Math 5555

Abstract Algebra II

Book 2

Induction takes class functions on H to class functions on G
characters .... characters on G
representations of H ...... representations of G.

Let χ be a class function on $H \le G$ i.e. $\chi : H \to \mathbb{C}$, $\chi(xhx^{-1}) = \chi(h)$ for all $x, h \in H$.
To get a class function on G, start with the trivial extension

$$\hat{\chi} : G \longrightarrow \mathbb{C}$$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H; \\ 0 & \text{if } g \notin H. \end{cases}$$

To make this into a class function, use an averaging over conjugates as we did before. This leads to $\chi^G : G \longrightarrow \mathbb{C}$:

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1}) \qquad (\chi^G \text{ is } \underline{\text{induced}} \text{ from } \chi, \quad \chi^G = \mathrm{Ind}_H^G \chi)$$

$w^{-1} = u^{-1}x^{-1}$
$w = xu$
$wu = x$

If $u \in G$ then $\chi^G(ugu^{-1}) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xugu^{-1}x^{-1}) = \frac{1}{|H|} \sum_{w \in G} \hat{\chi}(wgw^{-1})$
$= \chi^G(g)$.

So χ^G is a class function on G.

Note: Let T be a set of right coset representatives for H in G.
So every element $g \in G$ is uniquely expressible as $g = ht$, $h \in H$, $t \in T$
$|G| = |H||T|$ (Lagrange's Theorem), $|T| = \frac{|G|}{|H|} = [G:H]$.

$\chi: H \to \mathbb{C}$ is a class function on H; T is a right transversal for H in G (set of right coset representatives)

$T = \{t_1, \dots, t_\ell\}$

$G = Ht_1 \sqcup Ht_2 \sqcup \cdots \sqcup Ht_\ell$

$= \bigsqcup_{t\in T} Ht.$

$$\chi^G(g) = \frac{1}{|H|}\sum_{x\in G}\hat{\chi}(xgx^{-1})$$

$x = ht$, $h\in H$, $t\in T$

$$= \frac{1}{|H|}\sum_{t\in T}\sum_{h\in H}\hat{\chi}(htgt^{-1}h^{-1})$$

$htgt^{-1}h^{-1}\in H \iff tgt^{-1}\in H$

$$= \frac{1}{|H|}\sum_{t\in T}\underbrace{\sum_{h\in H}\hat{\chi}(tgt^{-1})}_{|H| \text{ terms equal to } \hat{\chi}(tgt^{-1})}$$

$$= \frac{1}{|H|}\sum_{t\in T}|H|\,\hat{\chi}(tgt^{-1}) = \sum_{t\in T}\hat{\chi}(tgt^{-1}) = \sum_{i=1}^{\ell}\hat{\chi}(t_i g t_i^{-1})$$

Special case: $\chi = \chi_1 =$ trivial (principal) character of H, $\chi(h) = 1$.

χ^G isn't the principal character of G unless $G = H$.

G permutes the right cosets of H by right multiplication giving a permutation representation $g\in G$ permutes $Ht_i \longmapsto Ht_i g = Ht_j$

$\chi^G = (1_H)^G$ is the perm. character of G acting on cosets of H.

$G \longrightarrow S_\ell$ $\quad \ell = [G:H] = |T|$.

Eg. Construct the character table of $G = S_4$ making use of the character table of $S_3 = H \le G$.

H:

$\lvert K_H(g)\rvert$	6	2	3
g	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$G = S_4$:

$\lvert C_G(g)\rvert$	24	4	8	3	4
g	$()$	(12)	$(12)(34)$	(123)	(1234)
ψ_1	1	1	1	1	1
ψ_2	1	-1	1	1	-1
ψ_3	2	0	2	-1	0
ψ_4	3				
ψ_5	3				
ψ	3	1	3	0	1

Check: $[\psi_3, \psi_3] = 1$

$[\psi_1, \psi_2] = 0$

$\underbrace{\psi(1)}_{3} = \underbrace{\psi_1(1)}_{1} + \underbrace{\psi_3(1)}_{2}$

$\psi = \sum_{i=1}^{5} a_i \psi_i$

$[\psi, \psi]_G = \frac{9}{24} + \frac{1}{4} + \frac{9}{8} + \frac{0}{3} + \frac{1}{4}$

$= \frac{3+2+9+0+2}{8} = \frac{16}{8}$

$= 2$

$= \sum_{i=1}^{5} a_i^2 = 1^2 + 1^2 + 0^2 + 0^2 + 0^2$

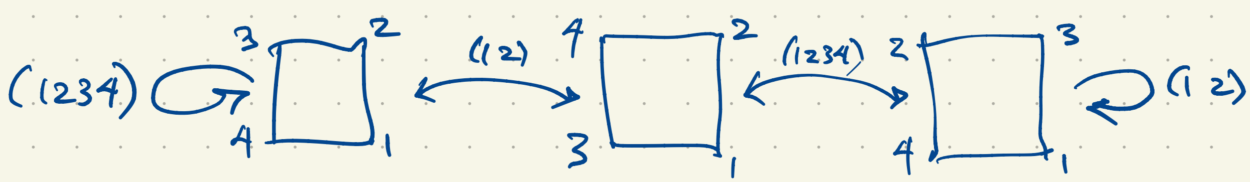
$\psi = \psi_1 + \psi_3$
(degree 1) (degree 2)

$a_1 = [\psi, \psi_1] = \frac{3}{24} + \frac{1}{4} + \frac{3}{8} + \frac{0}{3} + \frac{1}{4} = \frac{1+2+3+0+2}{8} = 1$

S_4 has $k=5$ conjugacy classes
... irreducible representations/characters of degree $n_1, n_2, \dots, n_5 \ge 1$, $n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = |G|$ $= 24$

S_4 has normal subgroups
1, S_4, A_4, $K = \langle (12)(34), (13)(24) \rangle = \{ (), (12)(34), (13)(24), (14)(23) \}$.

S_4 permutes the conjugacy class of $(12)(34)$ in all $3! = 6$ possible ways
There is a permutation action $S_4 \longrightarrow \mathrm{Sym}\{(12)(34), (13)(24), (14)(23)\} \cong S_3$
with kernel K of order 4.



$C_G((13)(24))$ $C_G((14)(23))$ $C_G((12)(34))$

This gives a permutation representation of S_4 of degree 3.
Its permutation character is
$\psi((1)) = 3$
$\psi(k) = 3$ for $k \in K$
$\psi((12)) = 1$
$\psi((1234)) = 1$ $\psi((123)) = 0$ $\psi((12)(34)) = 3$

$H = S_3$:

$\lvert C_H(g)\rvert$	6	2	3
g	$()$	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$\chi = \chi_2$

$G = S_4$:

$\lvert C_G(g)\rvert$	24	4	8	3	4
g	$()$	(12)	$(12)(34)$	(123)	(1234)
ψ_1	1	1	1	1	1
ψ_2	1	-1	1	1	-1
ψ_3	2	0	2	-1	0
ψ_4	3	-1	-1	0	1
ψ_5	3	1	-1	0	-1
χ^G	4	-2	0	1	0

$$[\chi^G, \chi^G]_G = \frac{16}{24} + \frac{4}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 2$$

$\Rightarrow \chi^G$ has irreducible constituents of multiplicity 1, 1, 0, 0, 0

$$[\chi^G, \psi_1] = \frac{4}{24} - \frac{2}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 0$$

$$[\chi^G, \psi_2] = \frac{4}{24} + \frac{2}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 1$$

$T = \langle (1234) \rangle = \{(), (1234), (13)(24), (1432)\}$ right transversal for H in G

$\chi^G = \chi_2^G$ is a character on G

$$\chi^G(g) = \sum_{t \in T} \hat{\chi}(tgt^{-1})$$

where $\hat{\chi} : G \longrightarrow \mathbb{C}$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

$$\chi^G(()) = 1 + 1 + 1 + 1 = 4$$

$$\chi^G((12)) = \hat{\chi}((12)) + \hat{\chi}((23)) + \underbrace{\cancel{\hat{\chi}((34))}}_{0} + \underbrace{\cancel{\hat{\chi}((14))}}_{0}$$

$$= -1 - 1 + 0 + 0 = -2$$

$$\chi^G((12)(34)) = \hat{\chi}((12)(34)) + \hat{\chi}((23)(41)) + \hat{\chi}((34)(12)) + \hat{\chi}((41)(23)) = 0 + 0 + 0 + 0 = 0$$

$$\chi^G((123)) = \hat{\chi}((123)) + \cancel{\hat{\chi}((234))} + \cancel{\hat{\chi}((341))} + \cancel{\hat{\chi}((412))}$$

$$= 1 + 0 + 0 + 0 = 1$$

$$\chi^G = \psi_2 + \psi_4$$

$$\psi_4 = \chi^G - \psi_2$$

$$[\psi_4, \psi_4]_G = \frac{9}{24} + \frac{1}{4} + \frac{1}{8} + \frac{0}{3} + \frac{1}{4} = \frac{3 + 2 + 1 + 0 + 2}{8} = 1$$

Frobenius Reciprocity Let χ be a class function on $H \leq G$ and let ψ be a class function on G. Then

$$[\psi_H, \chi]_H = [\psi, \chi^G]_G$$

$\psi_H = \psi|_H$

$$\chi^G(g) = \frac{1}{|H|}\sum_{x\in G}\hat{\chi}(xgx^{-1}) \qquad \hat{\chi}(g) = \begin{cases}\chi(g) & \text{if } g\in H\\ 0 & \text{if } g\notin H\end{cases}$$

Proof

$$[\chi^G, \psi]_G = \frac{1}{|G|}\sum_{g\in G}\chi^G(g)\overline{\psi(g)}$$

$$= \frac{1}{|G|}\sum_{g\in G}\sum_{x\in G}\frac{1}{|H|}\hat{\chi}(x^{-1}gx)\overline{\psi(g)}$$

$x\in G = HT$

$x = ht$, $h\in H$, $t\in T$

$|G| = |H||T|$

$$= \frac{1}{|G||H|}\sum_{g\in G}\sum_{x\in G}\hat{\chi}(xgx^{-1})\overline{\psi(g)}$$

$htgt^{-1}h^{-1}\in H \iff tgt^{-1}\in H$

$$= \frac{1}{|G||H|}\sum_{g\in G}\sum_{h\in H}\sum_{t\in T}\hat{\chi}(htgt^{-1}h^{-1})\overline{\psi(g)}$$

$$= \frac{1}{|G||H|}\sum_{g\in G}\sum_{h\in H}\sum_{t\in T}\underbrace{\hat{\chi}(tgt^{-1})\overline{\psi(g)}}_{|H| \text{ identical terms } \cancel{\text{for}}\ h\in H}$$

$$= \frac{1}{|G|}\sum_{g\in G}\sum_{t\in T}\hat{\chi}(tgt^{-1})\overline{\psi(g)}$$

$$= \frac{1}{|G|}\sum_{t\in T}\left(\sum_{g\in G}\hat{\chi}(tgt^{-1})\overline{\psi(g)}\right)$$

For each $t\in T$ reparameterize the inner sum, $u = tgt^{-1}$, $g = t^{-1}ut$

$$= \frac{1}{|G|}\sum_{t\in T}\left(\sum_{u\in G}\hat{\chi}(u)\psi(t^{-1}ut)\right)$$

$$= \frac{1}{|G|}\sum_{t\in T}\sum_{h\in H}\chi(h)\overline{\psi(t^{-1}ht)} = \frac{1}{|G|}\sum_{t\in T}\sum_{h\in H}\chi(h)\overline{\psi(h)}$$

$$= \frac{1}{|G|}|T|\sum_{h\in H}\chi(h)\overline{\psi(h)}$$

$$= \frac{1}{|H|}\sum_{h\in H}\chi(h)\overline{\psi_H(h)}$$

$$= [\chi, \psi_H]_H. \qquad \square$$

If ψ is a class function on G then so is $\overline{\psi}$ where $\overline{\psi}(g) = \overline{\psi(g)} = \psi(g^{-1})$.

Indeed if ψ is a character, $\psi(g) = \operatorname{tr} \pi(g)$, $\underset{\text{homomorphism}}{\pi : G \to GL_n(\mathbb{C})}$

$\pi(g) \underset{\text{similar to}}{\sim} \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ (actually $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ in the case of finite groups)

$\lambda_1, \dots, \lambda_n$ are m^{th} roots of unity where m = exponent of $G = \operatorname{lcm}(|g| : g \in G)$

$\overline{\lambda_i} = \lambda_i^{-1}$ since $\lambda_i^m = 1$, $\quad \overline{\lambda_i}\lambda_i = |\lambda_i|^2 = 1$

$|\lambda_i|^m = 1$

$|\lambda_i| = 1$

$\psi(g) = \sum \lambda_i$

$\psi(g^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda_i} = \overline{\psi(g)}$

For every finite group G, the irreducible representations of G can be chosen to be <u>unitary</u>.

$U_n(\mathbb{C}) = \{A \in \mathbb{C}^{n \times n} : AA^* = A^*A = I\}$, $\quad A^* = \overline{A}^T = \overline{A^T}$.

eg. for S_3, we have an irreducible representation of degree 2.

$$\pi_3 : S_3 \to GL_2(\mathbb{C}) \qquad \chi(g)$$

$$\begin{array}{lll}
() \longmapsto & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 2 \\
(12) \longmapsto & \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} & 0 \\
(123) \longmapsto & \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} & -1 \\
(132) \longmapsto & \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} & -1 \\
(13) \longmapsto & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0 \\
(23) \longmapsto & \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} & 0
\end{array}$$

This representation uses integer matrix entries.

left-to-right composition

$$(13) = (123)(12)(132)$$

Character values:

$$\chi(g) = \operatorname{tr} \pi(g)$$

A change of basis from $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ takes one representation to the other.

A different choice of basis for $\mathbb{C}^2$ yields an equivalent representation using unitary matrices:

$$\begin{array}{lll}
 & & \chi(g) \\
() \longmapsto & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 2 \\
(123) \longmapsto & \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix}, & -1 \\
(132) \longmapsto & \begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix} & -1 \\
(12) \longmapsto & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0 \\
(13) \longmapsto & \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ \bar{\omega} & 0 \end{bmatrix}\begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} 0 & \bar{\omega} \\ \omega & 0 \end{bmatrix} & 0 \\
(23) \longmapsto & \begin{bmatrix} 0 & \omega \\ \bar{\omega} & 0 \end{bmatrix} & 0
\end{array}$$

$$\omega = e^{2\pi i/3}$$

$$\bar{\omega} = \omega^2 = \omega^{-1}$$

root of $x^2 + x + 1$

$$\omega^2 + \omega + 1 = 0$$

$$\bar{\omega} + \omega = \omega^2 + \omega = -1$$

To prove the claim (that every finite group has its irred. repns. equivalent to equivalent to unitary representations) we use a fact from linear algebra: any inner product on $\mathbb{C}^n$ is equivalent (under change of basis) to standard inner product $B(x,y) = \sum x_i \overline{y_i}$.

Any inner product has the ~~form~~ $(x,y) \longmapsto xMy^*$ where $x=(x_1,\dots,x_n)$, $y^* = \begin{pmatrix} \overline{y_1} \\ \vdots \\ \overline{y_n} \end{pmatrix}$

$M^* = M$ $n\times n$ matrix Hermitian, $\det M \neq 0$.

If we perform an invertible change of variables $x \longmapsto xA$ (A invertible $n\times n$ matrix i.e. $A \in GL_n(\mathbb{C})$)

then $B(xA, yA) = (xA)M(yA)^* = x\underbrace{(AMA^*)}_{\text{congruent to } M}y^*$

For all nonsingular Hermitian M, we can find $A \in GL_n(\mathbb{C})$ such that $AMA^* = I$ (using Gram-Schmidt) i.e. any two inner products on $\mathbb{C}^n$ are equivalent by change of basis.

Given a representation $\pi : G \longrightarrow GL_n(\mathbb{C})$, we will find an inner product $B(x,y)$ on $\mathbb{C}^n$ such that all matrices $\pi(g)$, $g \in G$ preserve the inner product:

✓ $B(x,x) \geq 0$, equality iff $x=0$;

✓ $B(x,y)$ linear in x, conjugate linear in y; $B(y,x) = \overline{B(x,y)}$;

✓ $B(x\pi(g), y\pi(g)) = B(x,y)$.

To obtain the inner product $B(x,y)$:

$$B(x,y) = \sum_{g\in G} (x\pi(g))(y\pi(g))^* = \sum_{g\in G} x\pi(g)\pi(g)^* y^* = xMy^*$$

$$M = \sum_{g\in G} \pi(g)\pi(g)^* \quad \text{is Hermitian.}$$

$$B(x,x) = \sum_{g\in G}(x\pi(g))(x\pi(g))^* = \sum_{g\in G} \|x\pi(g)\|^2 \geq 0$$

$B(x,x) > 0$ unless $x=0$.

$$B(x\pi(g), y\pi(g)) = \sum_{u\in G}(x\underbrace{\pi(g)\pi(u)}_{\pi(w)})(y\underbrace{\pi(g)\pi(u)}_{\pi(w)})^* = \sum_{w\in G}(x\pi(w))(y\pi(w))^* = B(x,y)$$

$w = gu$

Suppose $H \leq G$; let T be a right transversal for H in G i.e. a set of right coset representatives. Every $g\in G$ can be uniquely factored as $g = ht$, $h\in H$, $t\in T$. (Lagrange's theorem) $|G| = |H||T|$

G permutes the right cosets of H by right-multiplication:

$$Ht \longmapsto Htg = Ht' \quad \text{for some } t'\in T.$$

This gives a permutation representation of G acting on the right cosets of H. If $d = |T| = [G:H]$ then we have a homomorphism $\pi: G \longrightarrow S_d \subset GL_d(\mathbb{C})$, with perm. character $\psi(g) = \operatorname{tr}\pi(g) = $ no. of fixed points of $\pi(g)$

Theorem $\psi = (1_H)^G$ i.e. the perm. character is the induced character obtained from 1_H = principal character, $1_H(h) = 1$ for all $h \in H$, induced up to G.

Proof $\hat{1}_H(g) = \begin{cases} 1, & \text{if } g \in H; \\ 0, & \text{if } g \in G, g \notin H. \end{cases}$

$$(1_H)^G(g) = \sum_{t \in T} \hat{1}_H(tgt^{-1}) = \text{number of } tgt^{-1} \ (t \in T) \text{ which are in } H$$

$$= |\{t \in T : tgt^{-1} \in H\}|$$

$tgt^{-1} \in H \iff Htgt^{-1} = H \iff Hg = Ht$
$\iff$ the coset Ht is fixed under right-multiplication by $g \in G$.

$$= \psi(g). \qquad \square$$

From the character table of G, we can see what all normal subgroups of G are.

$G = S_4$:

$\lvert C_G(g)\rvert$	24	4	8	3	4
g	$()$	(12)	$(12)(34)$	(123)	(1234)
ψ_1	1	1	1	1	1
ψ_2	1	-1	1	1	-1
ψ_3	2	0	2	-1	0
ψ_4	3	-1	-1	0	1
ψ_5	3	1	-1	0	-1

$\psi_2 : G \to \mathbb{C}^\times$ is a homomorphism

since $\psi_2(g) = \operatorname{tr} \pi_2(g)$ of degree $\psi_2(1) = 1$.

Its kernel is $\{g \in G : \psi_2(g) = 1\}$

$= \{$elements conjugate to $()$, $(12)(34)$ or $(123)\}$

$= A_4$.

Another normal subgp: all elements $g \in G$ s.t. $\psi_3(g) = \psi_3(1) = 2$

<u>Theorem</u> Let $\pi : G \to GL_n(\mathbb{C})$ be a representation with character $\chi(g) = \operatorname{tr} \pi(g)$. (Its degree is $n = \chi(1)$.) The kernel of χ defined by

$$\ker \chi = \ker \pi = \{g \in G : \pi(g) = I\} = \{g \in G : \chi(g) = \chi(1) = n\}$$

is a normal subgroup of G.

Note: π is a homomorphism; but χ is not a homomorphism unless $n=1$.

<u>Proof</u> If $g \in G$ then $\pi(g)$ is similar to $\begin{bmatrix} \varepsilon_1 & & 0 \\ & \varepsilon_2 & \\ 0 & & \ddots \ \varepsilon_n \end{bmatrix}$, where $\boxed{\varepsilon_i^m} = 1$ for all $i = 1, 2, \dots, n$.

$$\begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1}\alpha \\ 0 & \lambda^k \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

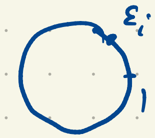
$\pi(g)$ is a root of $x^m - 1$

m is the <u>exponent of G</u> i.e. $g^m = 1$ for every $g \in G$

$\pi(g) \in GL_n(\mathbb{C})$ has order dividing m.

$\Rightarrow \chi(g) = \operatorname{tr} \pi(g) = \varepsilon_1 + \cdots + \varepsilon_n = n \iff \varepsilon_1 = \cdots = \varepsilon_n = 1 \iff \pi(g) = I$

$|\varepsilon_i| = 1 \Rightarrow \operatorname{Re} \varepsilon_i \leq 1$; equality iff $\varepsilon_i = 1$.



For $G = S_4$, there are four normal subgroups and they all arise as $\ker \chi$ for some χ.

$\ker \chi_5 = \{()\}$

$\ker \chi_3 = \{(),\ (12)(34),\ (13)(24),\ (14)(23)\}$

$\ker \chi_2 = A_4$

$\ker \chi_1 = S_4$.

Recall: Let $H \leq G$. Then H is normal in G ($H \trianglelefteq G$) iff H is a union of conjugacy classes, iff H is an intersection of kernels of irreducible characters.

In this way we "read off" all the normal subgroups of G from the char. table.

eg. $G = \{\pm 1\} \times \{\pm 1\} = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$ Klein

Char table

$\lvert C_G(g)\rvert$	4	4	4	4
g	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

Application to Frobenius groups.

Suppose $G \le S_n$ (G permutes $\{1,2,\cdots,n\}$). More generally, G permutes any set X, $G \le \mathrm{Sym}(X) = \{\text{permutations of } X\}$ X: any set. eg. $X=\{1,2,\cdots,n\}$

G is <u>transitive</u> if for all $x,y \in X$, there exists $g \in G$ mapping $x \longmapsto y$.

The <u>stabilizer</u> of a point $x \in X$ is $G_x = \{g \in G : g(x) = x\}$.

Of course $G_x \le G$.

The <u>orbit</u> of $x \in X$ is $x^G = \{g(x) : g \in G\} \subseteq X$.

or $G(x)$

$|x^G| = [G : G_x] = \dfrac{|G|}{|G_x|}$ (orbit-stabilizer formula).

$x^G = X$ iff G is transitive.

G is a <u>Frobenius group</u> if

(i) G is transitive

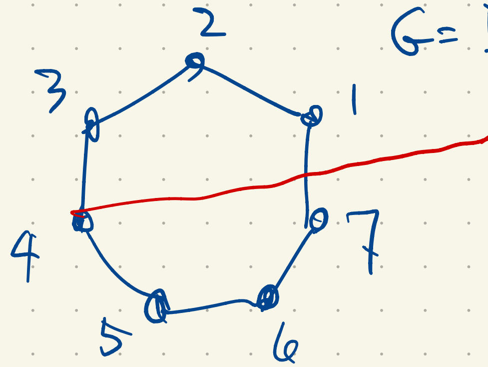
(ii) Point stabilizer is nontrivial $\left(|G_x| = \dfrac{|G|}{|X|} > 1 \right)$

(iii) The stabilizer of any two points is trivial ie. $G_x \cap G_y = 1$ for all $x \ne y$ in X.

If m is an odd positive integer then the symmetry group of a regular m-gon (i.e. the dihedral group of order $2m$) is a subgroup of S_m having

m rotations,
m reflections.

eg. $m = 7$



$G = D_7 = \langle (1234567), \ (17)(26)(35) \rangle$

$G_4 = \langle (17)(26)(35) \rangle$

$()$ = identity in G fixes 7 points;
nontrivial rotations fix 0 〃 ;
other elements fix 1 point.

This is a Frobenius group.

There are exactly two groups of order 21: the cyclic group, and a Frobenius group $\langle (1234567), (124)(365) \rangle < S_7$ transitive.

$\downarrow$ conjugate by $(124)(365)$

$$(2461357) = (1357246) = (1234567)^2.$$

Eg. G = direct isometries of $\mathbb{R}^2$ = {translations} $\cup$ {rotations}
(orientation-preserving)
is a Frobenius group. There are lots of finite analogues of this example.

e.g. $F = \mathbb{F}_{11} = \{\text{integers mod } 11\}$.

The affine general linear group on F^2 is a subgroup of S_{121} consisting of transformations $v \longmapsto Av+b$, $A \in GL_2(F)$, $b \in F^2$.

$GL_2(F) = \{\text{invertible linear transformations on } F^2\}$

$|GL_2(F)| = (11^2-1)(11^2-11) = 120 \cdot 110 = 13200$

$|AGL_2(F)| = 11^2 \cdot 13200$ transitive on F^2.

Stabilizer of $0 = \begin{bmatrix}0\\0\end{bmatrix} \in F^2$ is $GL_2(F)$ of order 13200.

This group $AGL_2(F)$ is not a Frobenius group e.g.

$v \longmapsto \begin{bmatrix}1&1\\0&3\end{bmatrix} v + \begin{bmatrix}0\\0\end{bmatrix}$ fixes all eleven vectors $\begin{bmatrix}a\\0\end{bmatrix} \longmapsto \begin{bmatrix}1&1\\0&3\end{bmatrix}\begin{bmatrix}a\\0\end{bmatrix} = \begin{bmatrix}a\\0\end{bmatrix}$

The two distinct points $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\end{bmatrix}$ are fixed by more than just identity

in fact the subgroup $\begin{bmatrix}1&b\\0&c\end{bmatrix}$ $b,c \in F$, $c \neq 0$ of order 110

Modify the example: $GL_2(F)$ has a subgroup isomorphic to $SL_2(\mathbb{F}_5)$ (order 120), sharply ~~transitive~~ transitive on the 120 nonzero vectors

The affine linear ~~transformations~~ transformations on F^2 of the form $v \longmapsto Av+b$, $A \in$ subgp of $GL_2(\mathbb{F}_{11})$ isomorphic to $SL_2(\mathbb{F}_5)$, $b \in \mathbb{F}_{11}^2$

forms a Frobenius group of order $121 \cdot 120$. Actually this example is sharply 2-transitive

On $\mathbb{R}^2$, the transformations $\mathbb{R}^2 \to \mathbb{R}^2$ which are direct similarities is a Frob. gp., sharply 2-trans.