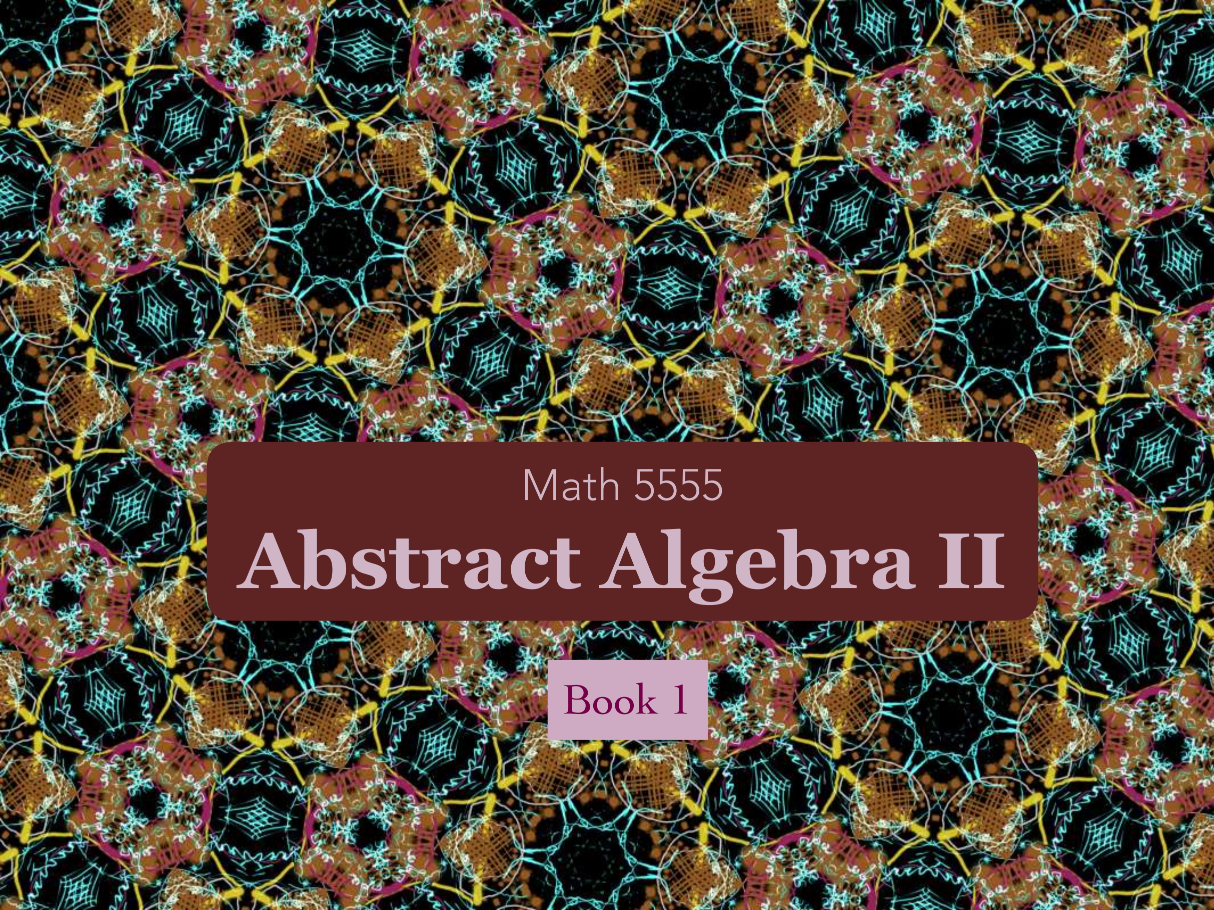


Math 5555

Abstract Algebra II

Book 1

In group theory we have

- permutation representations: homomorphisms $\pi : G \to S_n$ permutation representation of degree n
  (if π is 1-to-1 then π is a faithful representation: then $\underset{\cong G}{\pi(G)} \leq S_n$)
- linear representations: homomorphism $\pi : G \to GL_n(F)$ F: field
  linear representation of degree n over F

  If $F = \mathbb{C}$ (or $\mathbb{R}$ or ...) then π is an ordinary representation.

  If char $F = p$ (prime) then π is a modular representation.

  G: unless otherwise specified, G finite group. (until later...)

Usually $F = \mathbb{C}$ (or $\mathbb{R}$) and $|G| < \infty$.

If $\pi_i : G \to GL_{n_i}(\mathbb{C})$ $(i = 1, 2)$ then $\pi_1 \oplus \pi_2 : G \to GL_{n_1+n_2}(\mathbb{C})$, $g \mapsto \underbrace{\left[\begin{array}{c|c} \pi_1(g) & 0 \\ \hline 0 & \pi_2(g) \end{array}\right]}_{n_1 \quad n_2} \begin{matrix} \}n_1 \\ \}n_2 \end{matrix}$ is a representation of degree $n_1 + n_2$

$\pi : G \to GL_n(\mathbb{C})$ is decomposable if there is a decomposition $\mathbb{C}^n = U \oplus V$ such that $U, V \neq 0$
(dim $U = n_1$, dim $V = n_2$, n_1, n_2 positive integers, $n_1 + n_2 = n$)

U, V invariant under all matrices $\pi(g)$, $g \in G$.

$G = S_3$ acting naturally on $\mathbb{C}^3$ by permuting the standard basis vectors $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

via $\sigma : e_i \mapsto e_{\sigma(i)}$

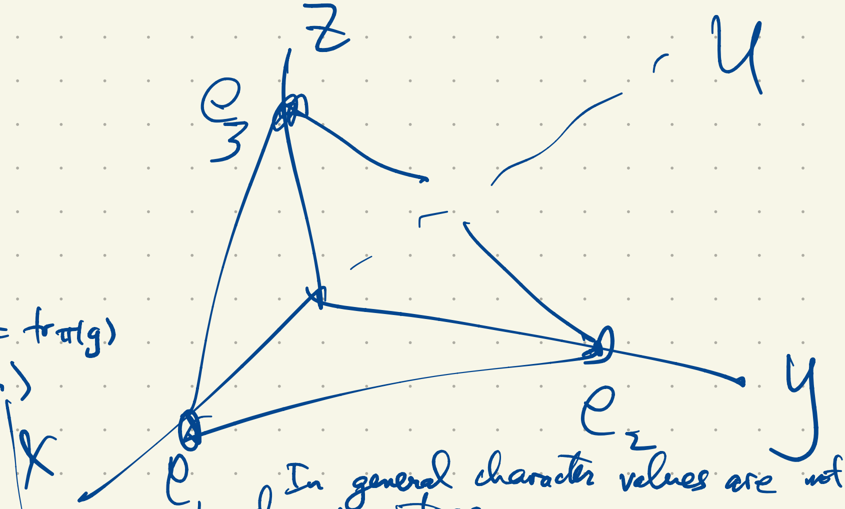
i.e. $(12) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$(123) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$() \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

This is a faithful representation of degree 3.

It is decomposable: $\mathbb{C}^3 = U \oplus V$, $U = \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle$, $V = U^{\perp} = \underbrace{\left\langle \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\rangle}_{\text{indecomposable.}}$



For a permutation representation

$\pi : G \to S_n \subset GL_n(\mathbb{C})$

the associated character $\chi(g) = \text{tr}\,\pi(g)$

(called the permutation character)

is $\chi(g)$ = no. of fixed points of g

$= |\{i : g(i) = i\}|$, $1 \leq i \leq n$.

In general character values are not always integers.

The representation $\pi : S_3 \to GL_3(\mathbb{C})$

has (affords) character

$\chi(g) = \text{tr}\,\pi(g)$

$\chi(1) = 3$ = degree of π = deg χ.

$\chi((12)) = \chi((13)) = \chi((23)) = 1$

$\chi((123)) = \chi((132)) = 0$

Given a representation $\pi : G \to GL_n(\mathbb{C})$, the character of π is

$$\chi(g) = \operatorname{tr} \pi(g) \in \mathbb{C}.$$

Character values $\chi(g) \in \mathbb{C}$ are always algebraic integers $(g \in G, \; |G| < \infty)$
and character values of S_n are ordinary integers.

$\chi(g)$ depends only on the conjugacy class of g.

If $g, h \in G$ then $g \sim h^{-1}gh$ (conjugate in G) so

$$\pi(h^{-1}gh) = \underbrace{\pi(h)^{-1}\pi(g)\pi(h)}_{\text{inverses in } GL_n(\mathbb{C})} \sim \pi(g) \quad \text{(similar in } GL_n(\mathbb{C}) \text{ i.e. conjugate)}$$

so $\operatorname{tr}\pi(g) = \operatorname{tr}(\pi(h^{-1})\pi(g)\pi(h)) = \operatorname{tr}\pi(h^{-1}gh)$

$\operatorname{tr}(AB) = \operatorname{tr}(BA)$

$\operatorname{tr}(B^{-1}AB) = \operatorname{tr}(AB \cdot B^{-1}) = \operatorname{tr} A$

$\chi(h^{-1}gh) = \chi(g)$.

If $\pi : G \to GL_n(\mathbb{C})$ is any represention
i.e. homomorphism, and $B \in GL_n(\mathbb{C})$, then
$\tilde{\pi}(g) = B^{-1}\pi(g)B \in GL_n(\mathbb{C})$ is also a representation
$\pi, \tilde{\pi}$ are equivalent (via a change of basis).

$$\tilde{\pi}(gh) = B^{-1}\pi(gh)B = B^{-1}\pi(g)\pi(h)B = B^{-1}\pi(g)B \cdot B^{-1}\pi(h)B = \tilde{\pi}(g)\tilde{\pi}(h).$$

They have the same character: the character of $\tilde{\pi}$ is

$$\tilde{\chi}(g) = \operatorname{tr}\tilde{\pi}(g) = \operatorname{tr}(B^{-1}\pi(g)B) = \operatorname{tr}\pi(g) = \chi(g).$$

It's not obvious but the converse is true: χ determines π up to equivalence.
Two representations have the same character iff they are equivalent.

Let $\pi : G \to GL_n(\mathbb{C})$ be a representation (i.e. homomorphism).
π is <u>reducible</u> if there exists nontrivial proper subspace $U < \mathbb{C}^n$ $(\dim U \in \{1, 2, \ldots, n-1\})$ such that
U is invariant under $\pi(g)$ for all $g \in G$ i.e. $\pi(g)U \subseteq U$ for all $g \in G$.

i.e. $$\pi(g) = \begin{array}{c} \left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right] \begin{array}{l} \}k \\ \}n-k \end{array} \\ \begin{array}{cc} \underbrace{}_{k} & \underbrace{}_{n-k} \end{array} \end{array} \qquad k = \dim U.$$
for all $g \in G$.

when $|G| < \infty$ and $\pi : G \to GL_n(F)$
char $F = 0$, π reducible $\Longleftrightarrow$ π decomposable
π irreducible $\Longleftrightarrow$ π indecomposable

In general, π decomposable $\Rightarrow$ π reducible

$$\left[\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right] \Rightarrow \left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right]$$

π indecomposable $\Leftarrow$ π irreducible.

The representation $\pi : \underset{\substack{\text{additive}\\ \text{group}}}{\mathbb{R}} \longrightarrow GL_2(\mathbb{R})$, $\pi(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, $\pi(a+b) = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \pi(a)\pi(b)$

is reducible: $\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$ is an invariant subspace. There is no complementary invariant subspace (in particular the complementary subspace $\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle$ is not invariant).

Maschke's Theorem Let G be a finite group and $F = \mathbb{C}$ (or more generally F is any field of characteristic not dividing $|G|$) Then any representation $\pi : G \to GL_n(F)$ is reducible iff it's decomposable (ie. π is irreducible iff π is indecomposable; ie. whenever every invariant subspace $U \leq F^n$ has a complementary subspace U' which is also invariant).

$U \leq V$ has a complementary subspace U' $\Leftrightarrow$ $V = U \oplus U'$. If $\dim V = n$ and $\dim U = k$ then U' is a complement to U iff $\dim U' = n-k$ and $U + U' = V$
iff $\dim U' = n-k$ and $U \cap U' = \{0\}$.

In this case every $v \in V$ is uniquely represented as $v = u + u'$, $u \in U$, $u' \in U'$.

In this, the projection $V \to U$ along U' is the map $P : v \mapsto Pv = u$. $P = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right] \begin{array}{l} \}k \\ \}n-k \end{array}$ with columns $\underbrace{}_{k}\ \underbrace{}_{n-k}$

P has image $PV = U$; $\ker P = U'$. Note: $I-P$ is the projection from V onto U' along U.

A linear transformation $P : V \to V$ is a projection iff $P^2 = P$. In this case P is a root of $x^2 - x$ so the eigenvalues of P are in $\{0,1\}$. Take $U = 1$-eigenspace of P $= \ker(I-P)$, $U' = \ker P$.

Proof of Maschke's Theorem: (in the nontrivial direction)

Suppose $\pi : G \to GL_n(\mathbb{C})$ is a representation having an invariant subspace $U \le \mathbb{C}^n$ i.e. $\pi(g)U \subseteq U$ for all $g \in G$. We want to find a complementary subspace $W \le \mathbb{C}^n$ which is also invariant. Start with any complementary subspace U' where $\mathbb{C}^n = U \oplus U'$, $\dim U = k$, $\dim U' = n-k$; every $v \in \mathbb{C}^n$ is uniquely expressible as $v = u + u'$, $u \in U$, $u' \in U'$. Unfortunately U' is not invariant in general. Let $P : \mathbb{C}^n \to U$ be the projection onto U along U' i.e. $P(v) = P(u+u') = u$, so $U = PV$, $U' = \ker P$. Consider the new map $\tilde{P} : V \to V$ defined by

$$\tilde{P} = \frac{1}{|G|}\sum_{g\in G} \underbrace{\pi^{-1}(g) P \pi(g)}_{n\times n \text{ matrix}} .$$

$\tilde{P} V \subseteq U$. Since $\tilde{P} v = \frac{1}{|G|}\sum_g \underbrace{\pi(g^{-1}) \underbrace{P \underbrace{\pi(g) v}_{\in V}}_{\in U}}_{\in U} \in U$.

For all $u \in U$, $\tilde{P} u = u$ i.e. $\tilde{P}|_U = id|_U$. Why?

$$\tilde{P} u = \frac{1}{|G|}\sum_g \pi(g^{-1}) \underbrace{P\pi(g) u}_{= \pi(g) u \text{ since } \pi(g) u \in U} = \frac{1}{|G|}\sum_g \cancel{\pi(g^{-1})\pi(g)}\, u = \frac{1}{\cancel{|G|}}\cancel{|G|}\, u = u.$$

Next: show $\tilde{P}$ commutes with all $\pi(g)$, $g \in G$. (P doesn't satisfy this in general !)

Then
$$\tilde{P}\pi(g) = \frac{1}{|G|}\sum_{h\in G} \pi(h^{-1}) P \pi(h)\pi(g) = \frac{1}{|G|}\sum_{h \in G} \pi(h^{-1}) P \pi(hg)$$

$$= \frac{1}{|G|}\sum_{x\in G} \underbrace{\pi(gx^{-1})}_{\pi(g)\pi(x)^{-1}} P\pi(x) = \pi(g)\cdot \frac{1}{|G|}\sum_{x\in G}\pi(x)^{-1} P \pi(x) = \pi(g)\tilde{P}.$$

$h \longleftrightarrow x = hg$

$x^{-1} = g^{-1}h^{-1}$

$gx^{-1} = h^{-1}$

Next: show $\tilde{P}^2 = \tilde{P}$. If $v \in V$ then $\tilde{P} v \in U$ so $\tilde{P}^2 v = \tilde{P}\underbrace{\tilde{P} v}_{\in U} = \tilde{P} v$. So $\tilde{P}$ is idempotent so $\tilde{P}$ is a projection on $\tilde{P}V = U$ along $W = \ker \tilde{P}$. Note: if $k = \dim U = \operatorname{rank}\tilde{P} = \operatorname{tr}\tilde{P}$, $\dim W = \dim \ker \tilde{P} = n-k$. W is also invariant under $\pi(g)$: if $v \in W$ then $\tilde{P} v = 0$ so $\underset{\tilde{P}\pi(g)v}{\underset{\shortparallel}{\pi(g)\tilde{P}v}} = \pi(g)\, 0 = 0$ so $\pi(g) v \in W$

$\square$

The irreducible representations of S_3 are

$\pi_1(g) = [1] \in GL_1(\mathbb{C})$ trivial representation

$\pi_2(g) = [\operatorname{sgn}(g)] \in GL_1(\mathbb{C})$ sign representation:

$= [\pm 1]$ according as g is an even or odd permutation.

$\operatorname{sgn}(()) = 1$ $\qquad \operatorname{sgn}((12)) = \operatorname{sgn}((13)) = \operatorname{sgn}((23)) = -1$

$\operatorname{sgn}((123)) = 1$

$\operatorname{sgn}((132)) = 1$

$\operatorname{sgn}(gh) = \operatorname{sgn}(g)\operatorname{sgn}(h)$

$\pi_3 : S_3 \longrightarrow GL_2(\mathbb{C})$

$() \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$(12) \longmapsto \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$

$(123) \longmapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$

$(132) \longmapsto \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$

$(13) \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$(23) \longmapsto \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\pi_3((13)) = \pi_3((123)(12)) = \pi_3((123))\,\pi_3((12)) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\pi_3((23)) = \pi_3((123)(13)) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

An irreducible character is a map $G \to \mathbb{C}$, $g \longmapsto \operatorname{tr}\pi(g)$

where π is an irreducible representation.

$\pi_3((123)) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ has char. poly. $x^2 + x + 1$

$\pi_3((13)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has char. poly. $x^2 - 1$

eigenspaces $\left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle$

(eigenvalue $1, -1$ respectively)

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$0, \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle, \mathbb{C}^2$ are the only subspaces invariant under $\pi_3((13))$; but the 1-dimensional invariant subspaces are not invariant under $\pi_3((123))$.

The character table of S_3 is

	$()$	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Given any finite group G, a <u>class function</u> on G is a function $f: G \to \mathbb{C}$ which only depends on conjugacy class, i.e. $f(g') = f(g)$ whenever $g, g' \in G$ are conjugate i.e. $g' = u^{-1}gu$ for some $u \in G$.

All characters of G (irreducible or otherwise) are class functions.

$V = \{\text{class functions on } G\}$ is a complex vector space

$$f, f' \in V \Rightarrow f + f' \in V, \quad (f+f')(x) = f(x) + f'(x)$$

V is in fact a complex inner product space:

$$[f, f']_G = [f, f'] = \frac{1}{|G|}\sum_{x\in G} f(x)\overline{f'(x)} \qquad [f', f] = \overline{[f, f']}$$

sesquilinear

$1\frac{1}{2}$- linear

$\dim V$ = number of conjugacy classes.

The irreducible characters of G always give an orthonormal basis for $V = \{\text{class functions on } G\}$.

Recall the permutation representation with permutation character

$$\pi : S_3 \longrightarrow GL_3(\mathbb{C})$$

$$\pi : e_i \longmapsto e_{\sigma(i)}$$

$$(12) \longmapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(123) \longmapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$() \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\chi(()) = 3$

$\chi((12)) = 1$

$\chi((123)) = 0$

Character table of S_3

	$()$	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1
χ	3	1	0

$$[\chi_1, \chi_1] = \frac{1}{6}\sum_{g\in S_3} \chi_1(g)\overline{\chi_1(g)}$$
$$= \frac{1}{6}(1\cdot1 + 1\cdot1 + 1\cdot1 + 1\cdot1 + 1\cdot1 + 1\cdot1)$$
$$= 1$$

$$[\chi_1, \chi_2] = \frac{1}{6}\sum_g \chi_1(g)\overline{\chi_2(g)}$$
$$= \frac{1}{6}(1\cdot1 - 1\cdot1 - 1\cdot1 - 1\cdot1 + 1\cdot1 + 1\cdot1)$$
$$= 0$$

$$[\chi_1, \chi_3] = \frac{1}{6}(1\cdot2 + 0 + 0 + 0 - 1 - 1) = 0$$

$$[\chi_2, \chi_2] = \frac{1}{6}(1+1+1+1+1+1) = 1$$

$$[\chi_2, \chi_3] = \frac{1}{6}(2+0+0+0-1-1) = 0$$

$$[\chi_3, \chi_3] = \frac{1}{6}(4+0+0+0+1+1) = 1$$

$$\left.\begin{aligned} [\chi, \chi_1] &= \frac{1}{6}(1\cdot3 + 1 + 1 + 1 + 0) = 1 \\ [\chi, \chi_2] &= \frac{1}{6}(3 - 1 - 1 - 1 + 0 + 0) = 0 \\ [\chi, \chi_3] &= \frac{1}{6}(6+0+0+0+0+0) = 1 \end{aligned}\right\} \Longrightarrow \pi \cong \pi_1 \oplus \pi_3$$

The long way to check the decomposition $\pi \simeq \pi_1 \oplus \pi_3$ is found in the course notes: with respect to new basis

$$\underbrace{v_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}}_{\text{basis for first invariant subspace}}, \quad \underbrace{v_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix}, \quad v_3 = \begin{bmatrix}1\\0\\-1\end{bmatrix}}_{\text{basis for second invariant subspace}}$$

$\pi(g)$ has matrix $\left[\begin{array}{c|cc}\pi_1(g) & 0 & 0\\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\pi_3(g)$}}\\ 0 & & \end{array}\right]$ with respect to v_1, v_2, v_3

(see p. 3 of course notes).

Character tables also have orthogonality of columns.

$\lvert C_G(g)\rvert$	6	2	3
g	()	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$C_G(g) = \{x \in G : xg = gx\}$

The conjugacy class of $g \in G$ is the index of the centralizer

$$[G : C_G(g)] = \frac{|G|}{|C_G(g)|}$$

The column orthogonality is expressed as: given $g, h \in G$,

$$\sum_{i=1}^{k} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} 0 & \text{if } g,h \text{ not conjugate} \\ |C_G(g)| & \text{if } g,h \text{ are conjugate}\end{cases}$$

(k = number of irreducible characters = no. of conjugacy classes)

Row orthogonality says: Let $g_1, \dots, g_k \in G$ be reps of conj. classes

$$[f, f']_G = \frac{1}{|G|}\sum_{a \in G} f(a)\overline{f'(a)} = \cancel{\frac{1}{|G|}}\sum_{j=1}^{k} \frac{\cancel{|G|}}{|C_G(g_j)|} f(g_j)\overline{f'(g_j)} = \sum_{j=1}^{k} \frac{1}{|C_G(g_j)|} f(g_j)\overline{f'(g_j)}$$

$$[\chi_i, \chi_j] = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j\end{cases}$$

Let's construct the character table of A_5, $|A_5| = 60$, the smallest nonabelian simple group.
$= 2^2 \cdot 3 \cdot 5$

$G = A_5$ has $k = 5$ conjugacy classes

$g_1 = ()$	identity	1A	size 1	$\lvert C_G(g_1)\rvert = 60$
$g_2 = (12)(34)$		2A	size 15	$\lvert C_G(g_2)\rvert = 4$
$g_3 = (123)$		3A	size 20	$\lvert C_G(g_3)\rvert = 3$
$g_4 = (12345)$		5A	size 12	$\lvert C_G(g_4)\rvert = 5$
$g_5 = (12354)$		5B	size $\underline{12}$	$\lvert C_G(g_5)\rvert = 5$
			60	

(12345) and (12354) are conjugate in S_5 since $(45)^{-1}(12345)(45) = (12354)$

$(1234)^{-1}(12345)(1234) = (23415) = (15234)$

but (12345) and (12354) are not conjugate in A_5. $(1235)^{-1}(12345)(1235) = (23541) = (12354)$

Character table of A_5:

$\lvert C_G(g)\rvert$	60	4	3	5	5
g	$()$	$(12)(34)$	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	3				
χ_3	3				
χ_4	4				
χ_5	5				
χ	5	1	2	0	0

principal character $\chi_1(g) = tr(\pi_1(g)) = tr([1]) = 1$

standard permutation character

$\sum_{j=1}^{k} |\chi_j(1)|^2 = |G|$ special case of $\sum_{j=1}^{k} |\chi_j(g)|^2 = |C_G(g)|$.

Write $60 = n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2$ where n_1, n_2, n_3, n_4, n_5 are positive integers.

$\chi = a_1\chi_1 + a_2\chi_2 + \cdots + a_5\chi_5$. $a_i \in \underline{\{0,1,2,\cdots\}}$ is the number of copies of χ_i in χ.

$[\chi, \chi_i] = a_i$

$[\chi, \chi] = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = \frac{25}{60} + \frac{1}{4} + \frac{4}{3} + \frac{0}{5} + \frac{0}{5} = \frac{25+15+80}{60} = \frac{120}{60} = 2$.