 1, \pm i, \pm j, \pm k\}$

D_4 has 2 elements of order 4
5 elements of order 2
1 element of order 1

Q has 6 elements of order 4
1 element of order 2
1 element of order 1

Both have $|Z(G)| = 2$
$G/Z(G) \cong$ Klein 4-group.
G has 5 conjugacy classes.

G (nonabel. of order 8) has char. table

$Z(G) = \langle -1 \rangle$

$\lvert C_G(g_i)\rvert$	8	8	4	4	4
g_i	1	-1	g_3	g_4	g_5
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

g_3, g_4, g_5 are $\pm i, \pm j, \pm k$ or $\{R, R^3\}$, $\{H, V\}$, $\{D, D'\}$

$\pm\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\pm\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\pm\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Character table for $G/Z(G)$ = Klein 4-group

	4	4	4	4
	1	g_2	g_3	g_4
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

$K \trianglelefteq G$ (normal subgroup)

If $\pi : G/K \longrightarrow GL_n(\mathbb{C})$ is a representation then composing with the canonical map gives a representation of G. If π is irreducible then lifting to G gives an irreducible representation of G.

$$G \longrightarrow G/K \longrightarrow GL_n(\mathbb{C})$$
$$g \longmapsto gK$$

S_5 character table

1	1	--	--	...	1
1	-1	-1	1	-1	...
3					
⋮					

$S_5/A_5 \cong C_2$ cyclic of order 2

	1	-1
χ_1	1	1
χ_2	1	-1

S_4 has a normal subgroup $K = \langle (12)(34), (13)(24) \rangle = \{ (), (12)(34), (13)(24), (14)(23) \}$

$S_4/K \cong S_3$. Three irred. representations of S_3 give three irr. repns of S_4 of degree 1, 1, 2.

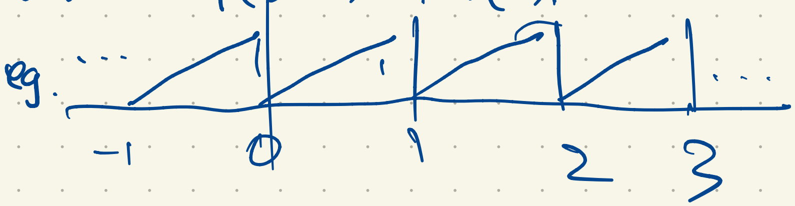
$G = S^1 = \{ z \in \mathbb{C} : |z| = 1 \} = \{ e^{2\pi i \theta} : \theta \in \mathbb{R} \} \cong \mathbb{R}/\mathbb{Z}$ (additively).

$\hat{G} = \mathrm{Hom}(S^1, \mathbb{C}^\times) = \{ \phi_n : n \in \mathbb{Z} \}$, $\phi_n(z) = z^n$. $\quad \phi_n(zz') = (zz')^n = z^n (z')^n$
$= \phi_n(z)\phi_n(z')$

S^1 acts on $L^2(S^1) =$ square-integrable functions $S^1 \to \mathbb{C}$.
$= \{ f : f: S^1 \to \mathbb{C}, \int_{S^1} |f|^2 < \infty \}$

f is defined on $S^1 \cong \mathbb{R}/\mathbb{Z}$ iff f is a periodic function $f: \mathbb{R} \to \mathbb{C}$
with $f(z+1) = f(z)$.

eg.



$f(e^{2\pi i \theta}) = \theta - \lfloor \theta \rfloor =$ fractional part of θ

$\int_0^1 \theta \, d\theta = \frac{1}{2}\theta^2 \Big|_0^1 = \frac{1}{2}$.

$\int_0^1 |f(\theta)|^2 d\theta = \int_0^1 \theta^2 d\theta = \frac{1}{3}\theta^3 \Big|_0^1 = \frac{1}{3}$

Functions $\sin(2\pi k\theta)$, $\cos(2\pi k\theta)$ $(k \in \mathbb{Z})$ span (a dense subspace of) $L^2(S^1)$.

S^1 acts on $L^2(S^1)$ by translation:

Alternatively: $\{e^{2\pi k i\theta} : k \in \mathbb{Z}\}$.