

Math 5555

Abstract Algebra II

Book 1

In group theory we have

• permutation representations: homomorphism $\pi: G \rightarrow S_n$ permutation representation of degree n
 (if π is 1-to-1 then π is a faithful representation; then $\pi(G) \leq S_n$)

• linear representations: homomorphism $\pi: G \rightarrow GL_n(F)$ F : field
 linear representation of degree n over F
 If $F = \mathbb{C}$ (or \mathbb{R} or ...) then π is an ordinary representation.

If $\text{char } F = p$ (prime) then π is a modular representation.

G : unless otherwise specified, G finite group. (Until later...)

Usually $F = \mathbb{C}$ (or \mathbb{R}) and $|G| < \infty$

If $\pi_i: G \rightarrow GL_{n_i}(\mathbb{C})$ ($i=1,2$) then $\pi_1 \oplus \pi_2: G \rightarrow GL_{n_1+n_2}(\mathbb{C})$, $g \mapsto \left[\begin{array}{c|c} \pi_1(g) & 0 \\ \hline 0 & \pi_2(g) \end{array} \right]$ is a representation of degree n_1+n_2

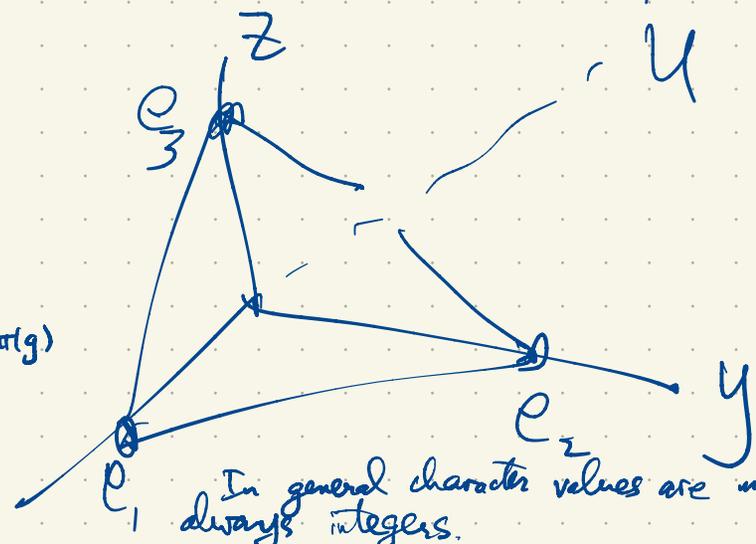
$\pi: G \rightarrow GL_n(\mathbb{C})$ is decomposable if there is a decomposition $\mathbb{C}^n = U \oplus V$ such that $U, V \neq 0$
 ($\dim U = n_1$, $\dim V = n_2$, n_1, n_2 positive integers, $n_1+n_2=n$)
 U, V invariant under all matrices $\pi(g)$, $g \in G$.

$G = S_3$ acting naturally on \mathbb{C}^3 by permuting the standard basis vectors $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

via $\sigma: e_i \mapsto e_{\sigma(i)}$
 i.e. $(12) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
 $(123) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$
 $(1) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

This is a faithful representation of degree 3.

It is decomposable: $\mathbb{C}^3 = U \oplus V$, $U = \langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rangle$, $V = U^\perp = \langle \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \rangle$
 indecomposable.



The representation $\pi: S_3 \rightarrow GL_3(\mathbb{C})$ has (affords) character

$$\chi(g) = \text{tr } \pi(g)$$

$$\chi(1) = 3 = \text{degree of } \pi = \text{deg } \chi$$

$$\chi((12)) = \chi((13)) = \chi((23)) = 1$$

$$\chi((123)) = \chi((132)) = 0$$

For a permutation representation $\pi: G \rightarrow S_n \subset GL_n(\mathbb{C})$ the associated character $\chi(g) = \text{tr } \pi(g)$ (called the permutation character) is $\chi(g) = \text{no. of fixed points of } g$
 $= |\{i: g(i) = i\}|$, $1 \leq i \leq n$

In general character values are not always integers.

Given a representation $\pi: G \rightarrow GL_n(\mathbb{C})$, the character of π is

$$\chi(g) = \text{tr } \pi(g) \in \mathbb{C}$$

Character values $\chi(g) \in \mathbb{C}$ are always algebraic integers ($g \in G$, $|G| < \infty$) and character values of S_n are ordinary integers.

$\chi(g)$ depends only on the conjugacy class of g .

If $g, h \in G$ then $g \sim h'gh$ (conjugate in G) so

$$\pi(h'gh) = \pi(h')\pi(g)\pi(h) \sim \pi(g) \text{ (similar in } GL_n(\mathbb{C}) \text{ i.e. conjugate)}$$

\uparrow
inverses in $GL_n(\mathbb{C})$

$$\text{so } \text{tr } \pi(g) = \text{tr } (\pi(h')\pi(g)\pi(h)) = \text{tr } \pi(h'gh)$$

$$\text{tr } (AB) = \text{tr } (BA)$$

$$\text{tr } (B^{-1}AB) = \text{tr } (AB \cdot B^{-1}) = \text{tr } A$$

$$\chi(h'gh) = \chi(g)$$

If $\pi: G \rightarrow GL_n(\mathbb{C})$ is any representation i.e. homomorphism, and $B \in GL_n(\mathbb{C})$, then

$\tilde{\pi}(g) = B^{-1}\pi(g)B \in GL_n(\mathbb{C})$ is also a representation

$\pi, \tilde{\pi}$ are equivalent (via a change of basis).

$$\tilde{\pi}(gh) = B^{-1}\pi(gh)B = B^{-1}\pi(g)\pi(h)B = B^{-1}\pi(g)B \cdot B^{-1}\pi(h)B = \tilde{\pi}(g)\tilde{\pi}(h)$$

They have the same character: the character of $\tilde{\pi}$ is

$$\tilde{\chi}(g) = \text{tr } \tilde{\pi}(g) = \text{tr } (B^{-1}\pi(g)B) = \text{tr } \pi(g) = \chi(g)$$

It's not obvious but the converse is true: χ determines π up to equivalence. Two representations have the same character iff they are equivalent.

Let $\pi: G \rightarrow GL_n(\mathbb{C})$ be a representation (i.e. homomorphism).

π is reducible if there exists nontrivial ^{proper} subspace $U \subset \mathbb{C}^n$ ($\dim U \in \{1, 2, \dots, n-1\}$) such that U is invariant under $\pi(g)$ for all $g \in G$ i.e. $\pi(g)U \subseteq U$ for all $g \in G$.

$$\text{i.e. } \pi(g) = \left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array} \right] \left. \begin{array}{l} \left. \vphantom{\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}} \right\}^k \\ \left. \vphantom{\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}} \right\}^{n-k} \end{array} \right\} \text{for all } g \in G$$

when $|G| < \infty$ and $\pi: G \rightarrow GL_n(\mathbb{C})$
 $\text{char } F = 0$, π reducible $\Leftrightarrow \pi$ decomposable
 π irreducible $\Leftrightarrow \pi$ indecomposable

In general, π decomposable $\Rightarrow \pi$ reducible
 $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \Rightarrow \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$
 π indecomposable $\Leftarrow \pi$ irreducible.

The representation $\pi: \mathbb{R} \rightarrow GL_2(\mathbb{R})$, $\pi(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, $\pi(a+b) = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \pi(a)\pi(b)$
additive group

is reducible: $\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$ is an invariant subspace. There is no complementary invariant subspace (in particular the complementary subspace $\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle$ is not invariant).

Maschke's Theorem Let G be a finite group and $F = \mathbb{C}$ (or more generally F is any field of characteristic not dividing $|G|$) then any representation $\pi: G \rightarrow GL_n(F)$ is reducible iff it's decomposable (i.e. π is irreducible iff π is indecomposable; i.e. whenever every invariant subspace $U \leq F^n$ has a complementary subspace U' which is also invariant).

$U \leq V$ has a complementary subspace $U' \iff V = U \oplus U'$. If $\dim V = n$ and $\dim U = k$ then U' is a complement to U iff $\dim U' = n-k$ and $U + U' = V$
 iff $\dim U' = n-k$ and $U \cap U' = \{0\}$.

In this case every $v \in V$ is uniquely represented as $v = u + u'$, $u \in U$, $u' \in U'$.

In this, the projection $V \rightarrow U$ along U' is the map $P: v \mapsto Pv = u$. $P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \} k \\ \} n-k \end{matrix}$
 P has image $PV = U$; $\ker P = U'$. Note: $I-P$ is the projection from V onto U' along U .

A linear transformation $P: V \rightarrow V$ is a projection iff $P^2 = P$. In this case P is a root of $x^2 - x$ so the eigenvalues of P are in $\{0, 1\}$. Take $U = 1$ -eigenspace of $P = \ker(I-P)$, $U' = \ker P$.

Proof of Maschke's Theorem: (in the nontrivial direction)

Suppose $\pi: G \rightarrow GL(\mathbb{C})$ is a representation having an invariant subspace $U \subseteq \mathbb{C}^n$ i.e. $\pi(g)U \subseteq U$ for all $g \in G$. We want to find a complementary subspace $W \subseteq \mathbb{C}^n$ which is also invariant. Start with any complementary subspace U' where $\mathbb{C}^n = U \oplus U'$, $\dim U = k$, $\dim U' = n-k$; every $v \in \mathbb{C}^n$ is uniquely expressible as $v = u + u'$, $u \in U$, $u' \in U'$.

Unfortunately U' is not invariant in general. Let $P: \mathbb{C}^n \rightarrow U$ be the projection onto U along U' i.e. $P(v) = P(u + u') = u$, so $U = PV$, $U' = \ker P$. Consider the new map $\tilde{P}: V \rightarrow V$ defined by

$$\tilde{P} = \frac{1}{|G|} \sum_{g \in G} \pi(g) P \pi(g) \quad \tilde{P}V \subseteq U \quad \text{Since } \tilde{P}v = \frac{1}{|G|} \sum_g \underbrace{\pi(g^{-1}) P \pi(g)}_{\substack{\in V \\ \in U}} v \in U.$$

For all $u \in U$, $\tilde{P}u = u$ i.e. $\tilde{P}|_U = \text{id}|_U$. Why?

$$\tilde{P}u = \frac{1}{|G|} \sum_g \pi(g^{-1}) P \pi(g) u = \frac{1}{|G|} \sum_g \pi(g^{-1}) \pi(g) u = \frac{1}{|G|} |G| u = u.$$

= $\pi(g)u$ since $\pi(g)u \in U$.

Next: show \tilde{P} commutes with all $\pi(g)$, $g \in G$. (P doesn't satisfy this in general!)

$$\begin{aligned} \text{Then } \tilde{P}\pi(g) &= \frac{1}{|G|} \sum_{h \in G} \pi(h^{-1}) P \pi(h) \pi(g) = \frac{1}{|G|} \sum_{h \in G} \pi(h^{-1}) P \pi(hg) \\ &= \frac{1}{|G|} \sum_{x \in G} \underbrace{\pi(g^{-1}x^{-1})}_{\pi(g^{-1})\pi(x^{-1})} P \pi(x) = \pi(g) \cdot \frac{1}{|G|} \sum_{x \in G} \pi(x^{-1}) P \pi(x) = \pi(g) \tilde{P}. \end{aligned}$$

$$h \leftrightarrow \begin{aligned} x &= hg \\ x^{-1} &= g^{-1}h^{-1} \\ g x^{-1} &= h^{-1} \end{aligned}$$

Next: show $\tilde{P}^2 = \tilde{P}$. If $v \in V$ then $\tilde{P}v \in U$ so $\tilde{P}^2v = \tilde{P}\tilde{P}v = \tilde{P}v$. So \tilde{P} is idempotent so

\tilde{P} is a projection on $\tilde{P}V = U$ along $W = \ker \tilde{P}$. Note: if $k = \dim U = \text{rank } \tilde{P} = \text{tr } \tilde{P}$, $\dim W = \dim \ker \tilde{P} = n - k$. W is also invariant under $\pi(g)$: if $v \in W$ then $\tilde{P}v = 0$

so $\pi(g)\tilde{P}v = \pi(g)0 = 0$ so $\pi(g)v \in W$

$\tilde{P}\pi(g)v$



The irreducible representations of S_3 are

$$\pi_1(g) = [1] \in GL_1(\mathbb{C}) \quad \text{trivial representation}$$

$$\pi_2(g) = [\text{sgn}(g)] \in GL_1(\mathbb{C}) \quad \text{sign representation:}$$

$$= [\pm 1] \text{ according as } g \text{ is an even}$$

$$\text{or odd permutation.}$$

$$\text{sgn}((1)) = 1$$

$$\text{sgn}((123)) = 1$$

$$\text{sgn}((132)) = 1$$

$$\text{sgn}((12)) = \text{sgn}((13)) = \text{sgn}((23)) = -1$$

$$\text{sgn}(gh) = \text{sgn}(g) \text{sgn}(h)$$

$$\pi_3: S_3 \rightarrow GL_2(\mathbb{C})$$

$$(1) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(12) \mapsto \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$$

$$(123) \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$(132) \mapsto \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$(13) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(23) \mapsto \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\pi_3((13)) = \pi_3((123)(12)) = \pi_3((123)) \pi_3((12))$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\pi_3((23)) = \pi_3((123)(13)) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

An irreducible character is a map $G \rightarrow \mathbb{C}$
 where π is an irreducible representation $g \mapsto \text{tr} \pi(g)$.

0 , $\langle [1] \rangle$, $\langle [-1] \rangle$, \mathbb{C}^2 are the only subspaces invariant under $\pi_3((13))$; but the 1-dimensional invariant subspaces are not invariant under $\pi_3((123))$.

$$\pi_3((123)) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \text{ has char. poly. } x^2 + x + 1$$

$$\pi_3((13)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ has char. poly. } x^2 - 1$$

↑ eigenspaces $\langle [1] \rangle$, $\langle [-1] \rangle$
 (eigenvalues 1, -1 respectively)

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

the character table of S_3 is

	(1)	(12)	(23)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Given any finite group G , a class function on G is a function $f: G \rightarrow \mathbb{C}$ which only depends on conjugacy class, i.e. $f(g') = f(g)$ whenever $g, g' \in G$ are conjugate i.e. $g' = ugu$ for some $u \in G$.

All characters of G (irreducible or otherwise) are class functions.

$V = \{ \text{class functions on } G \}$ is a complex vector space

$$f, f' \in V \Rightarrow f + f' \in V, \quad (f + f')(x) = f(x) + f'(x)$$

V is in fact a complex inner product space:

$$[f, f']_G = [f, f'] = \frac{1}{|G|} \sum_{x \in G} f(x) \overline{f'(x)} \quad [f', f] = \overline{[f, f']}$$

sesquilinear
1/2-linear

$\dim V = \text{number of conjugacy classes}$

The irreducible characters of G always give an orthonormal basis for $V = \{ \text{class functions on } G \}$.

Recall the permutation representation with permutation character

$$\pi: S_3 \rightarrow GL_3(\mathbb{C})$$

$$\pi: e_i \mapsto e_{\pi(i)}$$

$$(12) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(123) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$(1) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\chi(1) = 3$$

$$\chi((12)) = 1$$

$$\chi((123)) = 0$$

$$[\chi, \chi_1] = \frac{1}{6}(1 \cdot 3 + 1 + 1 + 1 + 0) = 1$$

$$[\chi, \chi_2] = \frac{1}{6}(3 - 1 - 1 + 0 + 0) = 0$$

$$[\chi, \chi_3] = \frac{1}{6}(6 + 0 + 0 + 0 + 0) = 1$$

$$\Rightarrow \pi \simeq \pi_1 \oplus \pi_3$$

Character table of S_3

	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1
χ	3	1	0

$$[\chi_1, \chi_1] = \frac{1}{6} \sum_{g \in S_3} \chi_1(g) \overline{\chi_1(g)} = \frac{1}{6}(1 \cdot 1 + 1 \cdot 1) = 1$$

$$[\chi_1, \chi_2] = \frac{1}{6} \sum_g \chi_1(g) \overline{\chi_2(g)} = \frac{1}{6}(1 \cdot 1 - 1 \cdot 1 - 1 \cdot 1 - 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1) = 0$$

$$[\chi_1, \chi_3] = \frac{1}{6}(1 \cdot 2 + 0 + 0 + 0 - 1 \cdot 1) = 0$$

$$[\chi_2, \chi_2] = \frac{1}{6}(1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1) = 1$$

$$[\chi_2, \chi_3] = \frac{1}{6}(2 + 0 + 0 + 0 - 1 \cdot 1) = 0$$

$$[\chi_3, \chi_3] = \frac{1}{6}(4 + 0 + 0 + 0 + 1 \cdot 1) = 1$$

The long way to check the decomposition $\pi \simeq \pi_1 \oplus \pi_3$ is found in the course notes: with respect to new basis

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

basis for first invariant subspace

basis for second invariant subspace

$\pi(g)$ has matrix $\left[\begin{array}{c|cc} \pi_1(g) & 0 & 0 \\ \hline 0 & \pi_3(g) & \end{array} \right]$ with respect to v_1, v_2, v_3

(see p. 3 of course notes).

Character tables also have orthogonality of columns.

$ C_G(g) $	6	2	3
g	(1)	(12)	(23)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$$C_G(g) = \{x \in G : xg = gx\}$$

The conjugacy class of $g \in G$ is the index of the centralizer

$$[G : C_G(g)] = \frac{|G|}{|C_G(g)|}$$

The column orthogonality is expressed as: given $g, h \in G$,
($k =$ number of irreducible characters = no. of conjugacy classes)

$$\sum_{i=1}^k \chi_i(g) \overline{\chi_i(h)} = \begin{cases} 0 & \text{if } g, h \\ & \text{not conjugate} \\ |C_G(g)| & \text{if } g, h \\ & \text{are conjugate} \end{cases}$$

Row orthogonality says: Let $g_1, \dots, g_k \in G$ be reps of conj. classes

$$[f, f']_G = \frac{1}{|G|} \sum_{a \in G} f(a) \overline{f'(a)} = \frac{1}{|G|} \sum_{j=1}^k \frac{|G|}{|C_G(g_j)|} f(g_j) \overline{f'(g_j)} = \sum_{j=1}^k \frac{1}{|C_G(g_j)|} f(g_j) \overline{f'(g_j)}$$

$$[\chi_i, \chi_j] = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Let's construct the character table of A_5 , $|A_5| = 60$, the smallest nonabelian simple group. $G = A_5$ has $k=5$ conjugacy classes $= 2^2 \cdot 3 \cdot 5$

$g_1 = ()$	identity	1A	size 1	$ C_G(g_1) = 60$
$g_2 = (12)(34)$		2A	size 15	$ C_G(g_2) = 4$
$g_3 = (123)$		3A	size 20	$ C_G(g_3) = 3$
$g_4 = (12345)$		5A	size 12	$ C_G(g_4) = 5$
$g_5 = (12354)$		5B	size $\frac{12}{60}$	$ C_G(g_5) = 5$

(12345) and (12354) are conjugate in S_5 since $(45)^{-1}(12345)(45) = (12354)$

$(1234)^{-1}(12345)(1234) = (23415) = (15234)$

but (12345) and (12354) are not conjugate in A_5 .

$(1235)^{-1}(12345)(1235) = (23541) = (12354)$

Character table of A_5 :

$ C_G(g) $	60	4	3	5	5
g	$()$	$(12)(34)$	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	3				
χ_3	3				
χ_4	4				
χ_5	5				
χ	5	1	2	0	0

principal character $\chi_1(g) = \text{tr}(\rho(g)) = \text{tr}([1]) = 1$

standard permutation character

$\sum_{j=1}^k |\chi_j(1)|^2 = |G|$ special case of $\sum_{j=1}^k |\chi_j(g)|^2 = |C_G(g)|$

Write $60 = n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2$ where n_1, n_2, n_3, n_4, n_5 are positive integers.

$\chi = a_1\chi_1 + a_2\chi_2 + \dots + a_5\chi_5$, $a_i \in \{0, 1, 2, \dots\}$ is the number of copies of χ_i in χ .

$$[\chi, \chi_i] = a_i$$

$$[\chi, \chi] = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = \frac{25}{60} + \frac{1}{4} + \frac{4}{3} + \frac{0}{5} + \frac{0}{5} = \frac{25+15+80}{60} = \frac{120}{60} = 2.$$

$\Rightarrow a_1, a_2, \dots, a_5 = 1, 1, 0, 0, 0$ in some order.

$\Rightarrow \chi = \chi_i + \chi_j$ for some $i \neq j$

$$[\chi, \chi_i] = \frac{1}{60} \cdot 5 \cdot 1 + \frac{1}{4} \cdot 1 \cdot 1 + \frac{1}{3} \cdot 2 \cdot 1 + \frac{1}{5} \cdot 0 \cdot 1 + \frac{1}{5} \cdot 0 \cdot 1$$

$$= \frac{1}{12} + \frac{1}{4} + \frac{2}{3} + 0 + 0$$

$$= \frac{1+3+8+0+0}{12} = \frac{12}{12} = 1$$

$$\chi = \chi_i + \chi_j, \quad j \in \{2, 3, 4, 5\}$$

$$\chi(1) = \chi_i(1) + \chi_j(1)$$

$$5 = 1 + \underbrace{\chi_j(1)}_4 \Rightarrow j=4$$

$$\chi = \chi_1 + \chi_4$$

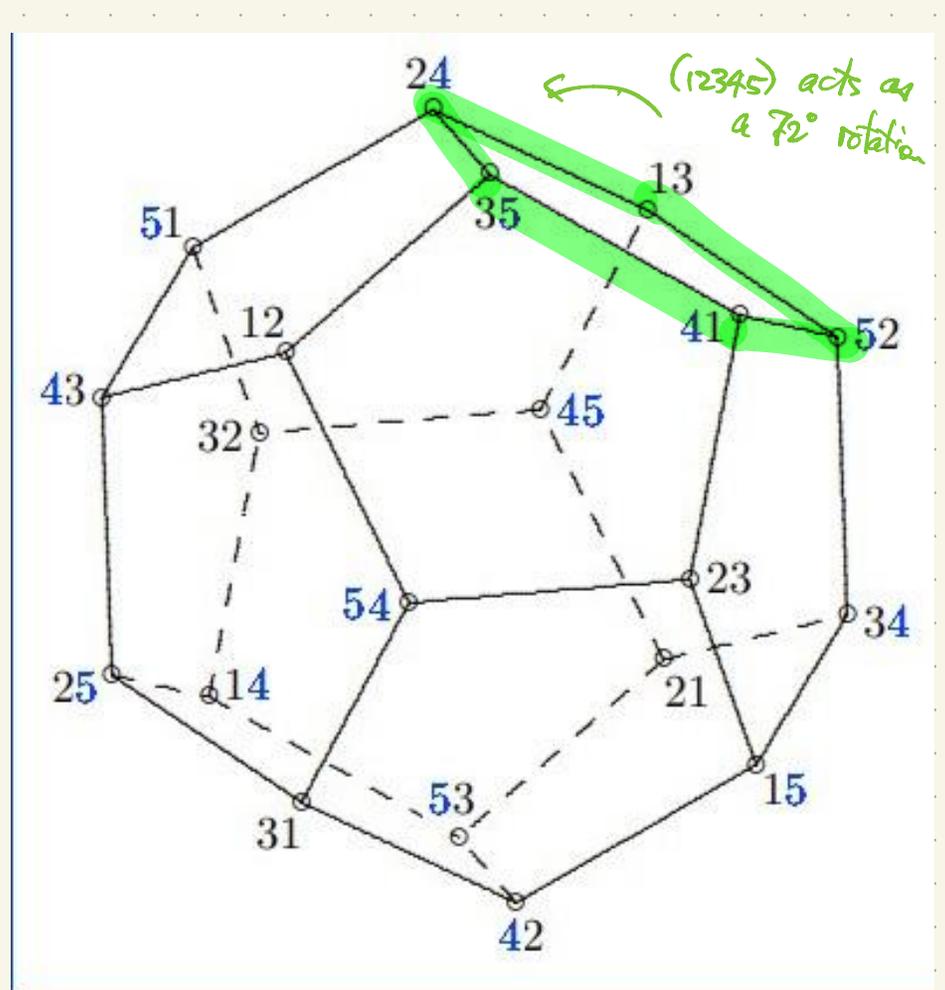
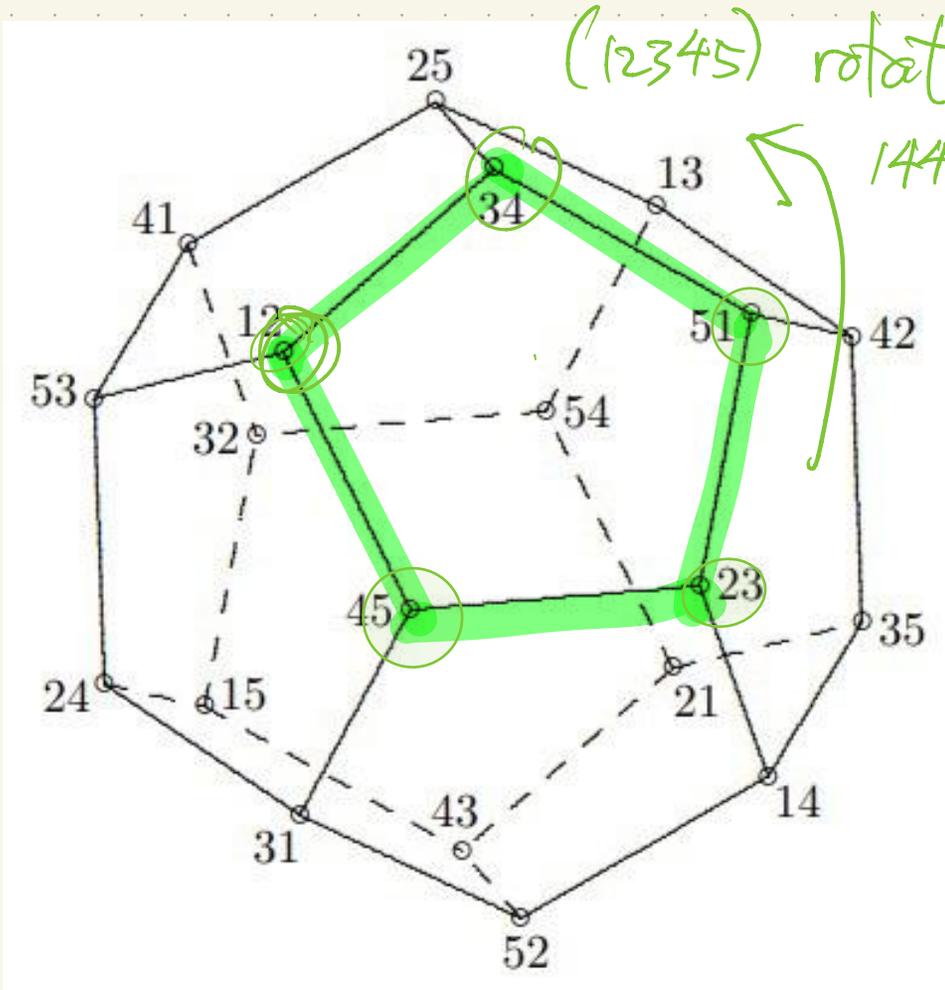
$(C_G(g))$	60	4	3	5	5
g	()	(12)(34)	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_3	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0
χ	5	1	2	0	0

$$1 + 3\frac{1+\sqrt{5}}{2} + 3\frac{1-\sqrt{5}}{2} - 4 + 5 \square = 0$$

$$[\chi_5, \chi_5] = \frac{5^2}{60} + \frac{1^2}{4} + \frac{(-1)^2}{3} + \frac{0^2}{5} + \frac{0^2}{5} = \frac{25}{60} + \frac{1}{4} + \frac{1}{3} + 0 + 0 = \frac{5+3+4+0+0}{12}$$

$$= \frac{12}{12} = 1.$$

Another way to get χ_5 is to use the transitive permutation action of $G = A_5$ of degree 6.



Dodecahedron with vertices (i,j) , $i \neq j$ in $\{1,2,3,4,5\}$. A_5 acts on the dodecahedron as a rotational symmetry group.

Every rotation matrix $A \in SO_3(\mathbb{R})$ is similar to $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix}$ where θ is the angle of the rotation.

$\cos 0 = 1$
 $\cos \pi = -1$
 $\cos \frac{2\pi}{5} = -\frac{1}{2}$
 $\cos \frac{4\pi}{5} = \cos 72^\circ = \frac{-1+\sqrt{5}}{4}$
 $\cos \frac{6\pi}{5} = \cos 144^\circ = \frac{-1-\sqrt{5}}{4}$



θ	$\text{tr } A$
0	3
$\pi = 180^\circ$	-1 (half turn)
$\frac{2\pi}{3} = 120^\circ$	0
$\frac{2\pi}{5} = 72^\circ$	$\frac{1+\sqrt{5}}{2}$ golden ratio ≈ 1.618
$\frac{4\pi}{5} = 144^\circ$	$\frac{1-\sqrt{5}}{2} \approx -0.618$

$G = A_5 \cong \text{PSL}_2(5)$ acting as fractional linear transformations on $\mathbb{F}_5 \cup \{\infty\} = \{0, 1, 2, 3, 4, \infty\}$

$$\text{PSL}_2(5) = \left\{ \pm \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad-bc=1, a, b, c, d \in \mathbb{F}_5 \right\} / \{\pm I\}$$

$$\pm \begin{bmatrix} a & b \\ c & d \end{bmatrix} : x \mapsto \frac{ax+b}{cx+d}$$

identity fixes all six points,
element of order 2:

$$g_2 = \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : x \mapsto \frac{1}{x} = -\frac{1}{x}$$

$$\chi(g_1) = 6$$

$$g_1 = \pm I$$

element of order 3:

$$\text{i.e. } (0, \infty)(1, 4)(2)(3) = (0, \infty)(1, 4), \quad \chi(g_2) = 2$$

$$g_3 = \pm \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} : x \mapsto \frac{-1}{x+1} = -\frac{1}{x+1}$$

$$\text{i.e. } (0, 4, \infty)(1, 2, 3), \quad \chi(g_3) = 0$$

element of order 5:

$$g_4 = \pm \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} : x \mapsto x+1 \quad \text{i.e. } (0, 1, 2, 3, 4)(\infty)$$

$$\chi(g_4) = 1$$

Another conjugacy class of elements of order 5:

$$g_5 = g_4^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} : x \mapsto x+2 \quad \text{i.e. } (0, 2, 4, 1, 3)$$

$$\chi(g_5) = 1$$

$$\begin{aligned} [\chi, \chi] &= \frac{6^2}{60} + \frac{2^2}{4} + \frac{0^2}{3} + \frac{1^2}{5} + \frac{1^2}{5} \\ &= \frac{3 + 5 + 0 + 1 + 1}{5} = \frac{10}{5} = 2 \end{aligned}$$

$ C_G(g) $	60	4	3	5	5
g	(1)	(12)(34)	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_3	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0
χ	6	2	0	1	1

$$\chi = a_1 \chi_1 + a_2 \chi_2 + \dots + a_5 \chi_5$$

$$[\chi, \chi] = \sum a_i^2 = 2$$

$\Rightarrow a_1, \dots, a_5: 1, 1, 0, 0, 0$ in some order

$$[\chi, \chi_1] = \frac{6}{60} + \frac{2}{4} + \frac{0}{3} + \frac{1}{5} + \frac{1}{5} = \frac{1+5+0+2+2}{10} = \frac{10}{10} = 1$$

$$\frac{\chi(1)}{6} = \frac{\chi_1(1)}{1} + \frac{\chi_5(1)}{5} \quad (j \neq 1)$$

$$\chi = \chi + \chi_5$$

Let G be a finite group with k conj. classes and $|G| = n$ then its char. table looks like

$ C_G(g) $	n	n_1	n_2	\dots	n_k
g	g_1	g_2	\dots	g_k	
χ_1	1	1	\dots	1	
χ_2	n_1				
\vdots					
χ_k	n_k				
χ	n	0	\dots	0	

$$n_i = \deg \chi_i = \chi_i(1)$$

Let χ be the perm. character of the regular representation of G : each $g \in G$ permutes G by $g: x \mapsto gx$. The perm. character of this representation is $\begin{cases} \chi(1) = n \\ \chi(g) = 0 \text{ for } g \neq 1 \end{cases}$

$$\chi = (?)\chi_1 + (?)\chi_2 + \dots + (?)\chi_k$$

\uparrow \uparrow \uparrow
 $[\chi, \chi_1]$ $[\chi, \chi_2]$ $[\chi, \chi_k]$

$$[\chi, \chi_i] = \frac{n n_i}{n} + 0 + \dots + 0 = n_i$$

$$\chi = n_1 \chi_1 + n_2 \chi_2 + \dots + n_k \chi_k, \quad n_i = \chi_i(1) \geq 1$$

Every irreducible character occurs with multiplicity $n_i \geq 1$ in the regular representation.

Evaluate at $1 \in G$:

$$\chi(1) = n = n_1 \chi_1(1) + n_2 \chi_2(1) + \dots + n_k \chi_k(1) = n_1^2 + n_2^2 + \dots + n_k^2$$

Character values are algebraic integers. If χ is a character of G with $\chi(g) = \text{tr } \pi(g)$ then