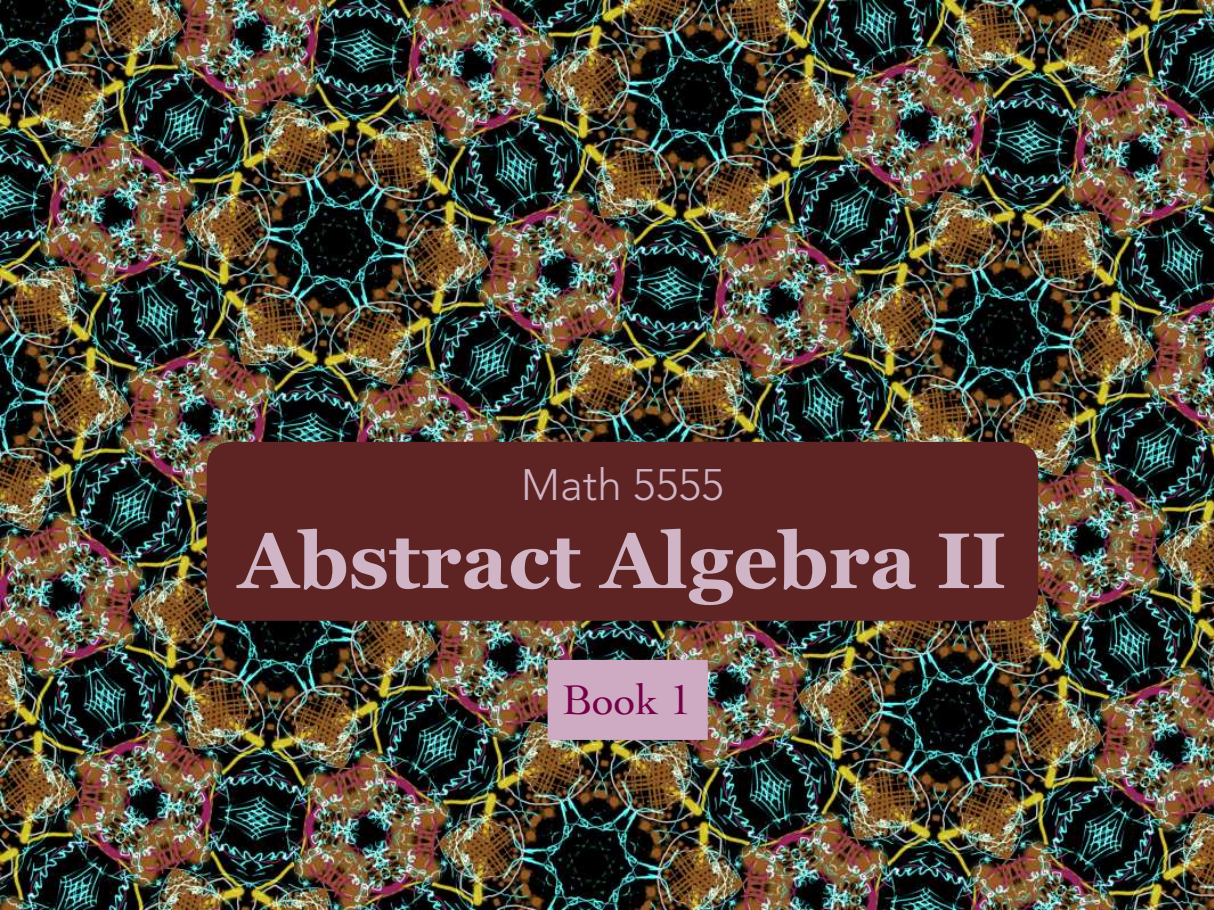




Math 5555

Abstract Algebra II

Book 1

In group theory we have

- permutation representations: homomorphisms $\pi: G \rightarrow S_n$ permutation representation of degree n
(if π is 1-to-1 then π is a faithful representation; then $\pi(G) \leq S_n$)

- linear representations: homomorphism $\pi: G \rightarrow GL_n(F)$ F : field
linear representation of degree n over F

If $F = \mathbb{C}$ (or $\mathbb{R}$ or ...) then π is an ordinary representation.

If $\text{char } F = p$ (prime) then π is a modular representation.

G : unless otherwise specified, G finite group. (until later...)

Usually $F = \mathbb{C}$ (or $\mathbb{R}$) and $|G| < \infty$

If $\pi_i : G \rightarrow GL_{n_i}(\mathbb{C})$ ($i=1,2$) then $\pi_1 \oplus \pi_2 : G \rightarrow GL_{n_1+n_2}(\mathbb{C})$, $g \mapsto \underbrace{\left[\begin{array}{c|c} \pi_1(g) & \mathbf{0} \\ \hline \mathbf{0} & \pi_2(g) \end{array} \right]}_{\substack{n_1 \\ n_2}} \}^{n_2}$ is a representation of degree n_1+n_2

$\pi: G \rightarrow GL_n(\mathbb{C})$ is decomposable if there is a decomposition $\mathbb{C}^n = U \oplus V$ such that $U, V \neq 0$
 $(\dim U = n_1, \dim V = n_2, \begin{matrix} n_1, n_2 \text{ positive} \\ \text{integers,} \\ n_1 + n_2 = n \end{matrix})$
 U, V invariant under all matrices $\pi(g), g \in G$.

$G = S_3$ acting naturally on $\mathbb{C}^3$ by permuting the standard basis vectors $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

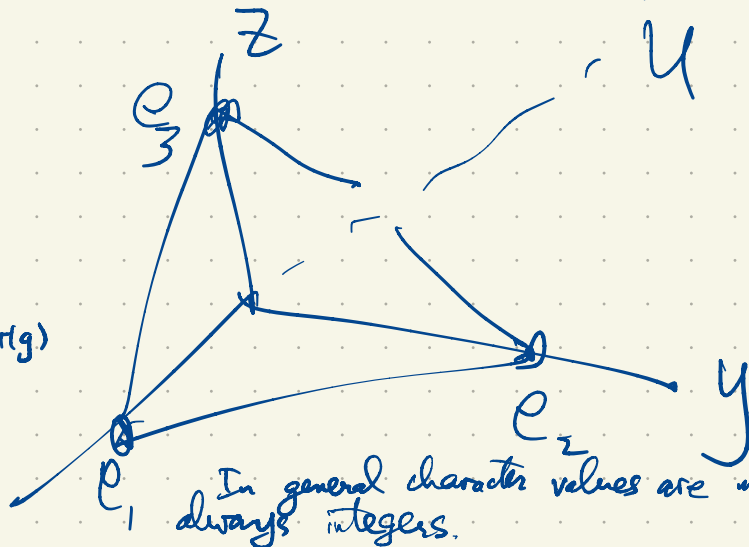
via $\sigma: e_i \mapsto e_{\sigma(i)}$

ie. $(12) \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$(123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
$$(1) \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is a faithful representation of degree 3.

$\mathbb{H}$ is decomposable: $\mathbb{C}^3 = U \oplus V$, $U = \langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rangle$, $V = \underbrace{U^\perp = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \rangle}_{\text{indecomposable}}.$



The representation $\pi: S_3 \rightarrow GL_3(\mathbb{C})$ has (affords) character

$$\chi(g) = \text{tr } \pi(g)$$

$$\chi(1) = 3 = \text{degree of } \pi = \deg \chi.$$

$$\chi((12)) = \chi((13)) = \chi((23)) = 1$$

$$\chi((123)) = \chi((132)) = 0$$

In general character values are not always integers.

for a permutation representation

$$\pi: G \rightarrow S_n \subset GL_n(\mathbb{C})$$

the associated character $\chi(g) = \text{tr} \pi(g)$
(called the permutation character)

is $\chi(g) = \text{no. of fixed points of } g$
 $= |\{i: g(i) = i\}|, \quad 1 \leq i \leq n.$

Given a representation $\pi: G \rightarrow GL_n(\mathbb{C})$, the character of π is

$$\chi(g) = \text{tr } \pi(g) \in \mathbb{C}.$$

Character values $\chi(g) \in \mathbb{C}$ are always algebraic integers ($g \in G$, $|G| < \infty$) and character values of S_n are ordinary integers.

$\chi(g)$ depends only on the conjugacy class of g .

If $g, h \in G$ then $g \sim h'gh$ (conjugate in G) so

$$\pi(h'gh) = \pi(h')\pi(g)\pi(h) \sim \pi(g) \text{ (similar in } GL_n(\mathbb{C}) \text{ i.e. conjugate)}$$

$\uparrow$
inverses in $GL_n(\mathbb{C})$

$$\text{so } \text{tr } \pi(g) = \text{tr } (\pi(h')\pi(g)\pi(h)) = \text{tr } \pi(h'gh)$$

$$\text{tr } (AB) = \text{tr } (BA)$$

$$\text{tr } (B^{-1}AB) = \text{tr } (AB \cdot B^{-1}) = \text{tr } A$$

$$\chi(h'gh) = \chi(g).$$

If $\pi: G \rightarrow GL_n(\mathbb{C})$ is any representation i.e. homomorphism, and $B \in GL_n(\mathbb{C})$, then

$\tilde{\pi}(g) = B^{-1}\pi(g)B \in GL_n(\mathbb{C})$ is also a representation.

$\pi, \tilde{\pi}$ are equivalent (via a change of basis).

$$\begin{aligned} \tilde{\pi}(gh) &= B^{-1}\pi(gh)B = B^{-1}\pi(g)\pi(h)B = B^{-1}\pi(g)B \cdot B^{-1}\pi(h)B \\ &= \tilde{\pi}(g)\tilde{\pi}(h). \end{aligned}$$

They have the same character: the character of $\tilde{\pi}$ is

$$\tilde{\chi}(g) = \text{tr } \tilde{\pi}(g) = \text{tr } (B^{-1}\pi(g)B) = \text{tr } \pi(g) = \chi(g).$$

It's not obvious but the converse is true: χ determines π up to equivalence. Two representations have the same character iff they are equivalent.

Let $\pi: G \rightarrow GL_n(\mathbb{C})$ be a representation (i.e. homomorphism).

π is reducible if there exists nontrivial ^{proper} subspace $U \subset \mathbb{C}^n$ ($\dim U \in \{1, 2, \dots, n-1\}$) such that U is invariant under $\pi(g)$ for all $g \in G$ i.e. $\pi(g)U \subseteq U$ for all $g \in G$.

$$\text{i.e. } \pi(g) = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \begin{matrix} \left\{ \begin{matrix} k \\ n-k \end{matrix} \right\} \\ \text{for all } g \in G. \end{matrix}$$

$$k = \dim U.$$

when $|G| < \infty$ and $\pi: G \rightarrow GL_n(\mathbb{C})$
 $\text{char } \mathbb{C} = 0$, π reducible $\iff \pi$ decomposable
 π irreducible $\iff \pi$ indecomposable

In general, π decomposable $\Rightarrow \pi$ reducible
 $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \Rightarrow \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$
 π indecomposable $\Leftarrow \pi$ irreducible.

The representation $\pi: \mathbb{R} \rightarrow GL_2(\mathbb{R})$, $\pi(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, $\pi(a+b) = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \pi(a)\pi(b)$
 additive group

is reducible: $\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$ is an invariant subspace. There is no complementary invariant subspace (in particular the complementary subspace $\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle$ is not invariant).

Maschke's Theorem Let G be a finite group and $F = \mathbb{C}$ (or more generally F is any field of characteristic not dividing $|G|$). Then any representation $\pi: G \rightarrow GL(F)$ is reducible iff it's decomposable (i.e. π is irreducible iff π is indecomposable; i.e. whenever every invariant subspace $U \leq V$ has a complementary subspace U' which is also invariant).

$U \leq V$ has a complementary subspace $U' \iff V = U \oplus U'$. If $\dim V = n$ and $\dim U = k$ then U' is a complement to U iff $\dim U' = n-k$ and $U + U' = V$
 iff $\dim U' = n-k$ and $U \cap U' = \{0\}$.

In this case every $v \in V$ is uniquely represented as $v = u + u'$, $u \in U$, $u' \in U'$.

In this, the projection $V \rightarrow U$ along U' is the map $P: v \mapsto Pv = u$. $P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \text{size } k \\ \text{size } n-k \end{matrix}$
 P has image $PV = U$; $\ker P = U'$. Note: $I-P$ is the projection from V onto U' along U .

A linear transformation $P: V \rightarrow V$ is a projection iff $P^2 = P$. In this case P is a root of $x^2 - x$ so the eigenvalues of P are in $\{0, 1\}$. Take $U = 1$ -eigenspace of $P = \ker(I-P)$, $U' = \ker P$.

Proof of Maschke's Theorem: (in the nontrivial direction)

Suppose $\pi: G \rightarrow GL(\mathbb{C})$ is a representation having an invariant subspace $U \leq \mathbb{C}^n$ i.e. $\pi(g)U \subseteq U$ for all $g \in G$. We want to find a complementary subspace $W \leq \mathbb{C}^n$ which is also invariant. Start with any complementary subspace U' where $\mathbb{C}^n = U \oplus U'$, $\dim U = k$, $\dim U' = n-k$; every $v \in \mathbb{C}^n$ is uniquely expressible as $v = u + u'$, $u \in U$, $u' \in U'$.

Unfortunately U' is not invariant in general. Let $P: \mathbb{C}^n \rightarrow U$ be the projection onto U along U' i.e. $P(v) = P(u + u') = u$, so $U = PV$, $U' = \ker P$. Consider the new map $\tilde{P}: V \rightarrow V$ defined by $\tilde{P} = \frac{1}{|G|} \sum_{g \in G} \pi(g) P \pi(g)$. $\tilde{P}V \subseteq U$. Since $\tilde{P}v = \frac{1}{|G|} \sum_g \pi(g^{-1}) P \underbrace{\pi(g)v}_{\substack{\in V \\ \in U}} \in U$.

For all $u \in U$, $\tilde{P}u = u$ i.e. $\tilde{P}|_U = \text{id}|_U$. Why?

$$\tilde{P}u = \frac{1}{|G|} \sum_g \pi(g^{-1}) P \pi(g) u = \frac{1}{|G|} \sum_g \pi(g^{-1}) \pi(g) u = \frac{1}{|G|} |G| u = u.$$

$= \pi(g)u$ since $\pi(g)u \in U$.

Next: show $\tilde{P}$ commutes with all $\pi(g)$, $g \in G$. (P doesn't satisfy this in general!)

$$\text{Then } \tilde{P}\pi(g) = \frac{1}{|G|} \sum_{h \in G} \pi(h^{-1}) P \pi(h) \pi(g) = \frac{1}{|G|} \sum_{h \in G} \pi(h^{-1}) P \pi(hg)$$

$$h \longleftrightarrow \begin{aligned} x &= hg \\ x^{-1} &= g^{-1}h^{-1} \\ g x^{-1} &= h^{-1} \end{aligned}$$

$$= \frac{1}{|G|} \sum_{x \in G} \underbrace{\pi(gx^{-1})}_{\pi(g)\pi(x)^{-1}} P \pi(x) = \pi(g) \cdot \frac{1}{|G|} \sum_{x \in G} \pi(x^{-1}) P \pi(x) = \pi(g) \tilde{P}.$$

Next: show $\tilde{P}^2 = \tilde{P}$. If $v \in V$ then $\tilde{P}v \in U$ so $\tilde{P}^2v = \tilde{P}\tilde{P}v = \tilde{P}v$. So $\tilde{P}$ is idempotent so

$\tilde{P}$ is a projection on $\tilde{P}V = U$ along $W = \ker \tilde{P}$. Note: if $k = \dim U = \text{rank } \tilde{P} = \text{tr } \tilde{P}$, $\dim W = \dim \ker \tilde{P} = n - k$. W is also invariant under $\pi(g)$: if $v \in W$ then $\tilde{P}v = 0$

so $\pi(g)\tilde{P}v = \pi(g)0 = 0$ so $\pi(g)v \in W$

$\tilde{P}\pi(g)v$



The irreducible representations of S_3 are

$$\pi_1(g) = [1] \in GL_1(\mathbb{C}) \quad \text{trivial representation}$$

$$\pi_2(g) = [\text{sgn}(g)] \in GL_1(\mathbb{C}) \quad \text{sign representation:}$$

$$= [\pm 1] \text{ according as } g \text{ is an even or odd permutation.}$$

$$\text{sgn}(1) = 1$$

$$\text{sgn}((123)) = 1$$

$$\text{sgn}((132)) = 1$$

$$\text{sgn}((12)) = \text{sgn}((13)) = \text{sgn}((23)) = -1$$

$$\text{sgn}(gh) = \text{sgn}(g) \text{sgn}(h)$$

$$\pi_3: S_3 \rightarrow GL_2(\mathbb{C})$$

$$(1) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(12) \mapsto \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$(123) \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$(132) \mapsto \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$(13) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(23) \mapsto \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\pi_3((13)) = \pi_3((123)(12)) = \pi_3((123)) \pi_3((12))$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\pi_3((23)) = \pi_3((123)(13)) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

An irreducible character is a map $G \rightarrow \mathbb{C}$
where π is an irreducible representation $g \mapsto \chi(g)$.

$0, \langle [1] \rangle, \langle [-1] \rangle, \mathbb{C}^2$ are the only subspaces invariant under $\pi_3((13))$; but the 1-dimensional invariant subspaces are not invariant under $\pi_3((123))$.

The character table of S_3 is

	(1)	(12)	(23)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$$\pi_3((123)) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \text{ has char. poly. } x^2 + x + 1$$

$$\pi_3((13)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ has char. poly. } x^2 - 1$$

↑ eigenspaces $\langle [1] \rangle, \langle [-1] \rangle$
(eigenvalues 1, -1 respectively)

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Given any finite group G , a class function on G is a function $f: G \rightarrow \mathbb{C}$ which only depends on conjugacy class, i.e. $f(g') = f(g)$ whenever $g, g' \in G$ are conjugate i.e. $g' = ugu$ for some $u \in G$.

All characters of G (irreducible or otherwise) are class functions.

$V = \{\text{class functions on } G\}$ is a complex vector space

$$f, f' \in V \Rightarrow f + f' \in V, \quad (f + f')(x) = f(x) + f'(x)$$

V is in fact a complex inner product space:

$$[f, f']_G = [f, f'] = \frac{1}{|G|} \sum_{x \in G} f(x) \overline{f'(x)} \quad [f', f] = \overline{[f, f']}$$

sesquilinear
 $\frac{1}{2}$ -linear

$\dim V = \text{number of conjugacy classes}$.

The irreducible characters of G always give an orthonormal basis for $V = \{\text{class functions on } G\}$.

Recall the permutation representation with permutation character

$$\pi: S_3 \rightarrow GL_3(\mathbb{C})$$

$$\pi: e_i \mapsto e_{\pi(i)}$$

$$(12) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(123) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$(1) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\chi((1)) = 3$$

$$\chi((12)) = 1$$

$$\chi((123)) = 0$$

$$[\chi, \chi_1] = \frac{1}{6}(1 \cdot 3 + 1 + 1 + 1 + 0) = 1$$

$$[\chi, \chi_2] = \frac{1}{6}(3 - 1 - 1 + 0 + 0) = 0$$

$$[\chi, \chi_3] = \frac{1}{6}(6 + 0 + 0 + 0 + 0) = 1$$

$$\Rightarrow \pi \simeq \pi_1 \oplus \pi_3$$

Character table of S_3

	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1
χ	3	1	0

$$[\chi_1, \chi_1] = \frac{1}{6} \sum_{g \in S_3} \chi_1(g) \overline{\chi_1(g)} = \frac{1}{6}(1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1) = 1$$

$$[\chi_1, \chi_2] = \frac{1}{6} \sum_g \chi_1(g) \overline{\chi_2(g)} = \frac{1}{6}(1 \cdot 1 - 1 \cdot 1 - 1 \cdot 1 - 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1) = 0$$

$$[\chi_1, \chi_3] = \frac{1}{6}(1 \cdot 2 + 0 + 0 + 0 - 1 \cdot 1) = 0$$

$$[\chi_2, \chi_2] = \frac{1}{6}(1 + 1 + 1 + 1 + 1 + 1) = 1$$

$$[\chi_2, \chi_3] = \frac{1}{6}(2 + 0 + 0 + 0 - 1 \cdot 1) = 0$$

$$[\chi_3, \chi_3] = \frac{1}{6}(4 + 0 + 0 + 0 + 1 \cdot 1) = 1$$

The long way to check the decomposition $\pi \simeq \pi_1 \oplus \pi_3$ is found in the course notes: with respect to new basis

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

basis for
first invariant
subspace

basis for
second invariant
subspace

$\pi(g)$ has matrix $\begin{pmatrix} \pi_1(g) & 0 & 0 \\ 0 & & \\ 0 & & \pi_3(g) \end{pmatrix}$ with respect to v_1, v_2, v_3

(see p. 3 of course notes).

Character tables also have orthogonality of columns.

$ C_g $	6	2	3
g	(1)	(12)	(23)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$$C_g(g) = \{x \in G: xg = gx\}$$

The conjugacy class of $g \in G$ is the index of the centralizer

$$[G : C_g(g)] = \frac{|G|}{|C_g(g)|}$$

The column orthogonality is expressed as: given $g, h \in G$,
(k = number of irreducible characters = no. of conjugacy classes)

$$\sum_{i=1}^k \chi_i(g) \overline{\chi_i(h)} = \begin{cases} 0 & \text{if } g, h \text{ not conjugate} \\ |C_g(g)| & \text{if } g, h \text{ are conjugate} \end{cases}$$

Row orthogonality says: Let $g_1, \dots, g_k \in G$ be reps of conj. classes

$$[f, f']_G = \frac{1}{|G|} \sum_{a \in G} f(a) \overline{f'(a)} = \frac{1}{|G|} \sum_{j=1}^k \frac{|G|}{|C_g(g_j)|} f(g_j) \overline{f'(g_j)} = \sum_{j=1}^k \frac{1}{|C_g(g_j)|} f(g_j) \overline{f'(g_j)}$$

$$[\chi_i, \chi_j] = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Let's construct the character table of A_5 , $|A_5| = 60$, the smallest nonabelian simple group. $G = A_5$ has $k=5$ conjugacy classes $= 2^2 \cdot 3 \cdot 5$

$g_1 = ()$	identity	1A	size 1	$ C_G(g_1) = 60$
$g_2 = (12)(34)$		2A	size 15	$ C_G(g_2) = 4$
$g_3 = (123)$		3A	size 20	$ C_G(g_3) = 3$
$g_4 = (12345)$		5A	size 12	$ C_G(g_4) = 5$
$g_5 = (12354)$		5B	size 12	$ C_G(g_5) = 5$
			<u>60</u>	

(12345) and (12354) are conjugate in S_5 since $(45)^{-1}(12345)(45) = (12354)$

$$(1234)^{-1}(12345)(1234) = (23415) = (15234)$$

but (12345) and (12354) are not conjugate in A_5 .

$$(1235)^{-1}(12345)(1235) = (23541) = (12354)$$

Character table of A_5 :

$ C_G(g) $	60	4	3	5	5
g	$()$	$(12)(34)$	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	3				
χ_3	3				
χ_4	4				
χ_5	5				
χ	5	1	2	0	0

principal character

$$\chi_i(g) = \text{tr}(\rho_i(g)) = \text{tr}([1]) = 1$$

standard permutation character

$$\sum_{j=1}^k |\chi_j(1)|^2 = |G| \quad \text{special case of}$$

$$\sum_{j=1}^k |\chi_j(g)|^2 = |C_G(g)|$$

Write $60 = n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2$ where n_1, n_2, n_3, n_4, n_5 are positive integers.

$\chi = a_1 \chi_1 + a_2 \chi_2 + \dots + a_5 \chi_5$, $a_i \in \{0, 1, 2, \dots\}$ is the number of copies of χ_i in χ .

$$[\chi, \chi_i] = a_i$$

$$[\chi, \chi] = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = \frac{25}{60} + \frac{1}{4} + \frac{4}{3} + \frac{0}{5} + \frac{0}{5} = \frac{25+15+80}{60} = \frac{120}{60} = 2.$$

$\Rightarrow a_1, a_2, \dots, a_5 = 1, 1, 0, 0, 0$ in some order.

$\Rightarrow \chi = \chi_i + \chi_j$ for some $i \neq j$

$$[\chi, \chi_1] = \frac{1}{60} \cdot 5 \cdot 1 + \frac{1}{4} \cdot 1 \cdot 1 + \frac{1}{3} \cdot 2 \cdot 1 + \frac{1}{5} \cdot 0 \cdot 1 + \frac{1}{5} \cdot 0 \cdot 1$$

$$= \frac{1}{12} + \frac{1}{4} + \frac{2}{3} + 0 + 0$$

$$= \frac{1+3+8+0+0}{12} = \frac{12}{12} = 1$$

$$\chi = \chi_1 + \chi_j, \quad j \in \{2, 3, 4, 5\}.$$

$$\chi(1) = \chi_1(1) + \chi_j(1)$$

$$5 = 1 + \underbrace{\chi_j(1)}_4 \Rightarrow j=4$$

$$\chi = \chi_1 + \chi_4$$

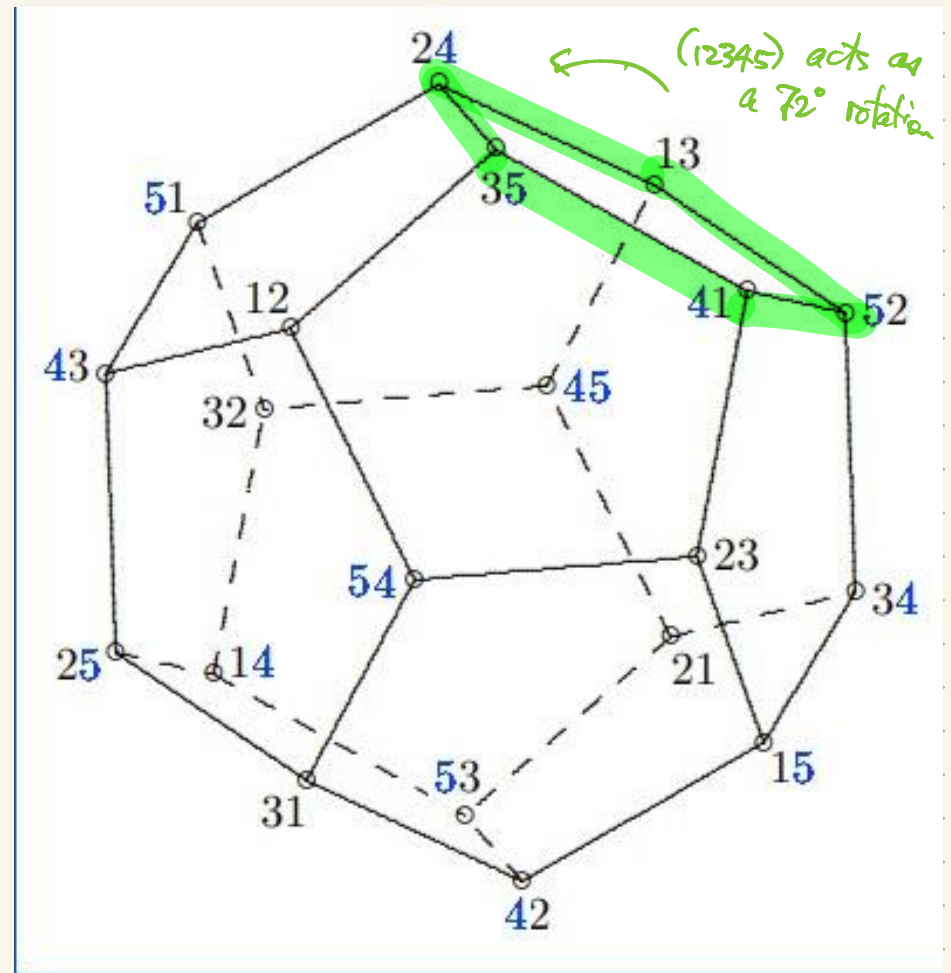
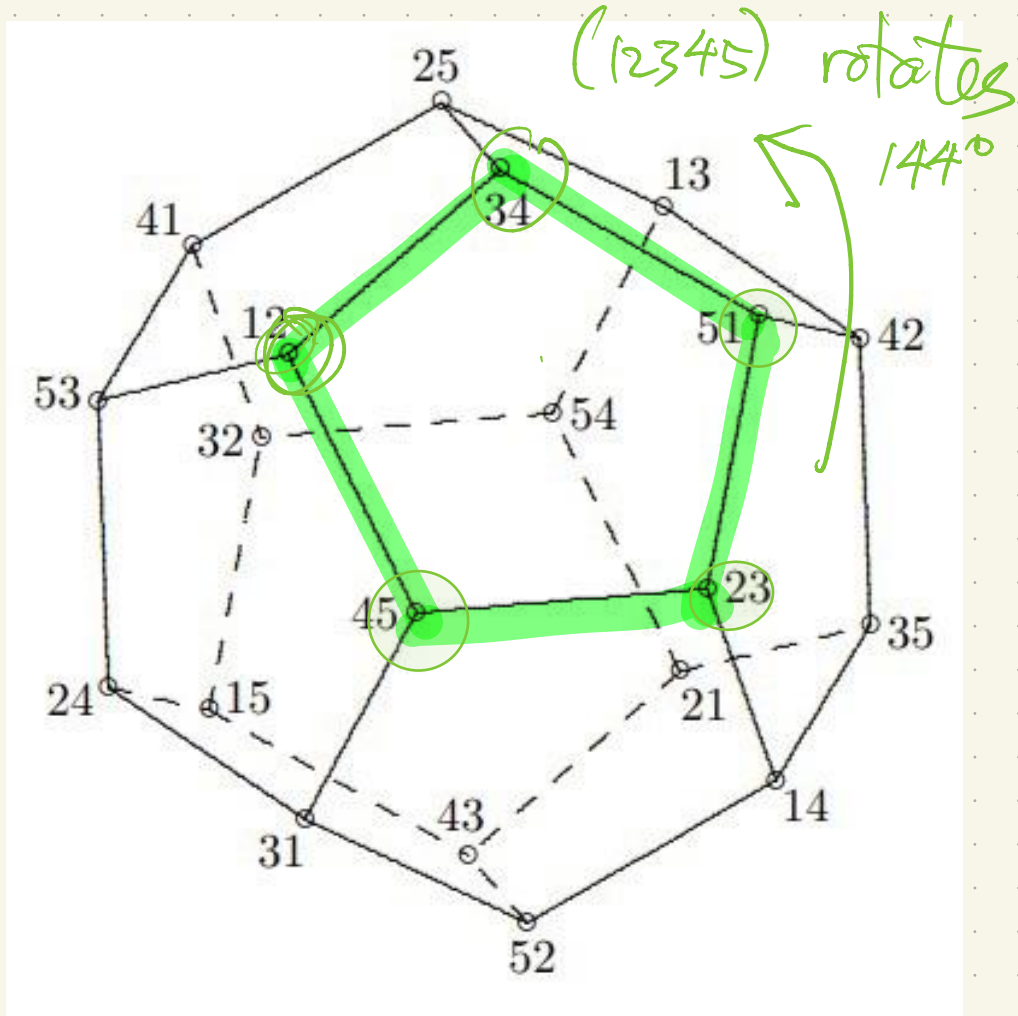
(C_g)	60	4	3	5	5
g	(1)	(12)(34)	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_3	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0
χ	5	1	2	0	0

$$1 + 3 \frac{1+\sqrt{5}}{2} + 3 \frac{1-\sqrt{5}}{2} - 4 + 5 \square = 0$$

$$[\chi_5, \chi_5] = \frac{5^2}{60} + \frac{1^2}{4} + \frac{(-1)^2}{3} + \frac{0^2}{5} + \frac{0^2}{5} = \frac{25}{60} + \frac{1}{4} + \frac{1}{3} + 0 + 0 = \frac{5+3+4+0+0}{12}$$

$$= \frac{12}{12} = 1.$$

Another way to get χ_5 is to use the transitive permutation action of $G = A_5$ of degree 6.



Dodecahedron with vertices (i,j) ,
 $i \neq j$ in $\{1,2,3,4,5\}$
 A_5 acts on the dodecahedron as a
 rotational symmetry group.

$$\begin{aligned}\cos 0 &= 1 \\ \cos \pi &= -1 \\ \cos \frac{2\pi}{5} &= -\frac{1}{2} \\ \cos \frac{2\pi}{5} &= \cos 72^\circ = \frac{-1+\sqrt{5}}{4} \\ \cos \frac{4\pi}{5} &= \cos 144^\circ = \frac{-1-\sqrt{5}}{4}\end{aligned}$$

Every rotation matrix $A \in SO_3(\mathbb{R})$ is similar to
 $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix}$ where θ is the angle of the rotation.



$$\text{tr } A = 1 + 2 \cos \theta.$$

θ	$\text{tr } A$
0	3
$\pi = 180^\circ$	-1 (half-turn)
$\frac{2\pi}{3} = 120^\circ$	0
$\frac{2\pi}{5} = 72^\circ$	$\frac{1+\sqrt{5}}{2}$ golden ratio ≈ 1.618
$\frac{4\pi}{5} = 144^\circ$	$\frac{1-\sqrt{5}}{2} \approx -0.618$

$G = A_5 \cong \text{PSL}_2(5)$ acting as fractional linear transformations on $\mathbb{F}_5 \cup \{\infty\} = \{0, 1, 2, 3, 4, \infty\}$

$$\text{PSL}_2(5) = \left\{ \pm \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad-bc=1, a, b, c, d \in \mathbb{F}_5 \right\} / \{\pm I\}$$

$$\pm \begin{bmatrix} a & b \\ c & d \end{bmatrix} : x \mapsto \frac{ax+b}{cx+d}$$

identity fixes all six points,
element of order 2:

$$g_2 = \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : x \mapsto -\frac{1}{x} = -\frac{1}{x}$$

$$\chi(g_2) = 6$$

$$g_1 = \pm I$$

element of order 3: i.e. $(0, \infty)(1, 4)(2)(3) = (0, \infty)(1, 4)$, $\chi(g_3) = 2$

$$g_3 = \pm \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} : x \mapsto \frac{-1}{x+1} = -\frac{1}{x+1} \quad \text{i.e. } (0, 4, \infty)(1, 2, 3), \quad \chi(g_3) = 0$$

element of order 5: $g_4 = \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} : x \mapsto x+1$ i.e. $(0, 1, 2, 3, 4)(\infty)$
 $= (0, 1, 2, 3, 4)$, $\chi(g_4) = 1$

Another conjugacy class of elements of order 5:

$$g_5 = g_4^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} : x \mapsto x+2 \quad \text{i.e. } (0, 2, 4, 1, 3) \quad \chi(g_5) = 1$$

$$\begin{aligned} [\chi, \chi] &= \frac{6^2}{60} + \frac{2^2}{4} + \frac{0^2}{3} + \frac{1^2}{5} + \frac{1^2}{5} \\ &= \frac{3 + 5 + 0 + 1 + 1}{5} = \frac{10}{5} = 2 \end{aligned}$$

$ C_G(g) $	60	4	3	5	5
g	(1)	(12)(34)	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_3	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0
χ	6	2	0	1	1

$$\chi = a_1 \chi_1 + a_2 \chi_2 + \dots + a_5 \chi_5$$

$$[\chi, \chi] = \sum a_i^2 = 2$$

$\Rightarrow a_1, \dots, a_5: 1, 1, 0, 0, 0$ in some order

$$[\chi, \chi_1] = \frac{6}{60} + \frac{2}{4} + \frac{0}{3} + \frac{1}{5} + \frac{1}{5} = \frac{1+5+0+2+2}{10} = \frac{10}{10} = 1$$

$$\frac{\chi(1)}{6} = \frac{\chi_1(1)}{1} + \frac{\chi_5(1)}{5} \quad (j \neq 1)$$

$$\chi = \chi_1 + \chi_5$$

Let G be a finite group with k conj. classes and $|G| = n$ then its char. table looks like

$ C_G(g) $	n	n_1	n_2	$\dots$	n_k
g	g_1	g_2	$\dots$	g_k	
χ_1	1	1	$\dots$	1	
χ_2	n_1				
$\vdots$					
χ_k	n_k				
χ	n	0	$\dots$	0	

$$n_i = \deg \chi_i = \chi_i(1)$$

Let χ be the perm. character of the regular representation of G : each $g \in G$ permutes G by $g: x \mapsto gx$. The perm. character of this representation is $\begin{cases} \chi(1) = n \\ \chi(g) = 0 \text{ for } g \neq 1 \end{cases}$

$$\chi = (?)\chi_1 + (?)\chi_2 + \dots + (?)\chi_k$$

$\uparrow \quad \quad \uparrow \quad \quad \uparrow$
 $[\chi, \chi_1] \quad [\chi, \chi_2] \quad [\chi, \chi_k]$

$$[\chi, \chi_i] = \frac{n n_i}{n} + 0 + \dots + 0 = n_i$$

$$\chi = n_1 \chi_1 + n_2 \chi_2 + \dots + n_k \chi_k, \quad n_i = \chi_i(1) \geq 1$$

Every irreducible character occurs with multiplicity $n_i \geq 1$ in the regular representation.

Evaluate at $1 \in G$:

$$\chi(1) = n = n_1 \chi_1(1) + n_2 \chi_2(1) + \dots + n_k \chi_k(1) = n_1^2 + n_2^2 + \dots + n_k^2$$

Character values are algebraic integers. If χ is a character of G with $\chi(g) = \text{tr } \pi(g)$ then $\pi(g)^m = \pi(g^m) = \pi(1) = I$ where m is the exponent of G (lcm of orders of elements of G). Eigenvalues of $\pi(g)$ are n^{th} roots of 1. $\uparrow$ identity matrix of size $\chi(1) = \deg \pi = n$

$\pi(g)$ has upper triangular form $\pi(g) \sim \begin{bmatrix} \lambda_1 & * \\ 0 & \ddots & \lambda_n \end{bmatrix}$, $\lambda_j \in \mathbb{C}$.

$\pi(g)^m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\lambda_j^m = 1$ for all $j=1, \dots, n$. Roots of $x^n - 1$ (roots of unity; in particular algebraic integers).

$\chi(g) = \lambda_1 + \dots + \lambda_n$ is an algebraic integer.

Why do we care?

An early success of character theory is Burnside's Theorem: If p, q are primes then every group of order $p^a q^b$ (a, b positive integers) is solvable.

$$|A_5| = 2^3 \cdot 3 \cdot 5$$

The proof of Burnside's Theorem uses the fact that character values are algebraic integers. (see handout)

Moreover the degree of each irreducible representation of G divides $|G|$.

eg. A_5 has irreducible representations of degree $1, 3, 3, 4, 5 \mid 60$.

For any ^{finite} abelian group G , all irreducible representations have degree 1.

If $|G| = n$ then G has n conjugacy classes (each of size 1) so G has n irreducible characters χ_i ($1 \leq i \leq n$) each of degree $n_i = \chi_i(1) = 1$. $\sum_{i=1}^n n_i^2 = n$.

If G is cyclic of order n , $G = \langle g \rangle = \{1, g, g^2, \dots, g^{n-1}\}$ then the char. table:

$\chi_i \backslash G(g^j) $	1	g	g^2	g^3	$\dots$	g^{n-1}
χ_1	1	1	1	1	$\dots$	1
χ_2	1	ζ	ζ^2	ζ^3	$\dots$	ζ^{n-1}
$\vdots$	$\vdots$	ζ^2	ζ^4	ζ^6	$\dots$	$\zeta^{2(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
χ_n	1	ζ^{n-1}	$\zeta^{2(n-1)}$	$\dots$	$\dots$	$\zeta^{(n-1)^2}$

$\zeta = e^{2\pi i/n}$ or any primitive n^{th} root of unity.

For $G = C_n \times C_m$ (direct product of cyclic groups of order n, m)
 the character table is a tensor product (Kronecker product) of char. tables
 of C_n and C_m .

If G is Klein 4-group $\{(\pm 1, \pm 1)\} = C_2 \times C_2$ under multiplication then

	4	4	4	4
	(1, 1)	(1, -1)	(-1, 1)	(-1, -1)
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

$$= \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\text{Char table of } C_2} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

If G is an abel. group then the irreducible characters of G are homomorphisms.
 (since $\chi(g) = \text{tr } \pi(g) = \pi(g)$ in this case). If $\chi, \chi' \in \text{Irr}(G) = \{\text{all irred. characters of } G\}$
 then $\chi\chi' \in \text{Irr } G$ and the irred. characters form a group $\hat{G} = \text{Hom}(G, \mathbb{C}^\times) \cong G$
 but not canonically.

Consider a nonabelian group of order 8, either $G = \text{dihedral of order 8} = D_4$

D_4 has 2 elements of order 4
 5 " " " " " 2
 1 " " " " " 1

or $G = Q = \{\pm 1, \pm i, \pm j, \pm k\}$
 Q has 6 elements of order 4
 1 " " " " 2
 1 " " " " 1

Both have $|Z(G)| = 2$

$G/Z(G) \cong \text{Klein 4-group}$

G has 5 conjugacy classes.

G (nonabel. of order 8) has char. table

$ C_G(g_i) $	8	8	4	4	4
g_i	1	-1	g_3	g_4	g_5
χ_1					
χ_2					
χ_3					
χ_4					
χ_5					

$Z(G) = \langle -1 \rangle$
 g_3, g_4, g_5 are $\pm i, \pm j, \pm k$ or $\{R, R^3\}, \{H, V\}, \{D, D'\}$