

Math 5555

Abstract Algebra II

Book 2

Induction takes class functions on H to class functions on G
 characters representations of H ——— characters on G representations of G

Let χ be a class function on $H \leq G$ i.e. $\chi: H \rightarrow \mathbb{C}$, $\chi(xhx^{-1}) = \chi(h)$ for all $x, h \in H$.
 To get a class function on G , start with the trivial extension

$$\hat{\chi}: G \rightarrow \mathbb{C}$$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H; \\ 0 & \text{if } g \notin H. \end{cases}$$

To make this into a class function, use an averaging over conjugates as we did before. This leads to $\chi^G: G \rightarrow \mathbb{C}$:

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1}) \quad (\chi^G \text{ is induced from } \chi, \quad \chi^G = \text{Ind}_H^G \chi)$$

$$\text{If } u \in G \text{ then } \chi^G(ugu^{-1}) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xug u^{-1} x^{-1}) = \frac{1}{|H|} \sum_{w \in G} \hat{\chi}(w g w^{-1})$$

$$\begin{aligned} w^{-1} &= u^{-1} x^{-1} \\ w &= x u \\ w u &= x \end{aligned}$$

So χ^G is a class function on G .

Note: Let T be a set of right coset representatives for H in G .

So every element $g \in G$ is uniquely expressible as $g = ht$, $h \in H, t \in T$.

$$|G| = |H| |T| \quad (\text{Lagrange's Theorem}), \quad |T| = \frac{|G|}{|H|} = [G:H].$$

$\chi: H \rightarrow \mathbb{C}$ is a class function on H ; T is a right transversal for H in G (set of right coset representatives)

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1})$$

$$x = ht, \quad \begin{matrix} h \in H \\ t \in T \end{matrix}$$

$$T = \{t_1, \dots, t_l\}$$

$$G = Ht_1 \sqcup Ht_2 \sqcup \dots \sqcup Ht_l = \bigsqcup_{t \in T} Ht$$

$$= \frac{1}{|H|} \sum_{t \in T} \sum_{h \in H} \hat{\chi}(htgt^{-1}h^{-1})$$

$$htgt^{-1}h^{-1} \in H$$

$$= \frac{1}{|H|} \sum_{t \in T} \sum_{h \in H} \hat{\chi}(tgt^{-1})$$

$$\iff tgt^{-1} \in H$$

$|H|$ terms equal to $\hat{\chi}(tgt^{-1})$

$$= \frac{1}{|H|} \sum_{t \in T} |H| \hat{\chi}(tgt^{-1}) = \sum_{t \in T} \hat{\chi}(tgt^{-1}) = \sum_{i=1}^l \hat{\chi}(t_i g t_i^{-1})$$

Special case: $\chi = \chi_1 =$ trivial (principal) character of H , $\chi(h) = 1$.

χ^G isn't the principal character of G unless $G=H$.

G permutes the right cosets of H by right multiplication giving a permutation representation $g \in G$ permutes $Ht_i \mapsto Ht_j = Ht_i g$.

$\chi^G = \left(\frac{1}{\#} \right)^G$ is the perm.

$$G \longrightarrow S_l$$

$$l = [G:H] = |T|.$$

character of G acting on cosets of H .

Ex. Construct the character table of $G = S_4$ making use of the character table of $S_3 = H \leq G$.

H:

$K_H(g)$	6	2	3
g	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

check: $[\chi_3, \chi_3] = 1$

$G = S_4$:

$(C_G(g))$	24	4	8	3	4
g	(1)	(12)	(12)(34)	(123)	(1234)
ψ_1	1	1	1	1	1
ψ_2	1	-1	1	1	-1
ψ_3	2	0	2	-1	0
ψ_4	3				
ψ_5	3				
ψ	3	1	3	0	1

$[\psi_1, \psi_2] = 0$
 $\psi(1) = \psi_1(1) + \psi_2(1)$
 $\psi = \sum_{i=1}^5 a_i \psi_i$
 $[\psi, \psi]_G = \frac{1^3}{24} + \frac{1}{4} + \frac{9}{8} + \frac{0}{3} + \frac{1}{4}$
 $= \frac{3 + 2 + 9 + 0 + 2}{8} = \frac{16}{8} = 2$
 $= \sum_{i=1}^5 a_i^2 = 1 + 1 + 0 + 0 + 0 = 2$
 $\psi = \psi_1 + \psi_3$
 (degree 1) (degree 2)
 $a_i = [\psi, \psi_i] = \frac{3}{24} + \frac{1}{4} + \frac{3}{8} + \frac{0}{3} + \frac{1}{4} = \frac{1+2+3+0+2}{8} = 1$
 $n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = 16 = 2$

S_4 has $k=5$ conjugacy classes
 ... irreducible representations/characters
 of degree $n_1, n_2, \dots, n_5 \geq 1$, $n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = 16 = 2$

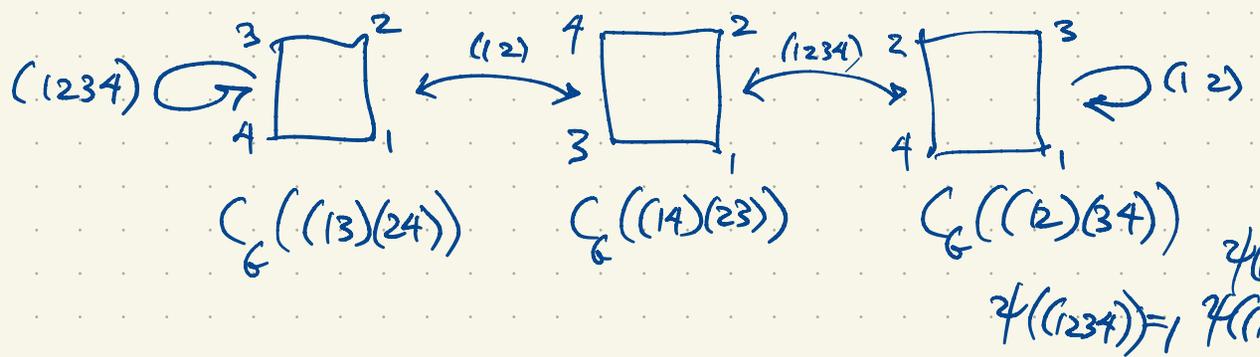
S_4 has normal subgroups

$1, S_4, A_4, K = \langle (12)(34), (13)(24) \rangle = \{ (1), (12)(34), (13)(24), (14)(23) \}$

S_4 permutes the conjugacy class of $(12)(34)$ in all $3! = 6$ possible ways
 there is a permutation action $S_4 \rightarrow \text{Sym} \{ (12)(34), (13)(24), (14)(23) \} \cong S_3$
 with kernel K of order 4.

This gives a permutation representation of S_4 of degree 3.

Its character is
 $\psi((1)) = 3$
 $\psi(k) = 3$ for $k \in K$
 $\psi((12)) = 1$
 $\psi((123)) = 0$
 $\psi((12)(34)) = 3$



$H = S_3$:

$K_H(g)$	6	2	3
g	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$G = S_4$:

(G)	24	4	8	3	4
g	(1)	(12)	(12)(34)	(123)	(1234)
ψ_1	1	1	1	1	1
ψ_2	1	-1	1	1	-1
ψ_3	2	0	2	-1	0
ψ_4	3	-1	-1	0	1
ψ_5	3	1	-1	0	-1
χ^G	4	-2	0	1	0

$$[\chi^G, \chi^G]_G = \frac{16}{24} + \frac{4}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 2$$

$\Rightarrow \chi^G$ has irreducible constituents of multiplicity $\neq 1, 0, 0, 0$

$$[\chi^G, \psi_1] = \frac{4}{24} - \frac{2}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 0$$

$$[\chi^G, \psi_2] = \frac{4}{24} + \frac{2}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 1$$

$T = \langle (1234) \rangle = \{ (1), (1234), (13)(24), (1432) \}$
right transversal for H in G

$\chi^G = \chi_2^G$ is a character on G

$$\chi^G(g) = \sum_{t \in T} \hat{\chi}(tgt^{-1})$$

where $\hat{\chi}: G \rightarrow \mathbb{C}$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

$$\chi^G = \psi_2 + \psi_4$$

$$\psi_4 = \chi^G - \psi_2$$

$$[\psi_4, \psi_4]_G = \frac{9}{24} + \frac{1}{4} + \frac{1}{8} + \frac{0}{3} + \frac{1}{4} = \frac{3+2+1+0+2}{8} = 1$$

$$\chi^G(1) = 1 + 1 + 1 + 1 = 4$$

$$\chi^G((12)) = \hat{\chi}((12)) + \hat{\chi}((23)) + \hat{\chi}((34)) + \hat{\chi}((14)) = -1 - 1 + 0 + 0 = -2$$

$$\chi^G((12)(34)) = \hat{\chi}((12)(34)) = 0 + 0 + 0 + 0 = 0$$

$$+ \hat{\chi}((23)(41))$$

$$+ \hat{\chi}((34)(12))$$

$$+ \hat{\chi}((41)(23))$$

$$\chi^G((123)) = \hat{\chi}((123)) + \hat{\chi}((234)) + \hat{\chi}((341)) + \hat{\chi}((412)) = 1 + 0 + 0 + 0 = 1$$

Frobenius Reciprocity let χ be a class function on $H \leq G$ and let ψ be a class function on G . Then

$$[\psi_H, \chi]_H = [\psi, \chi^G]_G$$

\uparrow
 $\psi_H = \psi|_H$

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1})$$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

Proof $[\chi^G, \psi]_G = \frac{1}{|G|} \sum_{g \in G} \chi^G(g) \overline{\psi(g)}$

$x \in G = HT$
 $x = ht, \quad h \in H, t \in T$
 $|G| = |H||T|$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{x \in G} \frac{1}{|H|} \hat{\chi}(xgx^{-1}) \overline{\psi(g)}$$

$$= \frac{1}{|G||H|} \sum_{g \in G} \sum_{x \in G} \hat{\chi}(xgx^{-1}) \overline{\psi(g)} \quad \leftarrow htgt^{-1}h^{-1} \in H \iff tgt^{-1} \in H$$

$$= \frac{1}{|G||H|} \sum_{g \in G} \sum_{h \in H} \sum_{t \in T} \hat{\chi}(htgt^{-1}h^{-1}) \overline{\psi(g)}$$

$$= \frac{1}{|G||H|} \sum_{g \in G} \sum_{h \in H} \sum_{t \in T} \hat{\chi}(tgt^{-1}) \overline{\psi(g)}$$

$|H|$ identical terms for $h \in H$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{t \in T} \hat{\chi}(tgt^{-1}) \overline{\psi(g)}$$

$$= \frac{1}{|G|} \sum_{t \in T} \left(\sum_{g \in G} \hat{\chi}(tgt^{-1}) \overline{\psi(g)} \right)$$

$$= \frac{1}{|G|} \sum_{t \in T} \left(\sum_{u \in G} \hat{\chi}(u) \overline{\psi(t^{-1}ut)} \right)$$

$$= \frac{1}{|G|} \sum_{t \in T} \sum_{h \in H} \chi(h) \overline{\psi(t^{-1}ht)} = \frac{1}{|G|} \sum_{t \in T} \sum_{h \in H} \chi(h) \overline{\psi(h)}$$

For each $t \in T$ reparameterize the inner sum, $u = tgt^{-1}$, $g = t^{-1}ut$

$$\begin{aligned} &= \frac{1}{|G|} |T| \sum_{h \in H} \chi(h) \overline{\psi(h)} \\ &= \frac{1}{|H|} \sum_{h \in H} \chi(h) \overline{\psi(h)} \\ &= [\chi, \psi_H]_H \quad \square \end{aligned}$$

If ψ is a class function on G then so is $\bar{\psi}$ where $\bar{\psi}(g) = \overline{\psi(g)} = \psi(g^{-1})$.
 Indeed if ψ is a character, $\psi(g) = \text{tr } \pi(g)$, $\pi: G \rightarrow GL_n(\mathbb{C})$
 $\pi(g) \sim$ similar to $\begin{bmatrix} \lambda_1 & * \\ 0 & \lambda_n \end{bmatrix}$ (actually $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix}$ in the case of finite groups) homomorphism

$\lambda_1, \dots, \lambda_n$ are m^{th} roots of unity where $m = \text{exponent of } G = \text{lcm}(|g| : g \in G)$
 $\bar{\lambda}_i = \lambda_i^{-1}$ since $\lambda_i^m = 1$, $\bar{\lambda}_i \lambda_i = |\lambda_i|^2 = 1$
 $|\lambda_i|^m = 1$
 $|\lambda_i| = 1$

$$\psi(g) = \sum \lambda_i$$

$$\psi(g^{-1}) = \sum \lambda_i^{-1} = \sum \bar{\lambda}_i = \overline{\psi(g)}$$

For every finite group G , the irreducible representations of G can be chosen to be unitary.

$$U_n(\mathbb{C}) = \{A \in \mathbb{C}^{n \times n} : AA^* = A^*A = I\}, \quad A^* = \bar{A}^T = \overline{A^T}$$

eg. for S_3 , we have an irreducible representation of degree 2.

$\pi_3: S_3 \rightarrow GL_2(\mathbb{C})$	$\chi(g)$
$(1) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	2
$(12) \mapsto \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$	0
$(123) \mapsto \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$	-1
$(132) \mapsto \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$	-1
$(13) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	0
$(23) \mapsto \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$	0

This representation uses integer matrix entries.

left-to-right composition

$$(13) = (123)(12)(132)$$

Character values:

$$\chi(g) = \text{tr } \pi(g)$$

A change of basis from $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ takes one representation to the other.

A different choice of basis for \mathbb{C}^2 yields an equivalent representation using unitary matrices:

	$\chi(g)$
$(1) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	2
$(123) \mapsto \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix}$	-1
$(132) \mapsto \begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix}$	-1
$(12) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	0
$(13) \mapsto \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} 0 & \bar{\omega} \\ \bar{\omega} & 0 \end{bmatrix} \begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} 0 & \bar{\omega} \\ \omega & 0 \end{bmatrix}$	0
$(23) \mapsto \begin{bmatrix} 0 & \omega \\ \bar{\omega} & 0 \end{bmatrix}$	0

$\omega = e^{2\pi i/3}$
 $\bar{\omega} = \omega^2 = \omega^{-1}$
 root of $x^2 + x + 1$
 $\omega^2 + \omega + 1 = 0$
 $\bar{\omega} + \omega = \omega^2 + \omega = -1$

To prove the claim (that every finite group has its irred. reps. equivalent to equivalent to unitary representations) we use a fact from linear algebra: any inner product on \mathbb{C}^n is equivalent (under change of basis) to standard inner product $B(x, y) = \sum x_i \bar{y}_i$.

Any inner product has the form $(x, y) \mapsto x M y^*$ where $x = (x_1, \dots, x_n)$
 $M^* = M$ $n \times n$ matrix Hermitian, $\det M \neq 0$.
 $y^* = \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{pmatrix}$

If we perform an invertible change of variables $x \mapsto xA$ (A invertible $n \times n$ matrix i.e. $A \in GL_n(\mathbb{C})$)
 then $B(xA, yA) = (xA)M(yA)^* = x \underbrace{(AMA^*)}_{\text{congruent to } M} y^*$

For all nonsingular Hermitian M , we can find $A \in GL_n(\mathbb{C})$ such that $AMA^* = I$ (using Gram-Schmidt) i.e. any two inner products on

\mathbb{C}^n are equivalent by change of basis.

Given a representation $\pi: G \rightarrow GL_n(\mathbb{C})$, we will find an inner product $B(x, y)$ on \mathbb{C}^n such that all matrices $\pi(g)$, $g \in G$ preserve the inner product:

- ✓ $B(x, x) \geq 0$, equality iff $x = 0$;
- ✓ $B(x, y)$ linear in x , conjugate linear in y ; $B(y, x) = \overline{B(x, y)}$;
- ✓ $B(x\pi(g), y\pi(g)) = B(x, y)$.

To obtain the inner product $B(x, y)$:

$$B(x, y) = \sum_{g \in G} (x \pi(g)) (y \pi(g))^* = \sum_{g \in G} x \pi(g) \pi(g)^* y^* = x M y^*$$

$$M = \sum_{g \in G} \pi(g) \pi(g)^* \text{ is Hermitian.}$$

$$B(x, x) = \sum_{g \in G} (x \pi(g)) (x \pi(g))^* = \sum_{g \in G} \|x \pi(g)\|^2 \geq 0$$

$B(x, x) > 0$ unless $x = 0$.

$$B(x \pi(g), y \pi(g)) = \sum_{u \in G} \underbrace{(x \pi(g) \pi(u))}_{\pi(w)} \underbrace{(y \pi(g) \pi(u))^*}_{\pi(w)} = \sum_{w \in G} (x \pi(w)) (y \pi(w))^* = B(x, y)$$

$w = gu$

Suppose $H \leq G$; let T be a right transversal for H in G i.e. a set of right coset representatives. Every $g \in G$ can be uniquely factored as $g = ht$, $h \in H$, $t \in T$. (Lagrange's theorem) $|G| = |H||T|$

G permutes the right cosets of H by right-multiplication:

$$Ht \mapsto Htg = Ht' \text{ for some } t' \in T.$$

This gives a permutation representation of G acting on the right cosets of H . If $d = |T| = [G : H]$ then we have a homomorphism $\pi: G \rightarrow S_d \subset GL_d(\mathbb{C})$ with perm. character $\psi(g) = \text{tr } \pi(g) = \text{no. of fixed points of } \pi(g)$

Theorem $\psi = (1_H)^G$ i.e. the perm. character is the induced character obtained from $1_H =$ principal character, $1_H(h) = 1$ for all $h \in H$, induced up to G .

Proof $\hat{1}_H(g) = \begin{cases} 1, & \text{if } g \in H; \\ 0, & \text{if } g \in G, g \notin H. \end{cases}$

$$(1_H)^G(g) = \sum_{t \in T} \hat{1}_H(tgt^{-1}) = \text{number of } tgt^{-1} \quad (t \in T) \text{ which are in } H$$

$$= |\{t \in T : \underbrace{tgt^{-1} \in H}\}|$$

$$tgt^{-1} \in H \iff Htgt^{-1} = H \iff Htg = Ht$$

\iff the coset Ht is fixed under right-multiplication by $g \in G$.

$$= \psi(g).$$

□

From the character table of G , we can see what all normal subgroups of G are.

$G = \Sigma_4$

(C_6)	24	4	8	3	4
g	(1)	(12)	(12)(34)	(123)	(1234)
ψ_1	1	1	1	1	1
ψ_2	1	-1	1	1	-1
ψ_3	2	0	2	-1	0
ψ_4	3	-1	-1	0	1
ψ_5	3	1	-1	0	-1

$\psi_2 : G \rightarrow \mathbb{C}^\times$ is a homomorphism
 Since $\psi_2(g) = \text{tr } \pi_2(g)$ of degree $\psi_2(1) = 1$.

Its kernel is $\{g \in G : \psi_2(g) = 1\}$
 $= \{ \text{elements conjugate to } (1), (12)(34) \text{ or } (123) \}$
 $= A_4$

Another normal subgroup: all elements $g \in G$
 s.t. $\psi_3(g) = \psi_3(1) = 2$