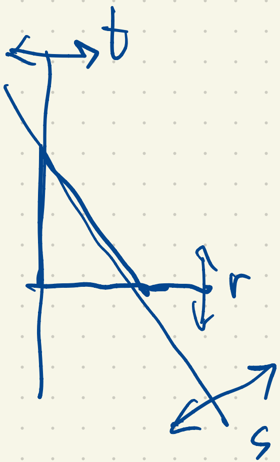
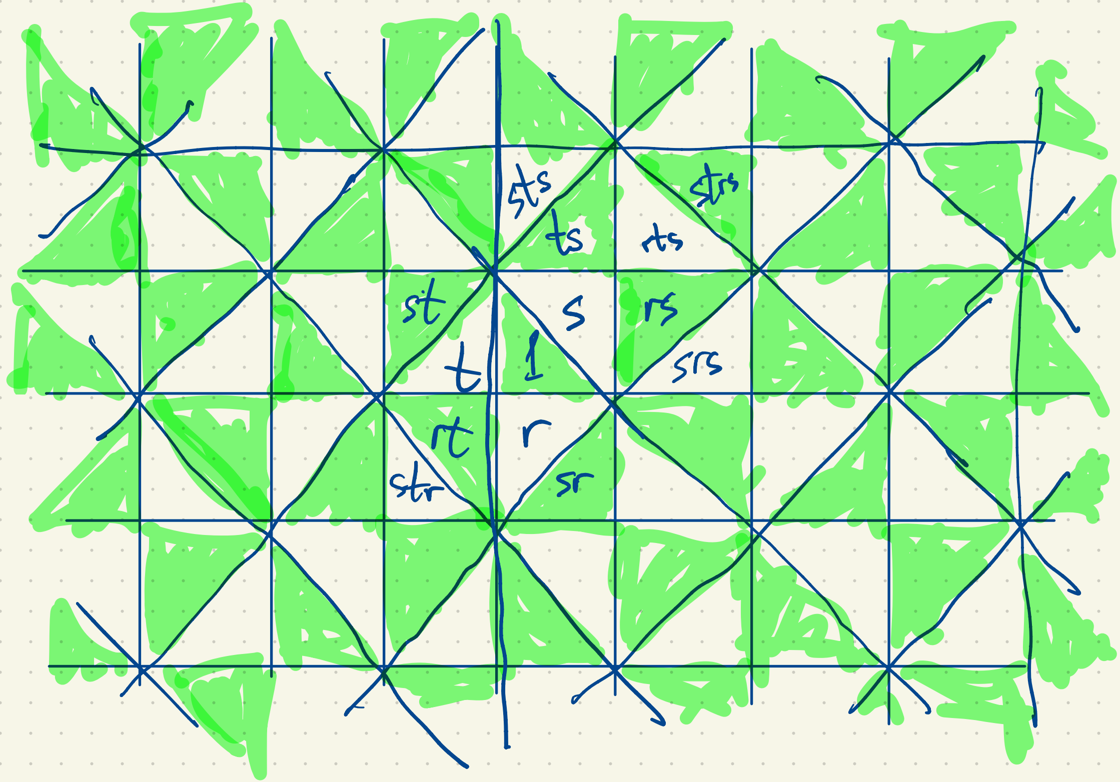
Group Theory

Book 2

reflections interchange white $\longleftrightarrow$ green triangles triangles

Elements of $G = G(4,4,2)$ map white? green?





$G = G(4,4,2)$ labels the green triangles
W labels all triangles

More generally if $l, m, n \geq 2$ satisfying $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1$ then $G = G(l,m,n)$ is a group of isometries of the Euclidean plane generated by rotations of order l, m, n:

$$G = \langle a, b, c \mid a^l, b^m, c^n, abc \rangle$$

Moreover $[W:G]$ where $W = \langle r, s, t \mid r^2, s^2, t^2, (rs)^l, (st)^m, (tr)^n \rangle$.

If $\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}>1$, $G=G(\ell,m,n)$ finite then in place of a tiling of the Euclidean plane, we get a tiling of S^2 (Euclidean sphere).

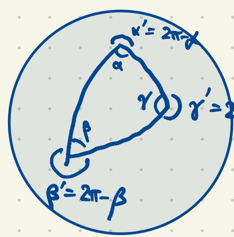
If $\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}<1$, then we get a tiling of the hyperbolic plane by congruent triangles.
$G=G(\ell,m,n)$ infinite

A spherical example: $G=G(2,3,4)$, $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{6+4+3}{12}=\frac{13}{12}>1$

Eine Kleine spherical geometry

Let $S\subset\mathbb{R}^3$ be a unit sphere; its surface area is $4\pi r^2=4\pi$. "Lines" on S are geodesics (great circles).

Triangles in S have area = angular excess = $\alpha+\beta+\gamma-\pi>0$



$$\underbrace{(\alpha+\beta+\gamma-\pi)}_{\text{angular excess of "inside"}}+\underbrace{(\alpha'+\beta'+\gamma'-\pi)}_{\text{angular excess "outside"}}=6\pi-2\pi=4\pi$$

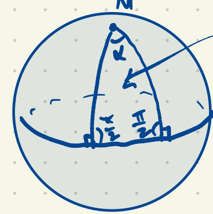
W consists of isometries of S preserving the tiling.

$[W:G]=2$, $G=G(2,3,4)$

$G=\langle \underset{rs}{a},\underset{st}{b},\underset{tr}{c}\mid a^2,b^3,c^4,abc\rangle$

N

equator



Area $=\alpha+\frac{\pi}{2}+\frac{\pi}{2}-\pi=\alpha$

$|G|=24$.

$G\cong S_4$

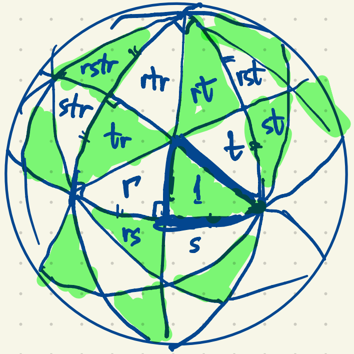
We subdivide S into 48 congruent spherical triangles each with angles $\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{4}$ and area $\frac{4\pi}{48}=\frac{\pi}{12}=\frac{\pi}{2}+\frac{\pi}{3}+\frac{\pi}{4}-\pi$

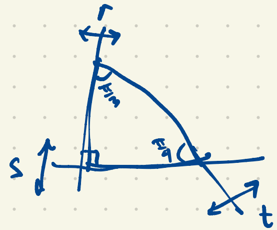
r,s,t: reflections of S in the "lines" bounding one triangle
i.e. reflections in the corresponding planes through the origin

$W=W(\overset{3}{\bullet\!-\!\!-}\overset{4}{\bullet\!=\!\!=\!\bullet})=\langle r,s,t\mid r^2,s^2,t^2,(rs)^2,(rt)^3,(st)^4\rangle$,

$W\cong C_2\times S_4$

$|W|=48$





Case $\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}<1$: $G(\ell,m,n)$ preserves a triangulation of the hyperbolic plane with triangles having angles $\frac{\pi}{\ell},\frac{\pi}{m},\frac{\pi}{n}$

In hyperbolic plane, a triangle (sides are lines = geodesics) having angular defect $\pi-(\alpha+\beta+\gamma)$ = area.