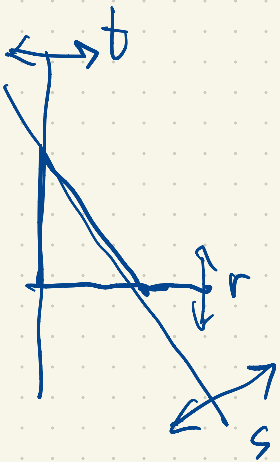
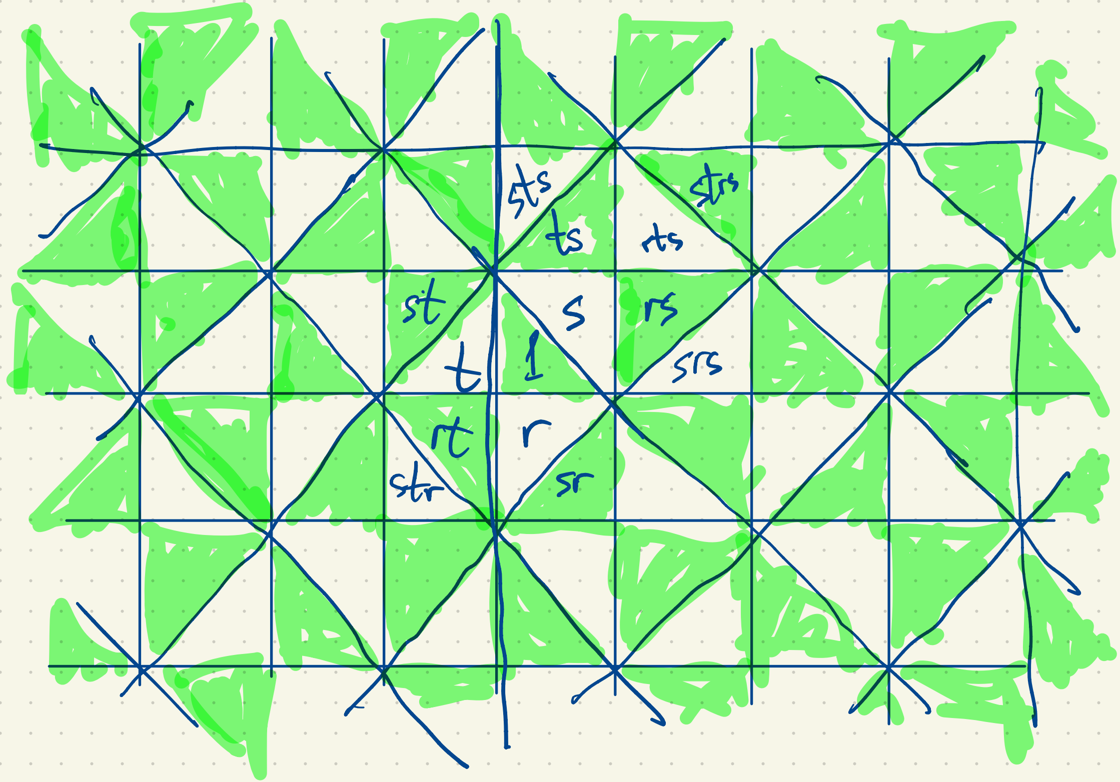
Group Theory

Book 2

reflections interchange
white triangles $\longleftrightarrow$ green triangles

Elements of $G = G_{map}(4,4,2)$
white? green?





$G = G\langle 4,4,2\rangle$ labels the green triangles
W labels all triangles

More generally if $l, m, n \geq 2$ satisfying $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1$ then $G = G(l,m,n)$ is a group of isometries of the Euclidean plane generated by rotations of order l, m, n:

$$G = \langle a, b, c \mid a^l, b^m, c^n, abc \rangle$$

Moreover $[W:G]$ where $W = \langle r, s, t \mid r^2, s^2, t^2, (rs)^l, (st)^m, (tr)^n \rangle$.

If $\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}>1$, $G=G(\ell,m,n)$ finite then in place of a tiling of the Euclidean plane, we get a tiling of S^2 (Euclidean sphere).

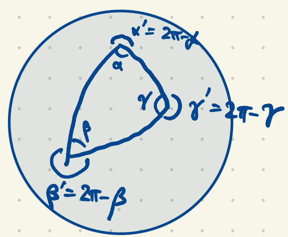
If $\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}<1$, then $G=G(\ell,m,n)$ infinite we get a tiling of the hyperbolic plane by congruent triangles.

A spherical example: $G=G(2,3,4)$, $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{6+4+3}{12}=\frac{13}{12}>1$

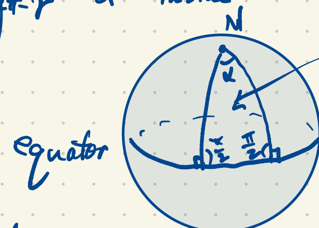
Eine Kleine spherical geometry

Let $S\subset\mathbb{R}^3$ be a unit sphere; its surface area is $4\pi r^2=4\pi$. "Lines" on S are geodesics (great circles).

Triangles in S have area = angular excess = $\alpha+\beta+\gamma-\pi>0$



$$\underbrace{(\alpha+\beta+\gamma-\pi)}_{\text{angular excess of "inside"}}+\underbrace{(\alpha'+\beta'+\gamma'-\pi)}_{\text{angular excess "outside"}}=6\pi-2\pi=4\pi$$



Area $=\alpha+\frac{\pi}{2}+\frac{\pi}{2}-\pi=\alpha$

$|G|=24$.

$G\cong S_4$

W consists of isometries of S preserving the tiling:

$[W:G]=2$, $G=G(2,3,4)$

$G=\langle \underset{rs}{a},\underset{st}{b},\underset{tr}{c}\mid a^2,b^3,c^4,abc\rangle$

We subdivide S into 48 congruent spherical triangles each with angles $\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{4}$ and area $\frac{4\pi}{48}=\frac{\pi}{12}=\frac{\pi}{2}+\frac{\pi}{3}+\frac{\pi}{4}-\pi$

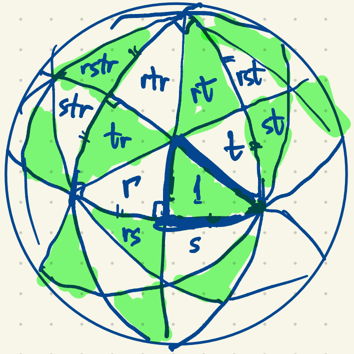
r,s,t: reflections of S in the "lines" bounding one triangle i.e. reflections in the corresponding planes through the origin

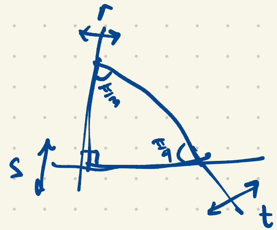
$W=W(\overset{3}{\bullet\!-\!\!-}\overset{4}{\bullet\!=\!=\!\bullet})=\langle r,s,t\mid r^2,s^2,t^2,(rs)^2,(rt)^3,(st)^4\rangle$,

$W\cong C_2\times S_4$

$|W|=48$

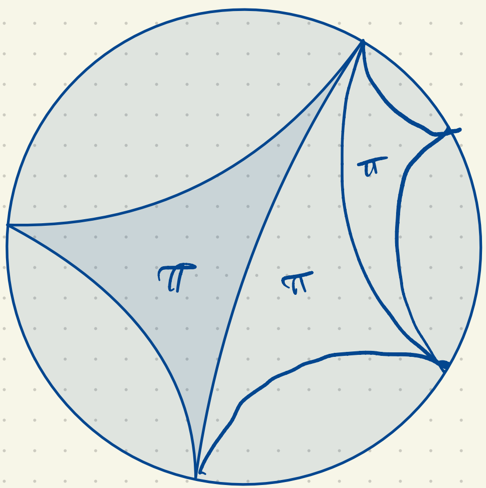
Starting Fri Oct 3 new room is BU24





Case $\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}<1$: $G(\ell,m,n)$ preserves a triangulation of the hyperbolic plane with triangles having angles $\frac{\pi}{\ell}, \frac{\pi}{m}, \frac{\pi}{n}$

In hyperbolic plane, a triangle (sides are lines = geodesics) having angular defect $\pi-(\alpha+\beta+\gamma)$ = area.

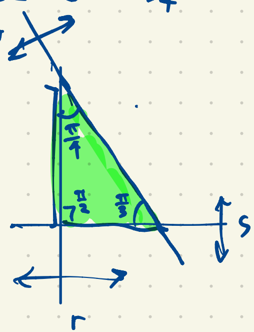


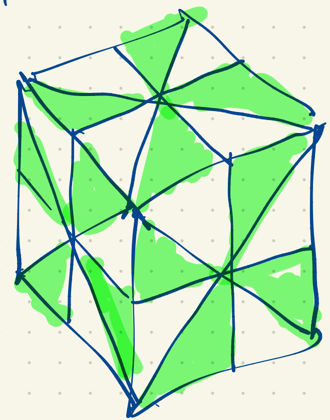
Today: $G = G(2,3,4) = \langle a,b,c : a^2, b^3, c^4, abc \rangle \cong S_4$

$W = W(\bullet\!\!-\!\!\!=\!\!\bullet) \cong C_2 \times S_4$ (B_2)

$= \langle r,s,t \mid r^2, s^2, t^2, (rs)^2, (rt)^3, (st)^4 \rangle$

To recognize $G \cong S_4$ without computer: $G = \langle rs, st, tr$: rs, st, tr are rotations by angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$ about vertices of triangle shown$\rangle$





G = Group of rotational symmetries of a cube $\cong S_4$.

G permutes the four "body diagonals" of the cube in all $4! = 24$ ways.
(A "body diagonal" joins two opposite vertices.)

G permutes the 24 green triangles transitively (and regularly).

If $W = HK$ where H and K are normal subgroups with $H \cap K = 1$ (so H and K commute with each other i.e. $hk = kh$ for all $h \in H$, $k \in K$) then $W \cong H \times K = \{(h,k) : h \in H, k \in K\}$ (direct product)

In our case $|W| = 48$, $|G| = 24$, $W = G \cup Gr$, W has a subgroup $H = \langle h \rangle$ of order 2, $H \triangleleft W$, $h = -I$

preserve orientation / reverse orientation

$H = Z(W)$

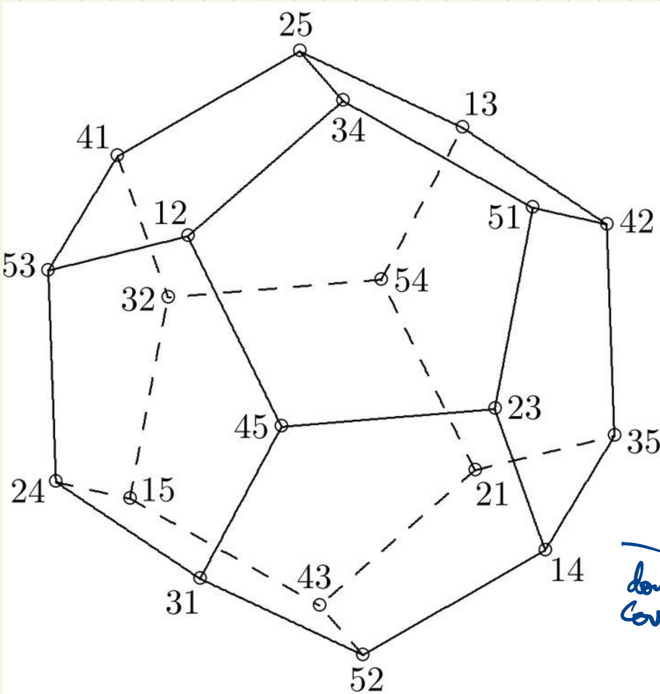
Cube has $3 + 6 = 9$ planes of symmetry but altogether 24 orientation-reversing symmetries

Similarly the regular dodecahedron (12 pentagonal faces) has rotational symmetry group G with $|G| = 60$, $G = A_5$ (Alt_5)

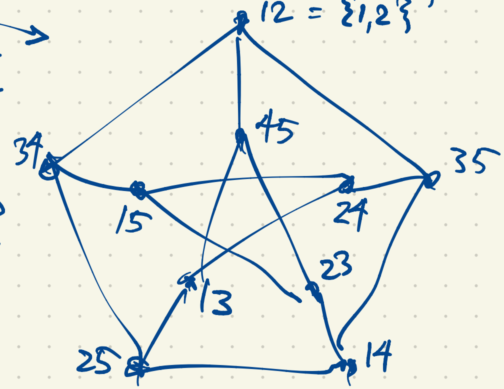
Elements of A_5 give rotational symmetries of the dodecahedron.

The full symmetry group of the regular dodecahedron has order 2×60 ($\underbrace{60 \text{ orientation-preserving}}_{\text{rotations}}$ and $\underbrace{60\ \text{orientation-reversing}}_{\text{reflections + other}}$)

The full group of symmetries has order 120 and it has a subgroup isomorphic to A_5 but the group is not S_5. S_5 is not a group of isometries of $\mathbb{R}^3$.
Instead the full group of isometries of the regular dodecahedron is isomorphic to $C_2 \times A_5$.



double cover



This graph, the Petersen graph, has isomorphism group S_5

$$(i\ j) \mapsto \{i, j\}$$
$$(j\ i) \mapsto$$

A presentation of a group G is an expression $G = \langle X : R \rangle$ where X is a set of symbols (letters)
and $R \subset F(X) =$ free group on $X = \{x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} : x_j \in X,\ i_j \in \mathbb{Z}\}$ $\quad x^j x^k = x^{j+k}$ $(j,k \in \mathbb{Z};\ x \in X)$

$x^0 = 1 =$ identity.

X = set of "generators"

R = set of "relators" (words in the generators)

If X is finite then G is <u>finitely generated</u>.

If X and R are both finite then G is <u>finitely presented</u>.

Burnside groups are finitely generated (usually) but not finitely presented.

$G = F/$subgroup of F generated by R and their conjugates

$=$ "largest" homomorphic image of $F = F(X)$ having R in kernel

<u>universal</u> as we'll discuss later — see handout.

Every group has a presentation. Given G, for every $g \in G$, introduce a generator x_g. So $X = \{x_g : g \in G\}$.
For every pair $g, h \in G$ we want to force $x_g x_h = x_{gh}$ but this doesn't happen in $F = F(X)$ so introduce relators $x_g x_h x_{gh}^{-1} \in R$. $\quad R = \{x_g x_h x_{gh}^{-1} : g,h \in G\}$. Then $F/\{\ldots R \ldots\} \cong G$.

If G is finitely generated then G is countable i.e. finite or countably infinite.

If $X = \{x_1, \ldots, x_n\}$ then $F = F(X) = \{$<u>products</u> of $x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\} = \bigcup_{\ell=0}^{\infty} S_\ell$ where

finite products

F is a countable union of finite sets, hence countable.

$S_\ell = \{x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_\ell}^{e_\ell} :$ $\quad i_1, \ldots, i_\ell \in \{1, \ldots, n\}$, $e_j \in \{\pm 1\}\}$

$|S_\ell| \le 2^\ell$

If G is countably generated (X countable) then $F = F(X)$ is countable.

$S_\ell = \{$words of length $\ell\}$ is countable.

If $|X| = m$ and $|R| = n$, m, n pos. integers, what can we say about $|G|$ where $G = \langle X \mid R \rangle$?
If $n < m$ then G is infinite. That is (contrapositive form) in order for $|G| < \infty$, we need at least as many relators as generators.

eg. $m = 1$. $X = \{x\}$. If $R = \emptyset$ then $G = \langle x \mid \emptyset \rangle = \{\dots, x^{-2}, x^{-1}, 1, x, x^2, \dots\}$

If $R = \{1\}$ then $G = \langle x \mid 1 \rangle = F(x) = \{\dots, x^{-2}, x^{-1}, 1, x, x^2, \dots\}$

If $R = \{x^{15}\}$ then $G = \langle x \mid x^{15} \rangle = \{1, x, x^2, \dots, x^{14}\} \cong C_{15}$

If $R = \{x^{15}, x^{40}\}$ then $G = \langle x \mid x^{15}, x^{40} \rangle = \langle x \mid x^5 \rangle = \{1, x, x^2, x^3, x^4\}$

$$x^5 = (x^{15})^3 (x^{40})^{-1}$$

If G is a group then a homomorphic image of G is the image of G under a homomorphism $G \xrightarrow{\phi} H$ (surjective, otherwise replace H by $\phi(G) \leq H$.)

Note: $H \cong G/K$ where $K = \ker \phi \trianglelefteq G$. A homomorphic image of G is the same thing as a quotient group G/K, $K \trianglelefteq G$.

In particular the abelianization of G is the largest homo. image of G which is abelian.

A normal subgroup $K \trianglelefteq G$ yields an abelian quotient group G/K iff $K \geq G'$ = derived subgroup of G

i.e. $G' = \langle [g,h] : g, h \in G \rangle$, $[g,h] = g^{-1}h^{-1}gh$.

If $K \geq G'$ then in G/K, take any two elements $gK, hK \in G/K$, we have

$$hg[g,h] = \cancel{hg}\,\cancel{g^{-1}h^{-1}}gh = gh$$

$$(hK)(gK) = hgK = ghK = (gK)(hK) \quad \text{and conversely.}$$

eg. the abelianization of an abelian gp. is itself.

eg. the abelianization of S_n, $n \geq 2$ is C_2 (group of order 2).

The abelianization of $GL_n(F)$ $(n \geq 1)$ is $F^\times = \{$ nonzero elements of $F\}$

$$GL_n(F) \xrightarrow{\det} F^\times \qquad \ker(\det) = SL_n(F)$$

If $F = F(X)$ free group on m generators $X = \{x_1, \cdots, x_m\}$ e.g. $m=3$

The abelianization of F $\quad w = x_1 x_2^3 x_1^{-4} x_3^2 x_1 x_2^{-1} \longmapsto (-2, 2, 2) \in \mathbb{Z}^3$

$F/F' \cong \mathbb{Z}^m$ where $F' = \{w \in F :$ every x_i has exponents in w adding to $0\}$ $i = 1, 2, \cdots, m$. (Multiplicatively, $x_1^{-2} x_2^2 x_3^2$.

(free abelian group on m generators)

$\mathbb{Z}^3 / \langle (1,2,3), (4,9,-1) \rangle$

Now consider $G = \langle x_1, \cdots, x_m \mid r_1, \cdots, r_n \rangle$, $\quad r_1, \cdots, r_n \in F = F(x_1, \cdots, x_m)$

$= F/K$, K is the normal subgp of F generated by $r_1, \cdots, r_n$ and their conjugates in F.

We want to show that if $n < m$ then $|G| = \infty$.

To prove this, first consider $F/F'K$ Here $F', K \trianglelefteq F$.

$F'K = \{ab : a \in F', b \in K\} \trianglelefteq F$.

$$F/F'K \cong \frac{F/F'}{F'K/F'}$$ (Third Isomorphism Theorem also sometimes called the Second Isomorphism theorem).

$$\begin{tikzcd} & F \arrow[dl] \arrow[dr] & \\ F/K \arrow[dr] & & F/F' \arrow[dl] \\ & F/F'K & \end{tikzcd}$$

$F/F'K$ is a homomorphic image of $F/F' \cong \mathbb{Z}^m$.

$F/F'K \cong \mathbb{Z}^m /$ subgp. generated by n elements.

$F'K/F' \cong K/K \cap F'$ (Second Iso. Thm.)

$$F/F'K \cong \frac{F/K}{F'K/K}$$

$$\mathbb{Z}^3 / \langle (1,0,0), (0,2,0) \rangle \cong \left(\mathbb{Z}/2\mathbb{Z}\right) \oplus \mathbb{Z}$$

$$\mathbb{Z}^3 / \langle (1,0,0), (0,2,0), (0,0,3) \rangle \cong \left(\mathbb{Z}/2\mathbb{Z}\right) \oplus \left(\mathbb{Z}/3\mathbb{Z}\right)$$

Coxeter-Todd coset enumeration is the best algorithm we have for deciding certain "word problems" in group theory but it doesn't work in all cases.

Group order: Given a presentation $G = \langle X \mid R \rangle$ (X, R finite)

what is $|G|$? Is it finite?

Is G trivial?

Given two words in F, $= F(X)$ do they yield the same element of G?

The word problem for groups is undecidable.

Matrix mortality problem

You are given a positive integer n and a list of $n \times n$ integer matrices $A_1, \dots, A_k$.

Is there a finite product of these A_i that equals the zero matrix? This is a decision problem.

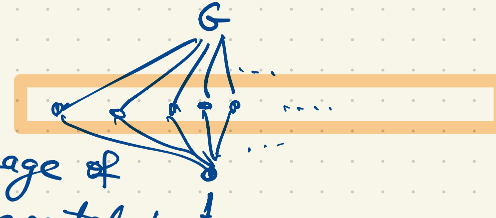
For $n=1$, this problem is decidable.

For $n \geq 3$, this problem is undecidable.

For $n=2$, we don't know whether the problem is decidable.

Suppose G is a Tarski monster for prime p i.e.

- G is an infinite nonabelian group
- $g^p = 1$ for all $g \in G$
- Given $x, y \in G$ with $x, y \neq 1$, either $\langle x, y \rangle$ is a subgroup of order p or $\langle x,y \rangle = G$.
- The <u>only</u> subgroups of G are



cyclic subgroups of order p.

Then $|B(2,p)| = \infty$.

This is because G is a homomorphic image of $B(2,p)$. $B(2,p)$ is the "largest" (universal) group generated by two elements a, b satisfying $g^p = 1$ for all g.

$$B(2,p) \twoheadrightarrow G \qquad \text{(epimorphism)}$$

$$B(2,p)/K \cong G \quad \text{for some normal subgroup } K \trianglelefteq B(2,p).$$

Since $|G| = \infty$, we must also have $|B(2,p)| = \infty$.

$B(2,p)$ has many subgroups not only cyclic of order p.

$B(2,p)$ is not simple. But G is simple: it has no normal subgroup other than 1 or G.

Proof: If G has a nontrivial normal subgroup N then it must be one of the cyclic subgroups of order p. There can't be more than one normal subgroup of order p. Why?

If N, N' are distinct normal subgroups (of order p) then $N \cap N' = 1$ and NN' is normal

$NN' \cong N \times N'$ For all $x \in N$, $y \in N'$, $x^{-1}y^{-1}xy = \underbrace{(x^{-1}y^{-1}x)}_{N'}\underbrace{y}_{N'} \in N'$ and $x^{-1}y^{-1}xy = \underbrace{x^{-1}}_{N}\underbrace{(y^{-1}xy)}_{N} \in N$

So $x^{-1}y^{-1}xy = 1 \Rightarrow yx = xy \Rightarrow NN' = N\times N'$ of order p^2. This cannot happen in a Tarski monster.

Let $N = \{1, x, \cdots, x^{p-1}\} \lhd G$

$N' = \{1, y, y^2, \cdots, y^{p-1}\} < G$

$y^{-1}xy \in N \Rightarrow y^{-1}xy = x^k$ for some $k \in \{1, 2, \cdots, p-1\}$. Let $l \in \{1, 2, \cdots, p-1\}$ be the inverse of k mod p i.e. $kl \equiv 1$ mod p. Then

Claim: $(y^l)^{-1}x y^l = x$ $\qquad (y^2)^{-1}x y^2 = y^{-1}(y^{-1}xy)y = y^{-1}x^k y = \underbrace{(y^{-1}xy)(y^{-1}xy)\cdots(y^{-1}xy)}_{k \text{ times}}$

If we replace $y \in N'$ by another generator of N'

then $y^{-1}xy = x$ i.e. $xy = yx$.

Then N and N' commute $\Rightarrow NN' \cong N\times N' \cong C_p \times C_p$, contradiction.

So G is simple.

Where do groups arise naturally? What are the natural examples of group operation?

- $+, \times$ in numbers
- functions under composition: permutation groups (subgroups of symmetric group)
  matrix groups (subgroups of $GL(V)$)
- fundamental groups in topology.
- group presentations $\langle X \mid R\rangle$

If F is a free group then every subgroup is free.

Nielsen-Schreier Theorem: Let F_n be a free group on n generators i.e. $F_n = \langle x_1, \cdots, x_n\rangle$.

If $H \leq F_n$ with $e = [F_n : H]$ then $H \cong F_{1+(n-1)e}$.

Let X be a path-connected topological space X: given any points $x, y \in X$ there exists a path in X from x to y i.e. a continuous map $[0,1] \longrightarrow X$

$$0 \longmapsto x$$
$$1 \longmapsto y$$

eg. $X = \mathbb{R}^2 - D$ is path-connected

$$D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$$

In this example, γ and γ' are not homotopic.

γ' is homotopic to the constant path $\gamma_0 : [0,1] \to X$, $\gamma_0(t) = x_0$ for all $t \in [0,1]$

$\mathbb{R}^2 - (x\text{-axis}) = \{(x,y) \in \mathbb{R}^2 : y \neq 0\}$ is not path-connected

Given a path-connected space X, and a point $x_0 \in X$, we can consider closed paths in X from x_0 to x_0 which are continuous maps $\gamma : [0,1] \longrightarrow X$ s.t. $\gamma(0) = \gamma(1) = x_0$.

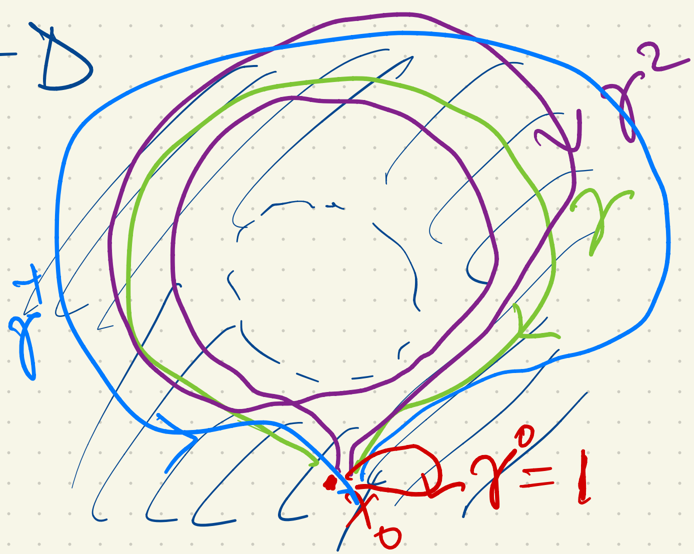
Two closed paths γ, γ' from x_0 to x_0 in X are homotopic if we can continuously distort γ to γ' while fixing the endpoints at x_0 and always staying inside X.

Write $[\gamma]$ as the equivalence class of γ under the equivalence relation of homotopy (with fixed base point x_0).

$\pi_1(X, x_0) = \{ [\gamma] : \gamma$ paths in X from x_0 to $x_0 \}$ is the fundamental group of X with base point x_0.

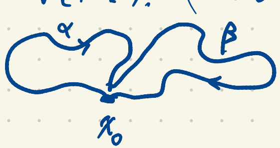
The group operation is "concatenation" of paths.

In $X = \mathbb{R}^2 - D$



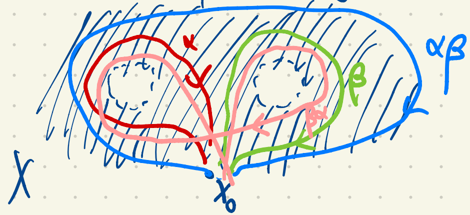
$\pi_1(\mathbb{R}^2 - D) = \langle \gamma \rangle$ Free gp on one generator
$\cong \mathbb{Z}$

In a top. space X, we fix a base point $x_0 \in X$ and consider the group $\pi_1(X, x_0)$ whose elements are flexible strings in X starting and ending at x_0. The identity element is a trivial string in X which "shrinks" to the point x_0. The inverse of any path/string is the same path/string in the reverse direction i.e. $\gamma^{-1}(t) = \gamma(1-t)$. $(0 \le t \le 1)$. The group operation is splicing/concatenation of strings.



$$(\alpha\beta)(t) = \begin{cases} \alpha(2t), & 0 \le t \le \frac{1}{2}; \\ \beta(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

eg. X is a plane region with two holes:



$\pi_1(X, x_0) = \langle \alpha, \beta \rangle$ is a free group on n generators.
(Van Kampen Theorem)

Theorem $\pi_1(X, x_0) \cong \pi_1(X)$ independent of choice of base point assuming X is path-connected.
$\pi_1(X)$ is the fundamental group of X or the first homotopy group of X. Its elements are (homotopy classes of) loops with base point (pointed copies of S^1).

$S^1 \times S^1$ = torus has $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$ free abelian group on two generators.

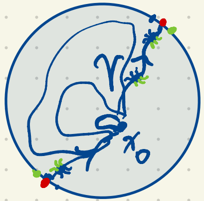
$P^2\mathbb{R}$ =  closed disk with antipodal boundary points identified.

real projective plane

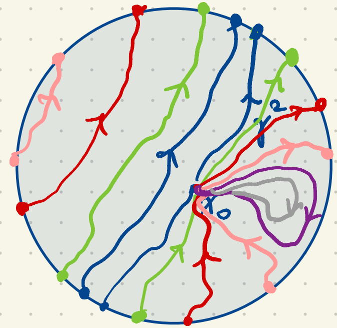
$\pi_1(P^2\mathbb{R})$ has order 2.

closed disk in $\mathbb{R}$

Möbius strip



γ^2 is homotopic to a trivial path
so $\pi_1(P^2\mathbb{R}) = C_2$ or $\mathbb{Z}/2\mathbb{Z}$.



π_1(real proj. plane) $\neq 1$
There is a path $\gamma \in \pi_1$(real proj. plane)
which is not homotopic to a trivial path
i.e. real proj. plane is not simply connected.

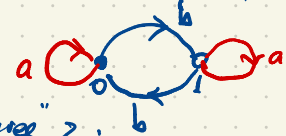
Example of a subgroup $H < F_2$, $[F_2 : H] = 2$, $\Rightarrow H \cong F_{1+2(2-1)} = F_3$
(Nielsen-Schreier)

$F = F_2 = \langle a, b \rangle$, consider the homomorphism $F_2 \to C_2 = \{0,1\}$, $a \mapsto 0$, $b \mapsto 1$.
$a^{e_1} b^{f_1} a^{e_2} b^{f_2} \cdots a^{e_k} b^{f_k} \mapsto \sum f_k \bmod 2$.

Let $H < F_2$ be the kernel of this homomorphism eg.
$a^3 bab^{-1} a^{-4} b^{-7} \notin H$
$b^3 a^{-3} b^{-1} a^4 \in H$ so $[F_2 : H] = 2$.
$F = H \sqcup Hb$

Claim: $H = \langle a, bab^{-1}, b^2 \rangle$ and moreover H is freely generated by $u = a$, $v = bab^{-1}$, $w = b^2$ i.e. the free group $F_3 = \langle u, v, w \rangle$
is isomorphic to H with isomorphism $F_3 \to H$, $u \mapsto a$, $v \mapsto bab^{-1}$, $w \mapsto b^2$.

Consider the digraph (directed graph)



Schreier graph on cosets of H.

Every vertex has "in-degree" 2 and "out-degree" 2; in each case one edge labelled a, the other b.

Every element $w \in F_2$ defines a word or walk in our graph starting at vertex 0.

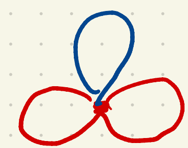
$a^3 b a b^{-1} a^{-4} b^{-7} \notin H$ defines a walk starting at 0, ending at 1.

$b^3 a^{-3} b^{-1} a^4 \in H$ defines a walk starting and ending at 0.

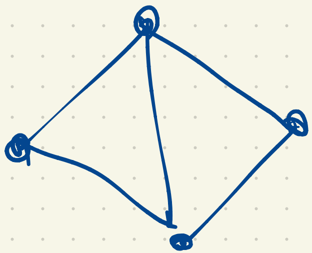
Given $w \in F_2$, we have $w \in H$ iff the walk of type w in our graph starting at 0 ends at 0. (Clear.)

$[F : H] = 2$ To see $H \cong F_3$, observe $H \cong \pi_1(X, x_0)$ where X is a path connected region in the plane with 3 "holes".

$\simeq$



has the same homotopy type $\square$



Given a graph Γ, the Euler characteristic of Γ is
(possibly with loops and multiple edges)
(undirected)

$$\chi(\Gamma) = \sum_i (-1)^i \dim C_i = \sum_i (-1)^i \dim H_i$$

$i \in \{0,1\}$ in the case of a graph $= \dim C_0 - \dim C_1 = \dim H_0 - \dim H_0$

C_0 = vector space freely spanned by 0-cells = vertices $\qquad \dim C_0$ = no. of vertices

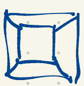
C_1 = ... 1-cells = edges $\qquad \dim C_1$ = . . . edges

$\dim H_0$ = no. of connected components $= 1$ if Γ is connected

$\dim H_1$ = . . . independent cycles = no. of edges that must be cut to make Γ into a tree (connected but no cycles)

1 − no. of independent cycles = no. of vertices − no. of edges.

$$1-2 = 4-5 = -1 = \chi(\Gamma).$$

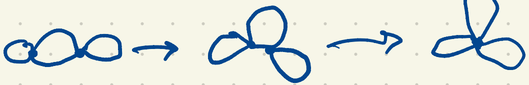


Nielsen-Schreier Theorem: Every subgroup of a free group is free. Moreover if $F = F_n$ (free of rank n) and $H \leq F$ is a subgroup of finite index e, then H is free of rank $1+(n-1)e$.

Given a connected graph Γ (which may have loops and multiple edges, but ignore arrows for now and view it as an undirected graph), what is the fundamental group of this graph?

If $\Gamma =$  then $\pi_1(\Gamma) \cong F_3 \cong \pi_1($  $)$

 and  have the same homotopy type



Having the $\underbrace{\text{same homotopy type}}_{\text{a coarser equiv. relation}}$ is not the same thing as $\underbrace{\text{homeomorphic}}_{\text{a finer equiv. relation}}$.

If two spaces are homeomorphic then they have the same homotopy type.

If X has the same homotopy type as Y then $H_i(X) \cong H_i(Y)$

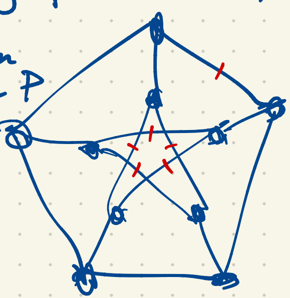
$\pi_1(X) \cong \pi_1(Y)$

$\chi(X) = \chi(Y)$

etc.

For a (connected) graph Γ as above, $\dim H_1 =$ "no. of independent cycles" $= 1 -$ no. of vertices $+$ no. of edges.

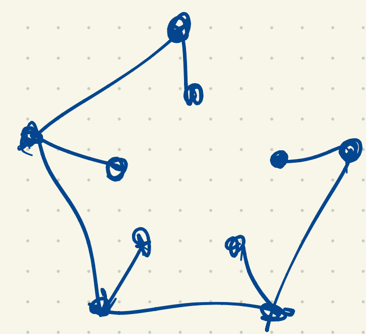
Petersen graph P



$\dim H_1 = 1 - 10 + 15 = 6.$ 6 independent cycles

$\Rightarrow P$ has same homotopy type as 

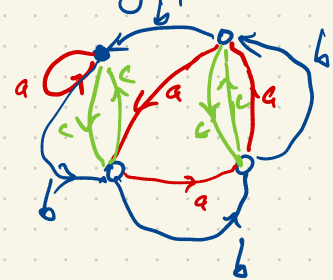
$\pi_1(P) \cong F_6$.



Imagine $F = F_3 = \langle a, b, c \rangle$ $\quad n=3$

$H \leq F$ of index $e = 4$

Schreier graph of H



H is a Free group of rank equal to the number of independent cycles in its Schreier graph

rank of H is $1 - 4 + 12 = 9$ $\qquad H \cong F_9$

$1 + (n-1)e = 1 + (3-1)\cdot 4 = 9.$

What does it mean to say $\{v_1, \cdots, v_n\}$ is a basis for a vector space V?

(Equivalently, V is freely spanned by $v_1, \cdots, v_n$) (The underlying field of scalars is fixed throughout)

V is freely spanned by $v_1, \cdots, v_n$ iff given any vector space W containing vectors $w_1, \cdots, w_n$, there is a unique linear transformation $V \to W$ mapping $v_i \mapsto w_i$ for all i.

$$\begin{tikzcd} & \{1, 2, \cdots, n\} \arrow[dl, "\iota"'] \arrow[dr, "\varphi"] & \\ V \arrow[rr, dashed, "\hat{\varphi}"'] \arrow[rr, phantom, bend left, "="] & & W \end{tikzcd}$$

This diagram commutes

i.e. $\hat{\varphi} \circ \iota = \varphi$

$\iota(j) = x_j, \quad \varphi(j) = w_j$

$\hat{\varphi}(x_j) = w_j.$

In the same way, we define free groups. Given a set X, and a map $\iota: X \to F$ where F is a group, we say F is <u>free</u> on X if the following universal property is satisfied:
Given any map $\varphi: X \to G$ where G is any group, then there is a unique group homomorphism $\hat{\varphi}: F \to G$ such that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "\iota"'] \arrow[dr, "\varphi"] & \\ F \arrow[rr, dashed, "\hat{\varphi}"'] & & G \end{tikzcd}$$

i.e. $\hat{\varphi} \circ \iota = \varphi$.

Although the definition does not explicitly require ("say") ι is injective, this must be the case in order for the conclusion to hold i.e. for free groups to exist.

Existence? Given any set X, there is an $\iota: X \to F$, F group free on X.
Uniqueness?
If $X = \{1, 2, \ldots, n\}$ take F = set of all words (strings of letters) generated by symbols $x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}$. The only simplifications in F are

$$x_i^j x_i^k = x_i^{j+k}, \quad x_i^0 = 1, \quad 1 x_i = x_i$$

If $\iota': X \to F'$ is also free, then $F' \cong F$. This follows from the definitions.

$$\begin{tikzcd} & X \arrow[dl, "\iota"'] \arrow[dr, "\iota'"] & \\ F \arrow[rr, dashed, shift left, "\varphi"] & & F' \arrow[ll, dashed, shift left, "\varphi'"] \end{tikzcd}$$

$\varphi: F \to F'$ homo. makes the diagram commute i.e.
$\varphi \circ \iota = \iota'$
$\varphi': F' \to F$ homo. $\varphi' \circ \iota' = \iota$

$$
\begin{tikzcd}
 & X \arrow[dl, "\iota"'] \arrow[d, "\iota'"] \arrow[dr, "\iota"] & \\
F \arrow[r, "\varphi"'] & F' \arrow[r, "\varphi'"'] & F
\end{tikzcd}
$$

$\varphi' \circ \varphi : F \to F$

$\varphi' \circ \varphi \circ \iota = \iota$

and $\varphi' \circ \varphi$ is the <u>only</u> homo. $F \to F$ which does this

Also $\mathrm{id}_F : F \to F$
$g \mapsto g$, $\mathrm{id}_F \circ \iota = \iota$

$$
\begin{tikzcd}
 & i \in X \arrow[dl, "\iota"'] \arrow[dr, "\iota"] & \\
x_i \arrow[rr] & & x_i
\end{tikzcd}
$$

Uniqueness

$\Longrightarrow \varphi' \circ \varphi = \mathrm{id}_F$

$\varphi \circ \varphi' = \mathrm{id}_{F'}$

$\Longrightarrow \varphi, \varphi'$ are isomorphisms.

Given groups G, H, we define the free product $G*H$ of G and H.

Concretely: Take symbols for elements of G and for elements of H and generate strings using these symbols i.e.

$$g_1h_1g_2h_2g_3h_3 \cdots g_kh_k$$

$$\text{or} \quad h_1g_1h_2g_2 \cdots h_kg_k \qquad\qquad h_i \in H$$

$$g_1h_1h_2g_2 \cdots h_{k-1}g_k \qquad\qquad g_j \in G.$$

$$h_1g_1h_2g_2 \cdots g_{k-1}h_k$$

Eg. a free product of n infinite cyclic groups is F_n (free group of rank n).

$$\underset{\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}}{\overset{\shortparallel}{\mathbb{Z}}} \cong F_1 \cong \{\ldots, a^{-3}, a^{-2}, a^{-1}, 1, a, a^2, a^3, \ldots\} \qquad \underset{\langle a\rangle}{\overset{\shortparallel}{\mathbb{Z}}} * \underset{\langle b\rangle}{\overset{\shortparallel}{\mathbb{Z}}} = F_2$$

$$\langle a\rangle * \langle b\rangle = \langle a, b\rangle$$

$$\left(\mathbb{Z}/2\mathbb{Z}\right) * \left(\mathbb{Z}/3\mathbb{Z}\right) = \{1, a, b, b^2, ab, ab^2, ba, b^2a, aba, ab^2a, bab, bab^2, b^2ab, b^2ab^2, \ldots\}$$

$$\underset{\substack{\{1,a\} \\ a^2=1}}{\overset{\shortparallel}{C_2}} * \underset{\substack{\{1,b,b^2\} \\ b^3=1}}{\overset{\shortparallel}{C_3}} = \langle a, b \mid a^2, b^3\rangle \text{ is a homo. image of } F_2$$

$$= F_2/K \qquad K = \text{subgp. of } F_2 \text{ generated by } a^2, b^3, \text{ and their conjugates } ba^2b^{-1}, \ldots$$

$$PSL_2(\mathbb{Z}) \cong \left(\mathbb{Z}/2\mathbb{Z}\right) * \left(\mathbb{Z}/3\mathbb{Z}\right) \cong \langle a, b \mid a^2, b^3 \rangle$$

$$SL_2(\mathbb{Z}) = \langle a, b \mid \underbrace{a^4 = b^6 = 1,\ a^2 = b^3}_{\text{relations}} \rangle = \langle a, b \mid \underbrace{a^4, b^6, a^2 b^{-3}}_{\text{relators}} \rangle$$