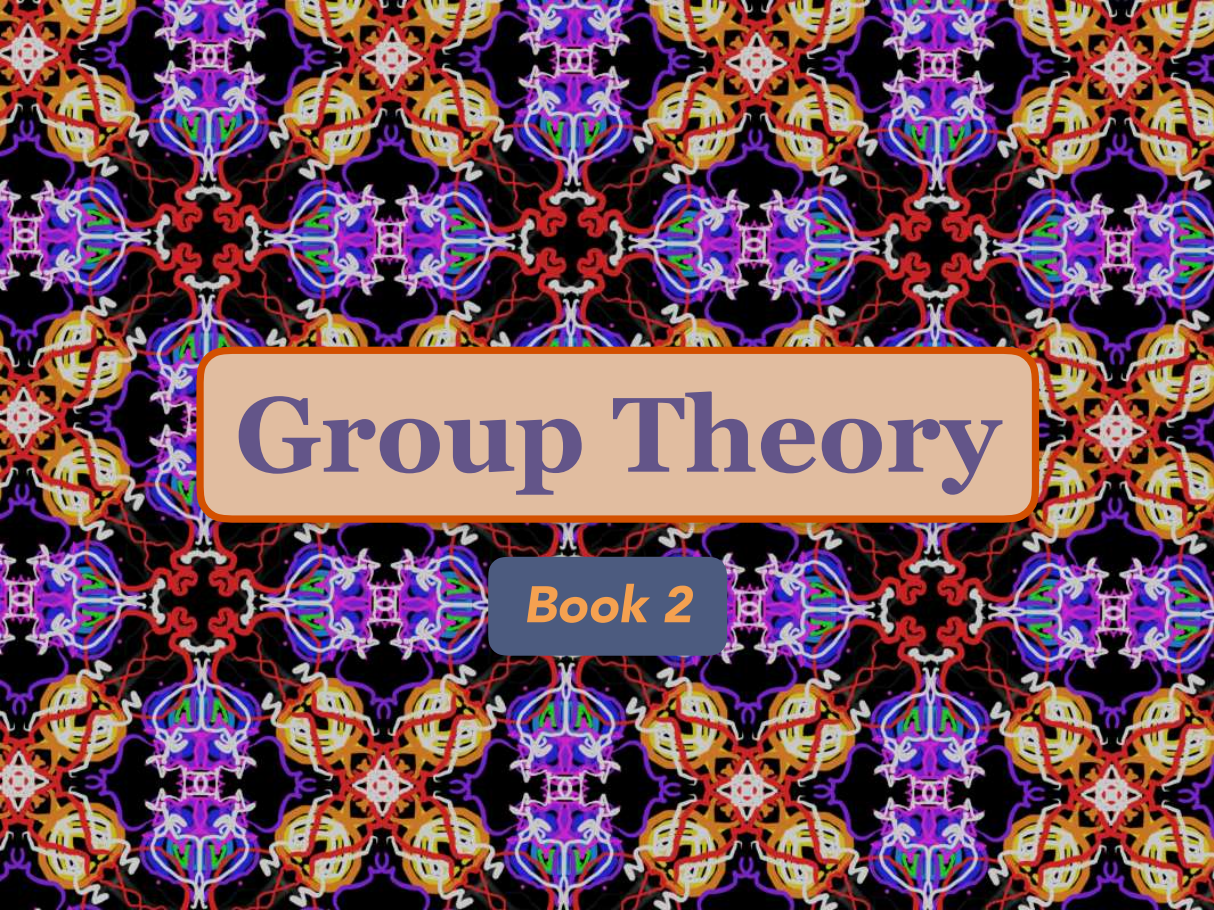


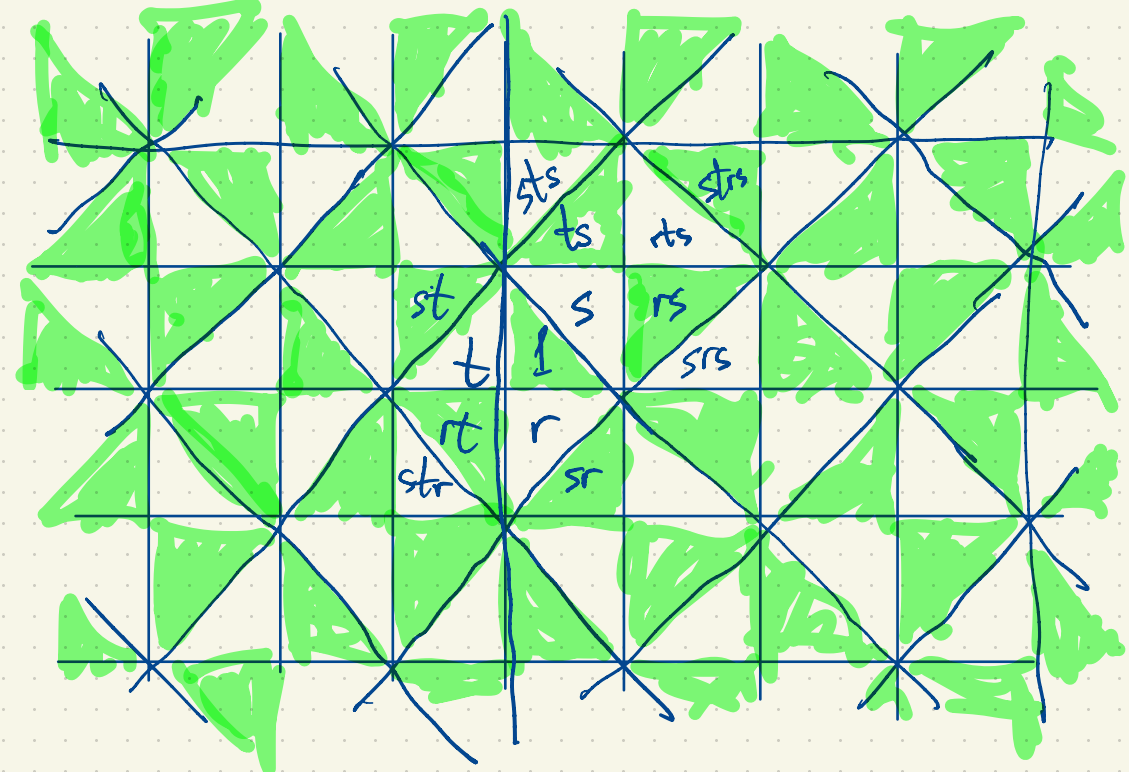
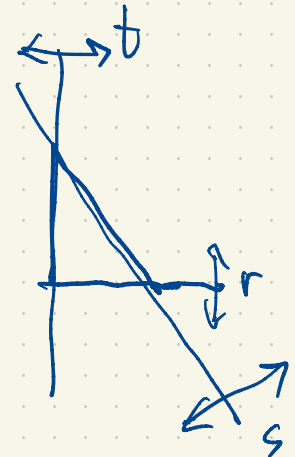


Group Theory

Book 2

reflections interchange
white $\leftrightarrow$ green
triangles

Elements of $G = G(4,4,2)$
white $\leftrightarrow$ map green $\rightarrow$



$G = G(4,4,2)$ labels the green triangles
W labels all triangles



More generally if $l, m, n \geq 2$ satisfying
 $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1$ then $G = G(l, m, n)$ is a group of
isometries of the Euclidean plane generated by rotations
of order l, m, n

$$G = \langle a, b, c \mid a^l, b^m, c^n, abc \rangle$$

Moreover $[W:G]$ where $W = \langle r, st \mid r^2, s^2, t^2, (rs)^l, (st)^m, (tr)^n \rangle$

If $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1$, $G = G(l, m, n)$ finite
 then in place of a tiling of the Euclidean plane, we get a
 tiling of S^2 (Euclidean sphere).

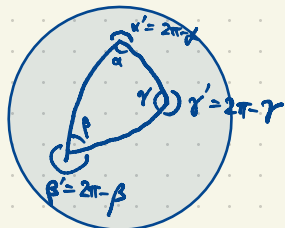
If $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$, then we get a tiling of the hyperbolic plane by congruent triangles.
 $C = G(l, m, n)$ infinite

A spherical example: $G = G(2, 3, 4)$, $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{6+4+3}{12} = \frac{13}{12} > 1$

Eine kleine spherical geometry

Let $S \subset \mathbb{R}^3$ be a unit sphere; its surface area is $4\pi r^2 = 4\pi$. "Lines" on S are geodesics (great circles).

Triangles in S have area = angular excess = $\alpha + \beta + \gamma - \pi > 0$



$$(\alpha + \beta + \gamma - \pi) + (\alpha' + \beta' + \gamma' - \pi) = 6\pi - 2\pi = 4\pi$$

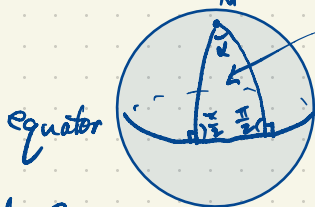
angular excess of "inside" angular excess outside"

W consists of isometries of S preserving the tiling.

$$[W:G] = 2, \quad G = G(2, 3, 4)$$

$$G = \langle a, b, c \mid a^2, b^3, c^4, abc \rangle$$

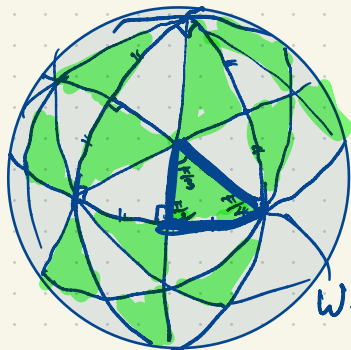
rs st tr



$$\text{Area} = \alpha + \frac{\pi}{2} + \frac{\pi}{2} - \pi = \alpha$$

$$|G| = 24.$$

$$G \cong S_4$$



We subdivide S

into 48 congruent spherical triangles, each with angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$

$$\text{and area } \frac{4\pi}{48} = \frac{\pi}{12} = \frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{4} - \pi$$

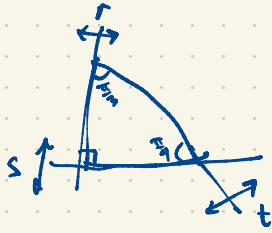
r, s, t : reflections of S in the "lines" bounding one triangle
 i.e. reflections in the planes through the origin

$$W = W\left(\begin{smallmatrix} 3 & 4 \\ \longrightarrow \end{smallmatrix}\right) = \langle r, s, t \mid r^2, s^2, t^2, (rs)^2, (rt)^2, (st)^2 \rangle,$$

$$W \cong C_2 \times S_4$$

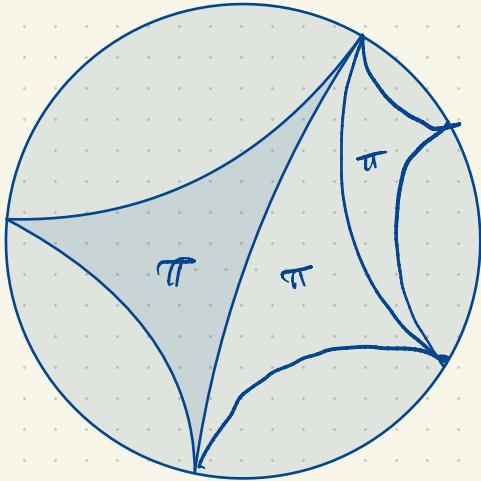
$$|W| = 48$$

Starting Fri Oct 3 new room is BU24



Case $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$: $G(l, m, n)$ preserves a triangulation of the hyperbolic plane with triangles having angles $\frac{\pi}{l}, \frac{\pi}{m}, \frac{\pi}{n}$

In hyperbolic plane, a triangle (sides are lines = geodesics) having angular defect $\pi - (\alpha + \beta + \gamma) = \text{area}$.

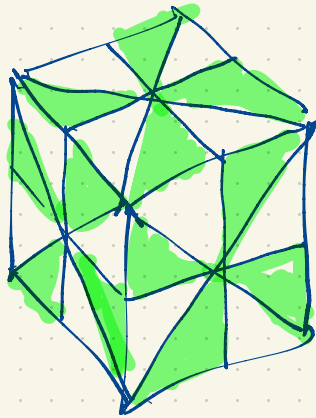
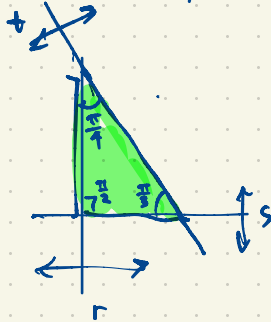


Today: $G = G(2,3,4) = \langle a, b, c : a^2, b^3, c^4, abc \rangle \cong S_4$

$$W = W(\overleftrightarrow{\quad}) \cong C_2 \times S_4$$

$$= \langle r, s, t \mid r^2, s^2, t^2, (rs)^2, (rt)^3, (st)^4 \rangle$$

To recognize $G \cong S_4$ without computer: $G = \langle rs, st, tr : rs, st, tr \text{ are rotations by angles } \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4} \text{ about vertices of triangle shown} \rangle$



$G =$ Group of rotational symmetries of a cube $\cong S_4$.

G permutes the four "body diagonals" of the cube in all $4! = 24$ ways. (A "body diagonal" joins two opposite vertices.)

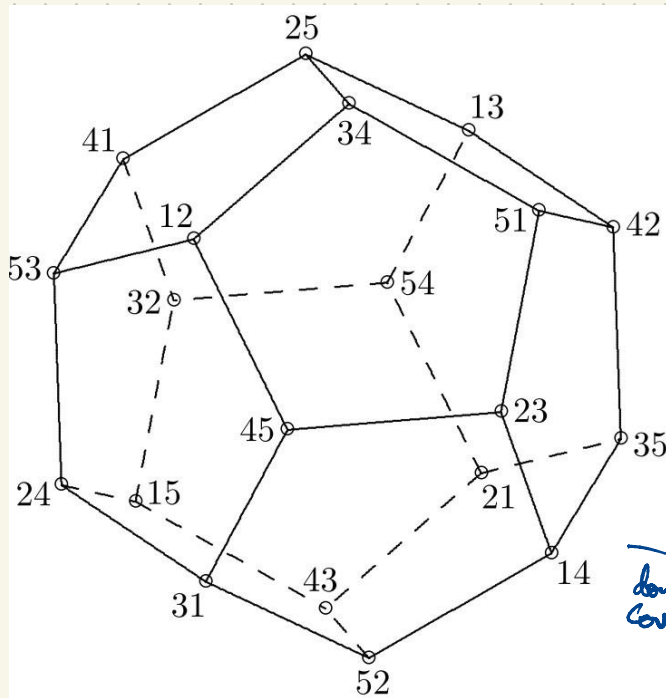
G permutes the 24 green triangles transitively (and regularly).

If $W = HK$ where H and K are normal subgroups with $H \cap K = 1$ (so H and K commute with each other i.e. $hk = kh$ for all $h \in H, k \in K$) then $W \cong H \times K = \{(h, k) : h \in H, k \in K\}$ (direct product)

In our case $|W| = 48$, $|G| = 24$, $W = G \cup Gr$, W has a subgroup $H = \langle h \rangle$ of order 2, $H \triangleleft W$, $h = -I$
 $H = Z(W)$
↑ preserve orientation
↑ reverse orientation

Cube has $3 + 6 = 9$ planes of symmetry
 but altogether 24 orientation-reversing symmetries

Similarly the regular dodecahedron (12 pentagonal faces) has rotational symmetry group G with $|G| = 60$, $G = A_5$ (Alt_5)

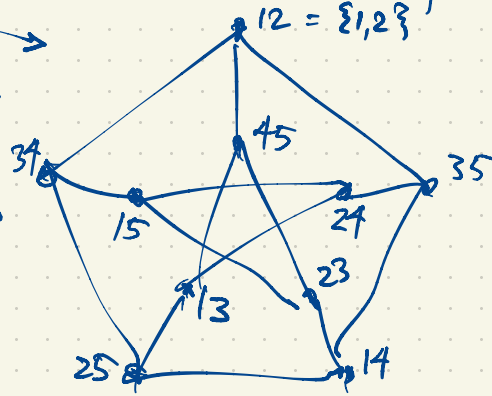


Elements of A_5 give rotational symmetries of the dodecahedron.

The full symmetry group of the regular dodecahedron has order 2×60
 (60 orientation-preserving rotations and 60 orientation-reversing reflections + other)

The full group of symmetries has order 120 and it has a subgroup isomorphic to A_5 but the group is not S_5 . S_5 is not a group of isometries of $\mathbb{R}^3$. Instead the full group of isometries of the regular dodecahedron is isomorphic to $C_2 \times A_5$.

double cover



This graph, the Petersen graph, has isomorphism group S_5

$(i, j) \mapsto \{i, j\}$
 $(j, i) \mapsto \{i, j\}$

A presentation of a group G is an expression $G = \langle X : R \rangle$ where X is a set of symbols (letters) and $R \subset F(X) = \text{free group on } X = \{x_1^{j_1} x_2^{j_2} \dots x_n^{j_n} : x_j \in X, j \in \mathbb{Z}\}$

X = set of "generators"

R = set of "relators" (words in the generators)

$$x_j^j x_k^{j_k} = x_k^{j_k} x_j^{j_j} \quad (j, k \in \mathbb{Z}; x \in X)$$

$$x^0 = 1 = \text{identity.}$$

If X is finite then G is finitely generated.

If X and R are both finite then G is finitely presented.

Burnside groups are finitely generated (usually) but not finitely presented.

$G = F / \text{subgroup of } F \text{ generated by } R \text{ and their conjugates}$

= "largest" homomorphic image of $F = F(X)$ having R in kernel
universal as we'll discuss later - see handout.

Every group has a presentation. Given G , for every $g \in G$, introduce a generator x_g . So $X = \{x_g : g \in G\}$.

For every pair $g, h \in G$ we want to force $x_g x_h = x_{gh}$ but this doesn't happen in $F = F(X)$ so introduce relators $x_g x_h x_{gh}^{-1} \in R$. $R = \{x_g x_h x_{gh}^{-1} : g, h \in G\}$. Then $F / \langle \dots R \dots \rangle \cong G$.

If G is finitely generated then G is countable i.e. finite or countably infinite.

If $X = \{x_1, \dots, x_n\}$ then $F = F(X) = \{\text{products of } x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}\} = \bigcup_{l=0}^{\infty} S_l$ where

F is a countable union of finite sets, hence countable.

$$S_l = \{x_{i_1}^{e_1} x_{i_2}^{e_2} \dots x_{i_l}^{e_l} : i_1, \dots, i_l \in \{1, \dots, n\}, e_j \in \{\pm 1\}\}$$

$$|S_l| \leq 2^l$$

If G is countably generated (X countable) then $F = F(X)$ is countable.

$S_1 = \{\text{words of length } 1\}$ is countable.

If $|X| = m$ and $|R| = n$, m, n pos. integers, what can we say about $|G|$ where $G = \langle X | R \rangle$?
 If $n < m$ then G is infinite. That is (contrapositive form) in order for $|G| < \infty$, we need at least as many relators as generators.

eg. $m=1$. $X = \{x\}$. If $R = \emptyset$ then $G = \langle x | \emptyset \rangle = \{\dots, x^2, x^{-1}, 1, x, x^2, \dots\}$

If $R = \{1\}$ then $G = \langle x | 1 \rangle = F(x) = \{\dots, x^2, x^{-1}, 1, x, x^2, \dots\}$

If $R = \{x^{15}\}$ then $G = \langle x | x^{15} \rangle = \{1, x, x^2, \dots, x^{14}\} \cong C_{15}$

If $R = \{x^{15}, x^{40}\}$ then $G = \langle x | x^{15}, x^{40} \rangle = \langle x | x^5 \rangle = \{1, x, x^2, x^3, x^4\}$

$$x^5 = (x^{15})^3 (x^{40})^{-1}$$

If G is a group then a homomorphic image of G is the image of G under a homomorphism

$$G \xrightarrow{\phi} H \quad (\text{surjective, otherwise replace } H \text{ by } \phi(G) \leq H.)$$

Note: $H \cong G/K$ where $K = \ker \phi \trianglelefteq G$. A homomorphic image of G is the same thing as a quotient group G/K , $K \trianglelefteq G$.

In particular the abelianization of G is the largest homo. image of G which is abelian.

A normal subgroup $K \trianglelefteq G$ yields an abelian quotient group G/K iff $K \supseteq G' =$ derived subgroup of G

$$\text{i.e. } G' = \langle [g, h] : g, h \in G \rangle, \quad [g, h] = g^{-1}h^{-1}gh.$$

If $K \supseteq G'$ then in G/K , take any two elements $gK, hK \in G/K$, we have

$$hg[g, h] = hg g^{-1}h^{-1}gh = gh$$

eg. the abelianization of an abelian gp. is itself.

$$(hk)(gK) = ghK = (gK)(hK) \quad \text{and conversely.}$$

eg. the abelianization of S_n , $n \geq 2$ is C_2 (group of order 2).

The abelianization of $GL_n(F)$ ($n \geq 1$) is $F^* = \{\text{nonzero elements of } F\}$
 $GL_n(F) \xrightarrow{\det} F^* \quad \ker(\det) = SL_n(F)$

If $F = F(X)$ free group on n generators $X = \{x_1, \dots, x_n\}$ eg. $n=3$

The abelianization of F

$F/F' \cong \mathbb{Z}^n$ where $F' = \{w \in F : \text{every } x_i \text{ has exponents in } w \text{ adding to } 0\}$
 (free abelian group on n generators)

$w = x_1 x_2^2 x_1^{-4} x_3^2 x_1 x_2^{-1} \mapsto (-2, 2, 2) \in \mathbb{Z}^3$

(Multiplicatively, $x_1^{-2} x_2^2 x_3^2$)

$\mathbb{Z}^3 / \langle (1, 2, 3), (4, 9, -1) \rangle$

Now consider $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle$, $r_1, \dots, r_n \in F = F(x_1, \dots, x_n)$

$= F/K$, K is the normal subgp of F generated by $r_1, \dots, r_n$ and their conjugates in F .

We want to show that if $n < m$ then $|G| = \infty$.

To prove this, first consider $F/F'K$. Here $F', K \trianglelefteq F$.

$F'K = \{ab : a \in F', b \in K\} \trianglelefteq F$.

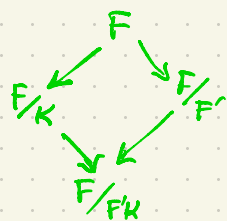
$F/F'K \cong \frac{F/F'}{F'K/F'}$ (Third Isomorphism Theorem also sometimes called the Second Isomorphism Theorem).

$F/F'K$ is a homomorphic image of $F/F' \cong \mathbb{Z}^m$.

$F/F'K \cong \mathbb{Z}^m / \text{subgp. generated by } n \text{ elements.}$

$F'K/F' \cong K/K \cap F'$ (Second Iso. Thm.)

$\frac{F/F'}{F'K/F'} \cong \frac{F/K}{F'K/K}$



$$\mathbb{Z}^3 / \langle (1,0,0), (0,3,0) \rangle \cong (\mathbb{Z}_{12}) \oplus \mathbb{Z}$$

$$\mathbb{Z}^3 / \langle (1,0,0), (0,2,0), (0,0,3) \rangle \cong (\mathbb{Z}_{12}) \oplus (\mathbb{Z}_{32})$$

Coxeter-Todd coset enumeration is the best algorithm we have for deciding certain "word problems" in group theory but it doesn't work in all cases.

Group orders: Given a presentation $G = \langle X | R \rangle$ (X, R finite)

What is $|G|$? Is it finite?

Is G trivial?

Given two words in $F_x = F(x)$ do they yield the same element of G ?

The word problem for groups is undecidable.

Matrix mortality problem

You are given a positive integer n and a list of $n \times n$ integer matrices $A_1, \dots, A_k$.

Is there a finite product of these A_i that equals the zero matrix? This is a decision problem.

For $n=1$, this problem is decidable.

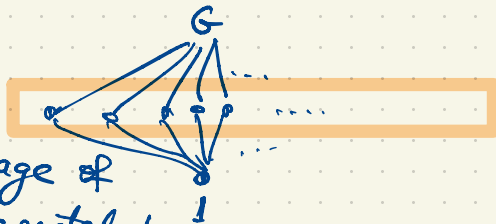
For $n \geq 3$, this problem is undecidable.

For $n=2$, we don't know whether the problem is decidable.

Suppose G is a Tarski monster for prime p i.e.

- G is an ^{infinite} nonabelian group
- $g^p = 1$ for all $g \in G$
- Given $x, y \in G$ with $x, y \neq 1$, either $\langle x, y \rangle$ is a subgroup of order p or $\langle x, y \rangle = G$.
- The only subgroups of G are

Then $|B(2, p)| = \infty$.



cyclic subgroups of order p .

This is because G is a homomorphic image of $B(2, p)$. $B(2, p)$ is the "largest" group generated by two elements a, b satisfying $g^p = 1$ for all g . (universal)

$$B(2, p) \twoheadrightarrow G \quad (\text{epimorphism})$$

$$B(2, p)/K \cong G \quad \text{for some normal subgroup } K \trianglelefteq B(2, p).$$

Since $|G| = \infty$, we must also have $|B(2, p)| = \infty$.

$B(2, p)$ has many subgroups not only cyclic of order p .

$B(2, p)$ is not simple. But G is simple: it has no normal subgroup other than 1 or G .

Proof: If G has a nontrivial normal subgroup N then it must be one of the cyclic subgroups of order p . There can't be more than one normal subgroup of order p . Why?

If N, N' are distinct normal subgroups (of order p) then $N \cap N' = 1$ and NN' is normal

$$NN' \cong N \times N' \quad \text{For all } x \in N, y \in N', \quad \overline{x'y'x}y = \underbrace{(\overline{x'y'x})}_N \underbrace{y}_{N'} \in N' \quad \text{and} \quad \overline{x'y'x}y = \underbrace{\overline{x'}}_N (\underbrace{y'xy}_N) \in N$$

So $x^{-1}y^{-1}xy = 1 \Rightarrow yx = xy \Rightarrow NN' = N \times N'$ of order p^2 . This cannot happen in a Tarski monster.

$$\text{Let } N = \{1, x, \dots, x^{p-1}\} \triangleleft G$$

$$N' = \{1, y, y^2, \dots, y^{p-1}\} < G$$

$y^{-1}xy \in N \Rightarrow y^{-1}xy = x^k$ for some $k \in \{1, 2, \dots, p-1\}$. Let $l \in \{1, 2, \dots, p-1\}$ be the inverse of k and p i.e. $kl \equiv 1 \pmod{p}$. Then

$$\text{Claim: } (y^l)^{-1}x y^l = x \quad (y^2)^{-1}x y^2 = y(y^{-1}xy)y = y^{-1}x^k y = \underbrace{(y^{-1}xy)(y^{-1}xy) \cdots (y^{-1}xy)}_{k \text{ times}}$$

If we replace $y \in N'$ by another generator of N'

then $y^{-1}xy = x$ i.e. $xy = yx$.

Then N and N' commute $\Rightarrow NN' \cong N \times N' \cong C_p \times C_p$, contradiction.

So G is simple.

Where do groups arise naturally? What are the natural examples of group operation?

- $+, \times$ in numbers
- functions under composition: permutation groups (subgroups of symmetric group), matrix groups (subgroups of $GL(V)$)
- fundamental groups in topology.
- group presentations $\langle X | R \rangle$

If F is a free group then every subgroup is free.

Nielsen-Schreier Theorem: Let F_n be a free group on n generators i.e. $F_n = \langle x_1, \dots, x_n \rangle$.

If $H \leq F_n$ with $e \neq [F_n : H]$ then $H \cong F_{1+(n-1)e}$.

Let X be a path-connected topological space X : given any points $x, y \in X$ there exists a path in X from x to y i.e. a continuous map $[0, 1] \rightarrow X$

$$\begin{array}{ccc} 0 & \mapsto & x \\ 1 & \mapsto & y \end{array}$$

eg. $X = \mathbb{R}^2 - \underline{D}$ is path-connected

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

$$\mathbb{R}^2 - (x\text{-axis}) = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$$

is not path-connected



In this example, γ and γ' are not homotopic.

γ' is homotopic to the constant path $\gamma_0: [0, 1] \rightarrow X$ for $\gamma_0(t) = x_0$ for all $t \in [0, 1]$

Given a path-connected space X , and a point $x_0 \in X$, we can consider closed paths in X from x_0 to x_0 which are continuous maps $\gamma: [0, 1] \rightarrow X$ st. $\gamma(0) = \gamma(1) = x_0$.

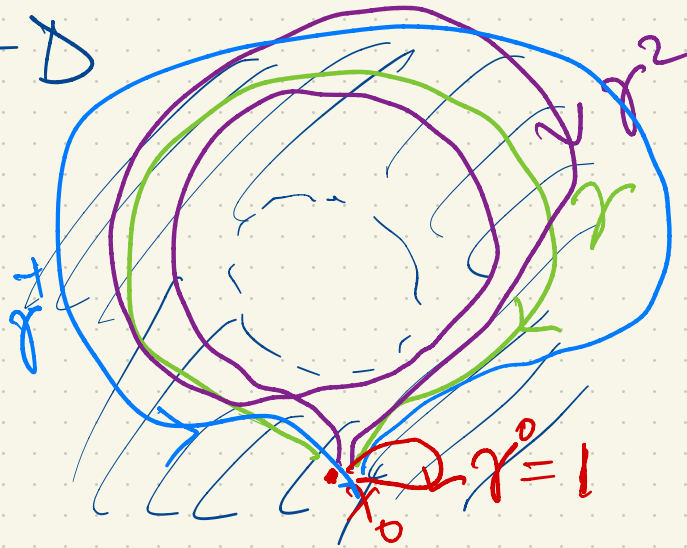
Two closed paths γ, γ' from x_0 to x_0 in X are homotopic if we can continuously distort γ to γ' while fixing the endpoints at x_0 and always staying inside X .

Write $[\gamma]$ as the equivalence class of γ under the equivalence relation of homotopy (with fixed base point x_0).

$\pi_1(X, x_0) = \{ [\gamma] : \gamma \text{ paths in } X \text{ from } x_0 \text{ to } x_0 \}$ is the fundamental group of X with base point x_0 .

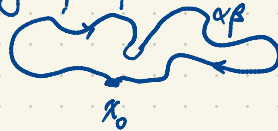
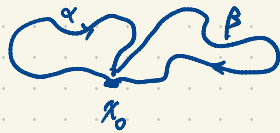
The group operation is "concatenation" of paths.

In $X = \mathbb{R}^2 - D$



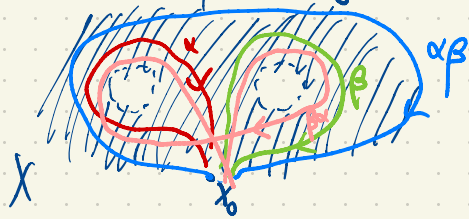
$$\pi_1(\mathbb{R}^2 - D) = \langle \gamma \rangle \text{ free gp on one generator} \cong \mathbb{Z}$$

In a top. space X , we fix a base point $x_0 \in X$ and consider the group $\pi_1(X, x_0)$ whose elements are flexible strings in X starting and ending at x_0 . The identity element is a trivial string in X which "shrinks" to the point x_0 . The inverse of any path/string is the same path/string in the reverse direction i.e. $\gamma^{-1}(t) = \gamma(1-t)$. ($0 \leq t \leq 1$). The group operation is splicing/concaterenation of strings.



$$(\alpha\beta)(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2}; \\ \beta(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

eg. X is a plane region with two holes:



$\pi_1(X, x_0) = \langle \alpha, \beta \rangle$ is a free group on n generators.
(Van Kampen Theorem)



Theorem $\pi_1(X, x_0) \cong \pi_1(X)$ independent of choice of base point assuming X is path-connected.
 $\pi_1(X)$ is the fundamental group of X or the first homotopy group of X . Its elements are (homotopy classes of) loops with base point (pointed copies of S^1).

$S^1 \times S^1 = \text{torus}$ has $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$ free abelian group on two generators.



$P^2\mathbb{R} =$  closed disk with antipodal boundary points identified.

real projective plane

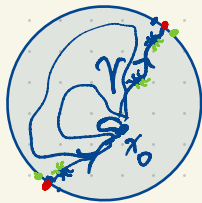
$\pi_1(P^2\mathbb{R})$ has order 2.



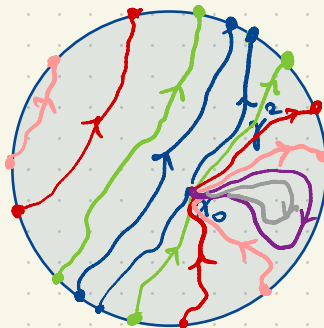
closed disk
in $\mathbb{R}$



Möbius
strip



γ^2 is homotopic to a trivial path
 so $\pi_1(P^2\mathbb{R}) = \mathbb{C}_2$ or $\mathbb{Z}/2\mathbb{Z}$.



$\pi_1(\text{real proj. plane}) \neq 1$
 there is a path $\gamma \in \pi_1(\text{real proj. plane})$
 which is not homotopic to a trivial path
 i.e. real proj. plane is not simply connected.

Example of a subgroup $H < F_2$, $[F_2 : H] = 2, \Rightarrow H \cong F_{1+2(2-1)} = F_3$
 (Nielsen-Schreier)

$F = F_2 = \langle a, b \rangle$, consider the homomorphism $F_2 \rightarrow C_2 = \{0, 1\}$, $a \mapsto 0, b \mapsto 1$.

Let $H < F_2$ be the kernel of this homomorphism eg.

$$a^0 b^1 a^0 b^1 \dots a^0 b^1 \mapsto \sum f_k \text{ mod } 2.$$

$$a^3 b a b^{-1} a^{-1} b^{-2} \notin H$$

$$F = H \sqcup Hb$$

$$b^3 a^{-3} b^{-1} a^1 \in H \text{ so } [F_2 : H] = 2.$$

Claim: $H = \langle a, b a b^{-1}, b^2 \rangle$ and moreover H is freely generated by $u = a, v = b a b^{-1}, w = b^2$ i.e. the free group $F_3 = \langle u, v, w \rangle$ is isomorphic to H with isomorphism $F_3 \rightarrow H$, $u \mapsto a, v \mapsto b a b^{-1}, w \mapsto b^2$.

Consider the digraph (directed graph)  Schreier graph on cosets of H .

Every vertex has "in-degree" 2 and "out-degree" 2;
in each case one edge labelled a , the other b .

Every element $w \in F_2$ defines a word or walk in our graph starting at vertex 0.

$a^3 b a b^{-1} a^{-1} b^{-2} \notin H$ defines a walk starting at 0, ending at 1.

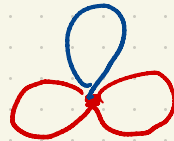
$b^3 a^{-3} b^{-1} a^4 \in H$ defines a walk starting and ending at 0.

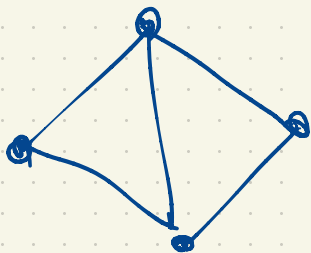
Given $w \in F_2$, we have $w \in H$ iff the walk of type w in our graph starting at 0 ends at 0. (Clear.)

$[F:H] = 2$ To see $H \cong F_2$, observe $H \cong \pi_1(X, x_0)$ where X is a "connected region in the plane with 3 'holes'".



$\sim$
has the
same homotopy
type $\square$





Given a graph Γ , the Euler characteristic of Γ is
(possibly with loops and multiple edges)
(undirected)

$$\chi(\Gamma) = \sum_i (-1)^i \dim C_i = \sum_i (-1)^i \dim H_i$$

$$i \in \{0, 1\} \text{ in the case of a graph} \quad = \dim C_0 - \dim C_1 = \underbrace{\dim H_0}_{1} - \underbrace{\dim H_1}_{1}$$

$C_0 =$ vector space freely spanned by 0-cells = vertices

$\dim C_0 =$ no. of vertices

$C_1 =$... 1-cells = edges

$\dim C_1 =$... edges

$\dim H_0 =$ no. of connected components = 1 if Γ is connected

$\dim H_1 =$... independent cycles = no. of edges that must be cut to make Γ into a tree (no cycles)

1 - no. of independent cycles = no. of vertices - no. of edges.

$$1 - 2 = 4 - 5 = -1 = \chi(\Gamma).$$



Nielsen-Schreier Theorem: Every subgroup of a free group is free. Moreover if $F = F_n$ (free of rank n) and $H \leq F$ is a subgroup of finite index e , then H is free of rank $1 + (n-1)e$.

Given a connected graph Γ (which may have loops and multiple edges, but ignore arrows for now and view it as an undirected graph), what is the fundamental group of this graph?

If $\Gamma =$  then $\pi_1(\Gamma) \cong F_2 \cong \pi_1(\text{figure 8})$

 and  have the same homotopy type



Having the same homotopy type is not the same thing as homeomorphic.

a coarser equiv. relation

a finer equiv. relation.

If two spaces are homeomorphic then they have the same homotopy type.

If X has the same homotopy type as Y then $H_i(X) \cong H_i(Y)$

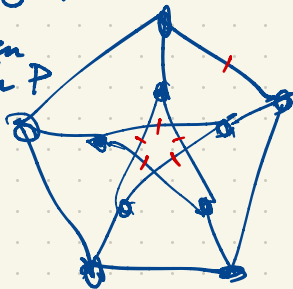
$$\pi_i(X) \cong \pi_i(Y)$$

$$\chi(X) = \chi(Y)$$

etc.

(connected)
For a graph Γ as above, $\dim H_1 =$ "no. of independent cycles" $= 1 - \text{no. of vertices} + \text{no. of edges}$.

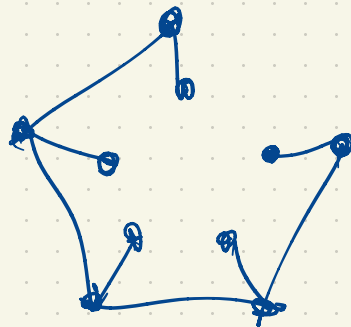
Petersen graph P



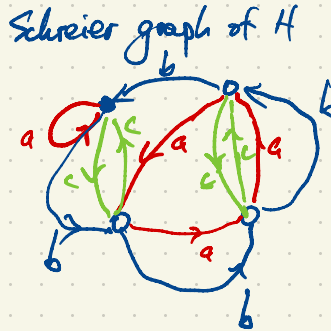
$$\dim H_1 = 1 - 10 + 15 = 6. \quad 6 \text{ independent cycles}$$

$\Rightarrow P$ has same homotopy type as

$$\pi_1(P) \cong F_6.$$



Imagine $F = F_3 = \langle a, b, c \rangle$ $n=3$
 $H \leq F$ of index $e=4$



H is a free group of rank equal to the number of independent cycles in its Schreier graph

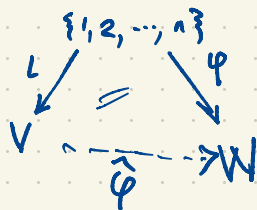
$$\text{rank of } H = 1 - 4 + 12 = 9$$

$$H \cong F_9$$

$$1 + (n-1)e = 1 + (3-1) \cdot 4 = 9.$$

What does it mean to say $\{v_1, \dots, v_n\}$ is a basis for a vector space V ?
 (Equivalently, V is freely spanned by $v_1, \dots, v_n$) (the underlying field of scalars is fixed throughout.)

V is freely spanned by $v_1, \dots, v_n$ iff given any vector space W containing vectors $w_1, \dots, w_n$, there is a unique linear transformation $V \rightarrow W$ mapping $v_i \mapsto w_i$ for all i .



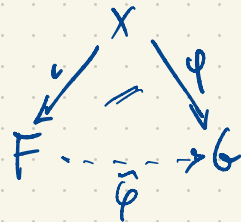
This diagram commutes
 i.e. $\hat{\varphi} \circ \iota = \varphi$

$$\iota(j) = x_j, \quad \varphi(j) = w_j$$

$$\hat{\varphi}(x_j) = w_j.$$

In the same way, we define free groups. Given a set X , and a map $i: X \rightarrow F$ where F is a group, we say F is free on X if the following universal property is satisfied:

Given any map $\varphi: X \rightarrow G$ where G is any group, then there is a unique group homomorphism $\hat{\varphi}: F \rightarrow G$ such that the following diagram commutes:



$$\text{i.e. } \hat{\varphi} \circ i = \varphi.$$

Although the definition does not explicitly require ("say") i is injective, this must be the case in order for the conclusion to hold i.e. for free groups to exist.

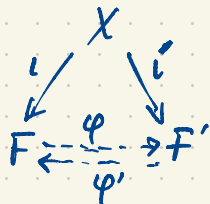
Existence? Given any set X , there is an $i: X \rightarrow F$, F group free on X .

Uniqueness?

If $X = \{1, 2, \dots, n\}$ take $F =$ set of all words (strings of letters) generated by symbols $x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}$. The only simplifications in F are

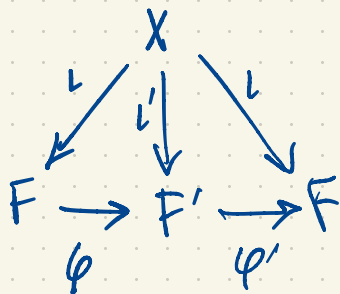
$$x_i^j x_i^k = x_i^{j+k}, \quad x_i^0 = 1, \quad |x_i = x_i^{-1}$$

If $i: X \rightarrow F'$ is also free, then $F' \cong F$. This follows from the definitions.



$\varphi: F \rightarrow F'$ homo. makes the diagram commute i.e.

$$\varphi' \circ i' = i$$

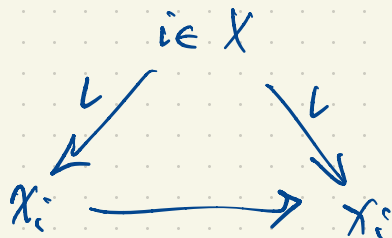


$$\varphi' \circ \varphi : F \rightarrow F$$

$$\varphi' \circ \varphi \circ L = L$$

and $\varphi' \circ \varphi$ is the only homs. $F \rightarrow F$
which does this

Also $\text{id}_F : F \rightarrow F$
 $g \mapsto g, \text{id}_F \circ L = L$



Uniqueness

$$\Rightarrow \varphi' \circ \varphi = \text{id}_F$$

$$\varphi \circ \varphi' = \text{id}_{F'}$$

$\Rightarrow \varphi, \varphi'$ are isomorphisms.

Given groups G, H , we define the free product $G * H$ of G and H .

Concretely: Take symbols for elements of G and for elements of H and generate strings using these symbols i.e.

$$\begin{aligned} & g_1 h_1 g_2 h_2 g_3 h_3 \dots g_k h_k \\ \text{or} & h_1 g_1 h_2 g_2 \dots h_k g_k \\ & g_1 h_1 h_2 g_2 \dots h_k g_k \\ & h_1 g_1 h_2 g_2 \dots g_k h_k \end{aligned} \quad \begin{aligned} & h_i \in H \\ & g_j \in G. \end{aligned}$$

Eg. a free product of n infinite cyclic groups is F_n (free group of rank n).

$$\begin{aligned} \mathbb{Z} &\cong F_1 \cong \{ \dots, a^3, a^2, a^1, 1, a, a^2, a^3, \dots \} \\ &\{ \dots, -3, -2, -1, 0, 1, 2, 3, \dots \} \\ \mathbb{Z} * \mathbb{Z} &= F_2 \\ &\langle a \rangle * \langle b \rangle = \langle a, b \rangle \end{aligned}$$

$$(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z}) = \{ 1, a, b, b^2, ab, ab^2, ba, b^2a, aba, ab^2a, bab, bab^2, b^2ab, b^2ab^2, \dots \}$$

$$\begin{aligned} C_2 * C_3 &= \langle a, b \mid a^2, b^3 \rangle \text{ is a homo. image of } F_2 \\ \{1, a\} &\quad \{1, b, b^2\} \\ a^2=1 &\quad b^3=1 \\ &= F_2 / K \quad K = \text{subgp. of } F_2 \text{ generated by } a^2, b^3, \\ &\quad \text{and their conjugates } ba^2b^{-1}, \dots \end{aligned}$$

$$\mathrm{PSL}_2(\mathbb{Z}) \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right) * \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right) \cong \langle a, b \mid a^2, b^3 \rangle$$

$$\mathrm{SL}_2(\mathbb{Z}) = \langle a, b \mid \underbrace{a^4 = b^6 = 1}_{\text{relations}}, \underbrace{a^2 = b^3}_{\text{relators}} \rangle = \langle a, b \mid \underbrace{a^4, b^6, a^2 b^{-3}}_{\text{relators}} \rangle$$

$$a = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

↙ $G_1 \amalg G_2$

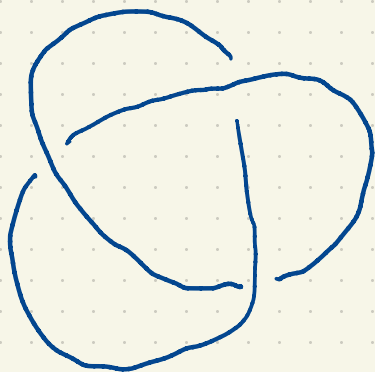
Given two groups G_1, G_2 , a free product of G_1 and G_2 , denoted $G_1 * G_2$, is a group together with homomorphisms $i_1: G_1 \rightarrow G_1 * G_2$, $i_2: G_2 \rightarrow G_1 * G_2$ such that for every group G and every pair of homomorphisms $f_1: G_1 \rightarrow G$, $f_2: G_2 \rightarrow G$ there exists a unique group homo. $f: G_1 * G_2 \rightarrow G$ such that the following diagram commutes:

$$\begin{array}{ccccc} G_1 & \xrightarrow{i_1} & G_1 * G_2 & \xleftarrow{i_2} & G_2 \\ & \searrow f_1 & \downarrow f & \swarrow f_2 & \\ & & G & & \end{array}$$

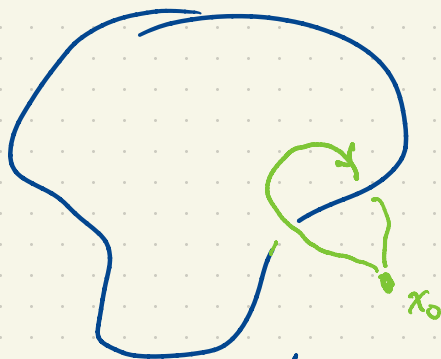
$$\begin{aligned} \text{i.e. } f \circ i_1 &= f_1 \\ f \circ i_2 &= f_2 \end{aligned}$$

Fact: i_1, i_2 must be injective.

$\mathrm{SL}_2(\mathbb{Z})$ is almost the same as $\left(\frac{\mathbb{Z}}{4\mathbb{Z}}\right) * \left(\frac{\mathbb{Z}}{6\mathbb{Z}}\right)$ but we introduce amalgamation by way of the additional relation $a^2 = b$.



$\neq$

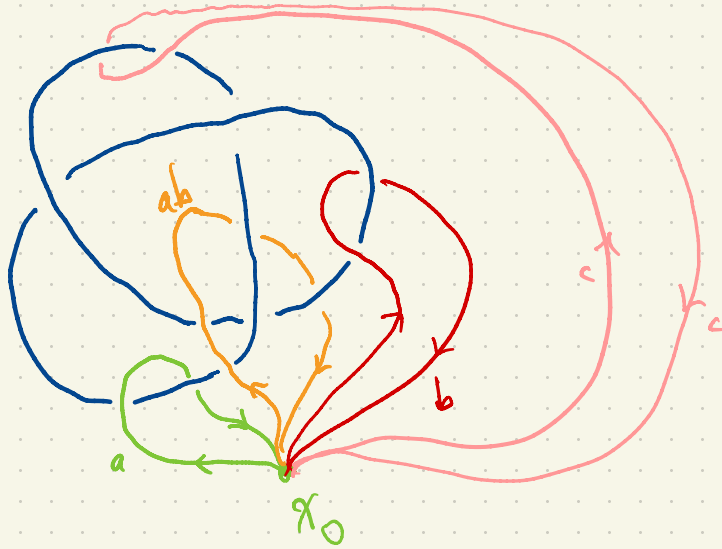


K : trefoil.

unknot

A knot is an embedding of S^1 into $\mathbb{R}^3$.

$$\pi_1(\mathbb{R}^3 - K) \not\cong \pi_1(\mathbb{R}^3 - \text{unknot}) = \mathbb{Z}$$



$$ab = bc = ca$$

$$\Rightarrow c = b^{-1}ab = aba^{-1}$$

$$\begin{aligned} \pi_1(\mathbb{R}^3 - K) &= \langle a, b, c : ab = bc = ca \rangle \\ &= \langle x, y : x^3 = y^2 \rangle \end{aligned}$$

$$\left. \begin{aligned} x &= ab = bc = ca \\ y &= abc \end{aligned} \right\} \Rightarrow \begin{aligned} x^3 &= ab \cdot ca \cdot bc \\ &= abc \cdot abc = y^2 \end{aligned}$$