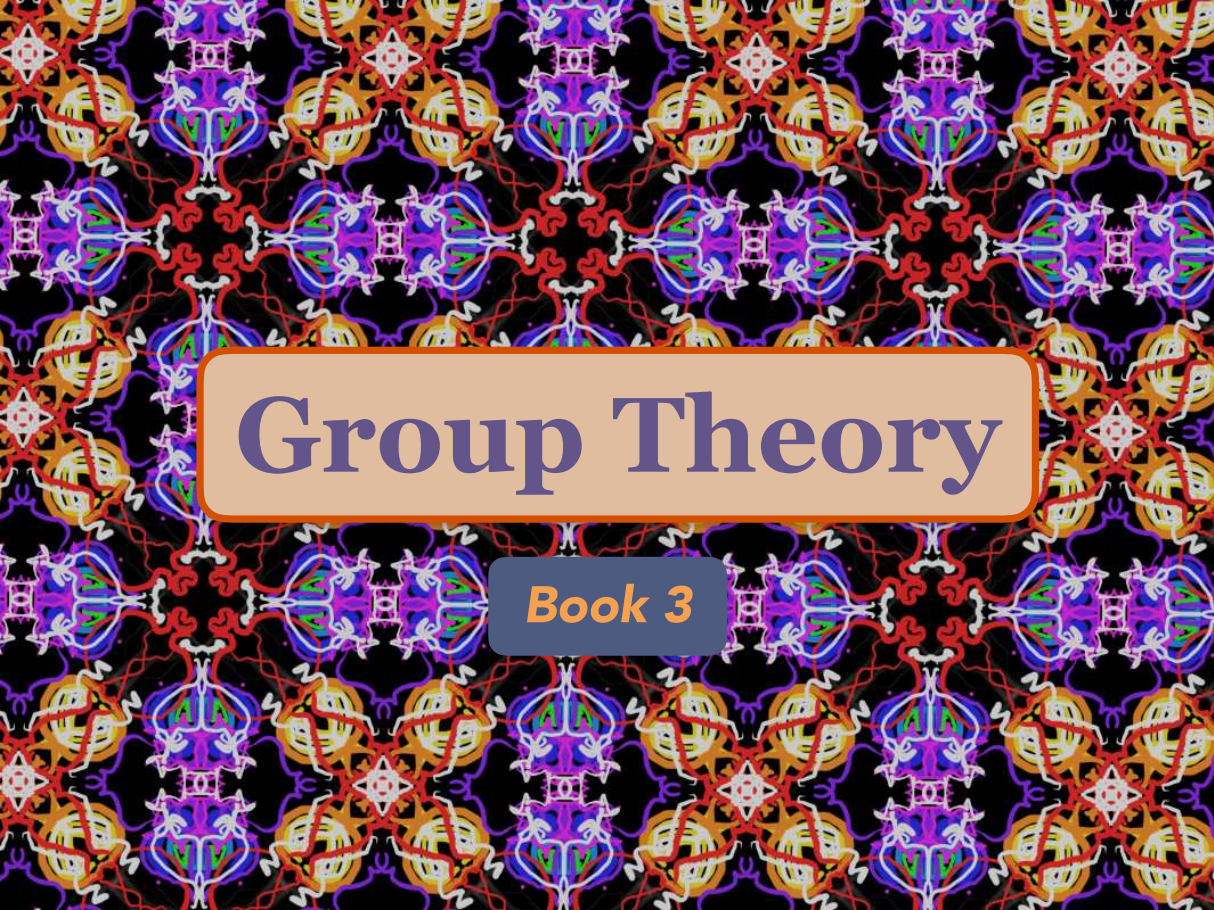
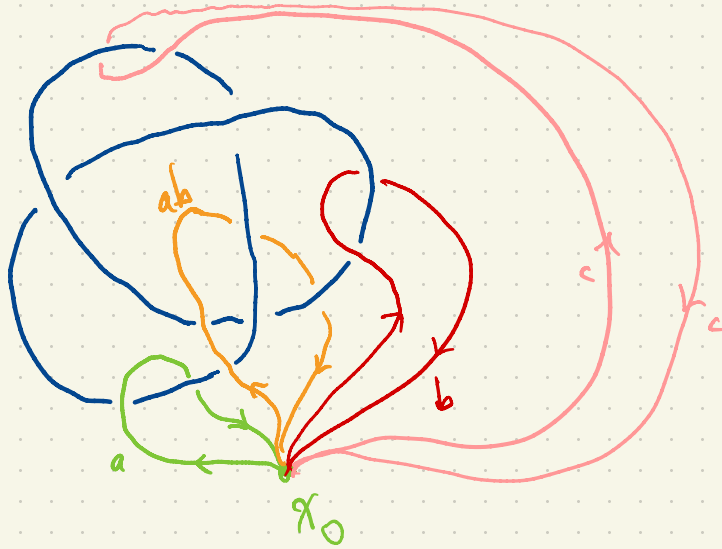




Group Theory

Book 3



$$ab = bc = ca$$

$$\Rightarrow c = b^{-1}ab = aba^{-1}$$

$$\begin{aligned}\pi_1(\mathbb{R}^3 - K) &= \langle a, b, c : ab = bc = ca \rangle \\ &= \langle x, y : x^3 = y^2 \rangle\end{aligned}$$

$$\begin{aligned}x &= ab = bc = ca \\ y &= abc\end{aligned} \quad \Rightarrow \quad \begin{aligned}x^3 &= ab \cdot ca \cdot bc \\ &= abc \cdot abc = y^2\end{aligned}$$

Observation : $\pi_1(\mathbb{R}^3 - K)$ has no ^(nontrivial) torsion elements (a "torsion" element in a group is an element of finite order)

If X is a path-connected subset of $\mathbb{R}^2$ then $\pi_1(X)$ has no (nontrivial) torsion elements. (known for a few decades, this considered folklore). Intuitive!

The real proj. plane $\mathbb{P}^2\mathbb{R}$ has torsion (its fund. gp. $\pi_1(\mathbb{P}^2\mathbb{R})$ has order 2)

(top. spaces) $\nwarrow$ a subspace of $\mathbb{R}^4$, not embeddable in $\mathbb{R}^3$.
For subsets $X \subset \mathbb{R}^4$, $\pi_1(X)$ can have nontrivial torsion elements.

What about $X \subset \mathbb{R}^3$? Can $\pi_1(X)$ have nontrivial torsion elements? Famous open problem.

We will show $SL_2(\mathbb{Z})$ has a subgroup isomorphic to F_2 :

$$\langle \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \rangle < SL_2(\mathbb{Z})$$

use the Ping-Pong Lemma:

One version is as follows.

Let G be a group acting on a set X . We can think of G as a subgroup of

$$\text{Sym } X = \{ \text{bijections } X \rightarrow X \}$$

i.e. permutations of X

Here I'm using left action, so we compose "right-to-left"

More generally, to say G acts on X means: For every $g \in G$ we have a permutation of X , $x \mapsto gx$ such that for all $g, h \in G$, $g(hx) = (gh)x$ i.e. the map $G \rightarrow \text{Sym } X$ is a group homomorphism.

If the map $G \rightarrow \text{Sym } X$ is one-to-one, we say G acts faithfully on X .
(So G is identified with a subgroup of $\text{Sym } X$.)

Eg. $SL_2(F)$ acts on P.F. as fractional linear transformations.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{array}{c} F \cup \{\infty\} \\ \downarrow \\ x \end{array}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(x) = \frac{ax+b}{cx+d}$$

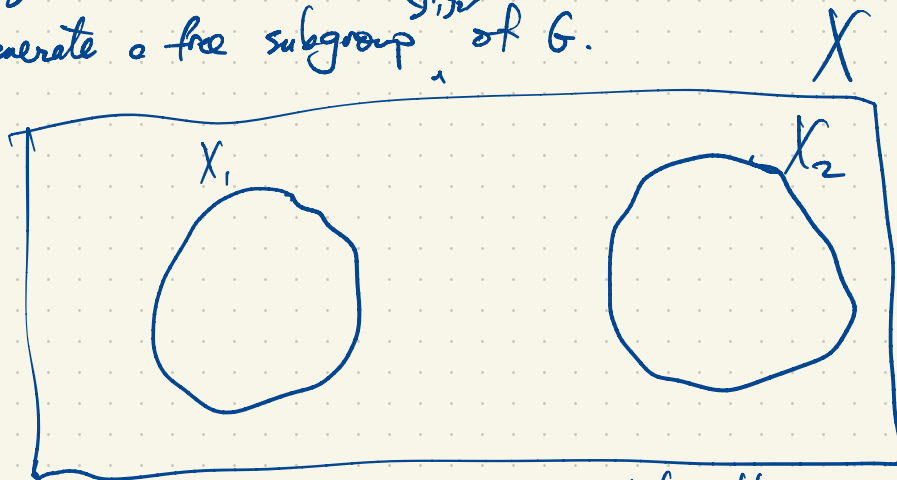
But this action is not faithful: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $-\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ give the same frac. lin. transf.
 $PSL_2(F)$ acts faithfully on P.F.

Given two subsets $X_1, X_2 \subset X$ which are disjoint and two elements $g_1, g_2 \in G$ such that

$$\left. \begin{aligned} g_1^k(X_2) &\subseteq X_1 \\ g_2^k(X_1) &\subseteq X_2 \end{aligned} \right\} \text{ for all integers } k \neq 0$$

and

then g_1, g_2 generate a free subgroup $\langle g_1, g_2 \rangle$ of G .



Proof A nontrivial word in two generators looks like

$$w = g_1^{k_1} g_2^{l_1} g_1^{k_2} g_2^{l_2} \dots g_1^{k_r} g_2^{l_r} g_1^{k_r}$$

k_i, l_i nonzero integers up to conjugacy in the free group.

Then $w \neq 1$ since it maps $X \rightarrow X_1$.

Distinct words in the generators w_1, w_2 must give distinct permutations of X by considering $w = w_1 w_2$. □

Application: $SL_2(\mathbb{Z})$ has $F_2 = \langle a, b \rangle$ (free group of rank 2) as a subgroup.
 Try $u = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. This doesn't work. These generate $\langle u, v \rangle = SL_2(\mathbb{Z})$
 but this is not a free group since

$$uv^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad (uv^{-1})^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (uv^{-1})^6 = 1.$$

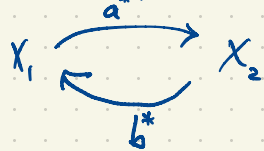
Next try $a = u^2 = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$, $b = v^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. These generate $\langle a, b \rangle \cong F_2$.

To prove this, use the Ping-Pong Lemma.

$G = SL_2(\mathbb{Z})$ acts on $P^1\mathbb{Q} = \mathbb{Q} \cup \{\infty\}$ by fractional linear transformations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} (x) = \frac{ax+b}{cx+d}$$

Let $X_1 = \{x \in P^1\mathbb{Q} : |x| < 1\}$, $X_2 = \{x \in P^1\mathbb{Q} : |x| > 1\}$. Note: $\infty \in X_2$.



$$b^n = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix}, \quad a^n = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix} \quad n \neq 0$$

$$a^n(x) = \frac{x + 2n}{0x + 1} = x + 2n$$

If $x \in X_1$ then $|x| < 1$ i.e. $-1 < x_1 < 1$
 then $a^n(x) \in (2n-1, 2n+1)$

so $|a^n(x)| > 1$ i.e. $a^n(x) \in X_2$.
 for $n \neq 0$.

If $x \in X_2$ then $|x| > 1$

$$b^n(x) = \frac{1x + 0}{2x + 1} = \frac{1}{2n + \frac{1}{x}}, \quad \left|\frac{1}{x}\right| < 1, \quad \left|2n + \frac{1}{x}\right| > 1,$$

$$|b^n(x)| = \left|\frac{1}{2n + \frac{1}{x}}\right| < 1, \quad b^n(x) \in X_1. \quad \square$$

In $G = SL_2(\mathbb{Z})$, $\langle a, b \rangle \cong F_2$.

Similarly in $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) / \{\pm I\}$, $\langle a, b \rangle \cong F_2$.

$$\langle a, b \rangle = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} : a_1, a_2, a_3, a_4 \in \mathbb{Z}; a_1 a_4 - a_2 a_3 = 1; a_1, a_4 \text{ odd}; a_2, a_3 \text{ even} \right\}$$

In $SL_2(\mathbb{Z})$ elements have one of the forms

$$\begin{bmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{bmatrix}, \begin{bmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{bmatrix}, \begin{bmatrix} \text{odd} & \text{odd} \\ \text{even} & \text{odd} \end{bmatrix}, \begin{bmatrix} \text{odd} & \text{even} \\ \text{odd} & \text{odd} \end{bmatrix}, \begin{bmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{odd} \end{bmatrix}, \begin{bmatrix} \text{odd} & \text{odd} \\ \text{odd} & \text{even} \end{bmatrix}$$

$$\cancel{\begin{bmatrix} \text{even} & \text{odd} \\ \text{even} & \text{odd} \end{bmatrix}}, \dots$$

ten choices of parity are excluded in $SL_2(\mathbb{Z})$.

$$|SL_2(\mathbb{F}_2)| = 2(2^2 - 1) = 6$$

$$SL_2(\mathbb{F}_2) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

Reducing matrix entries mod 2 gives an epimorphism (surjective homomorphism) $SL_2(\mathbb{Z}) \rightarrow SL_2(\mathbb{F}_2)$.

The kernel of this map is $\Gamma(2) = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in SL_2(\mathbb{Z}) : a_1, a_4 \text{ odd}; a_2, a_3 \text{ even} \right\} = \langle a, b \rangle$
(a principal congruence subgroup) $[SL_2(\mathbb{Z}) : \Gamma(2)] = 6$.

Tits Alternative: G linear group over $F \Rightarrow$ either
 G virtually solvable
or G has F_2 as a subgroup (exclusive 'or').

conjectured originally by Bass & Serre

A linear group is a matrix group over a field F , in particular $GL_n(F)$ and its subgroups such as $SL_n(F)$, $O_n(F)$, $U_n(F)$, $Sp_n(F)$, ...

Tits proved that the alternative holds in GL_n .

G is solvable if it has a composition series with abelian factors i.e.

$$G_0 = 1 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \dots \trianglelefteq G_n = G \quad \text{with } G_k/G_{k-1} \text{ abelian.}$$

eg. S_3 is solvable: $1 \trianglelefteq A_3 \trianglelefteq S_3$

G is virtually solvable if it has a solvable subgroup of finite index.

Every finite group is virtually solvable by considering the trivial subgroup.

Eg. the linear group $O_2(\mathbb{R}) = \{ \text{orthogonal } 2 \times 2 \text{ real matrices} \} = \{ A \in GL_2(\mathbb{R}) : A^T A = I \}$

If $A \in O_n(\mathbb{R})$ then $AA^T = A^T A = I$, so $(\det A)^2 = 1 \Rightarrow \det A = \pm 1$.

$SO_n(\mathbb{R}) = \{ A \in O_n(\mathbb{R}) : \det A = 1 \} = \text{special orthogonal group} = \{ \text{rotations fixing } 0 \in \mathbb{R}^n \}$

$[O_n(\mathbb{R}) : SO_n(\mathbb{R})] = 2$

$SO_2(\mathbb{R}) \cong S^1$ as Lie groups (abelian)

$$\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a^2 + b^2 = 1 \right\}$$

$O_2(\mathbb{R})$ is nonabelian but solvable.

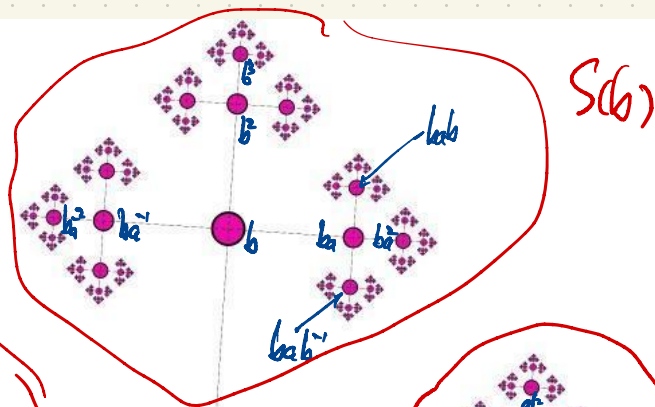
$$1 \triangleleft SO_2(\mathbb{R}) \triangleleft O_2(\mathbb{R})$$

$O_2(\mathbb{R})$ cannot have a subgroup isomorphic to F_2 .

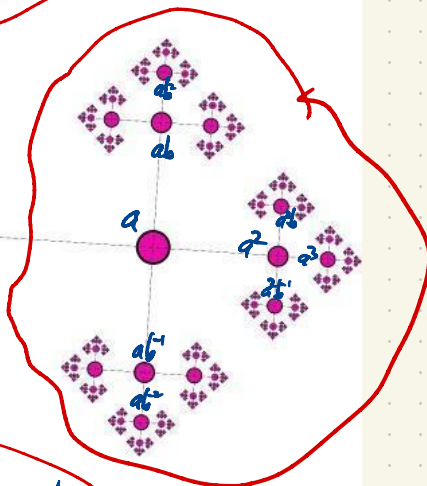
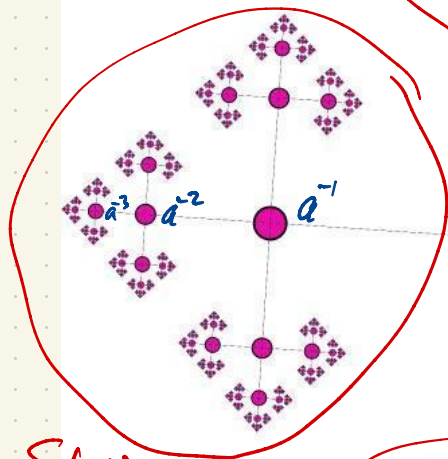
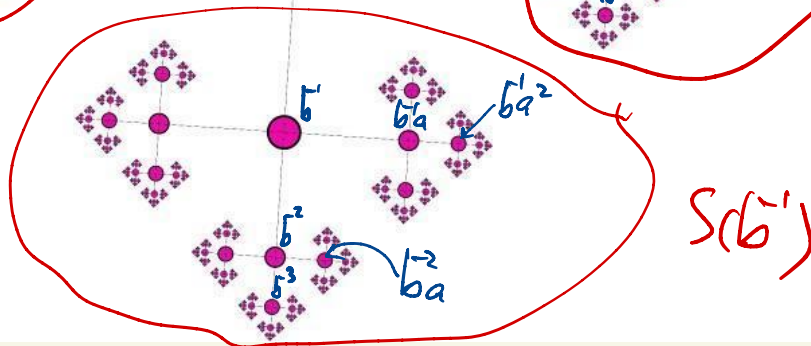
$SO_3(\mathbb{R})$ has a subgroup $\cong F_2$. See handout on free groups.

F_2 has a "paradoxical decomposition" $F_2 = A \sqcup B \sqcup C \sqcup D$ (disjoint union i.e. partition)

$$F_2 = g_1 A \sqcup g_2 B = g_3 C \sqcup g_4 D \quad g_i \in F_2.$$

F_2


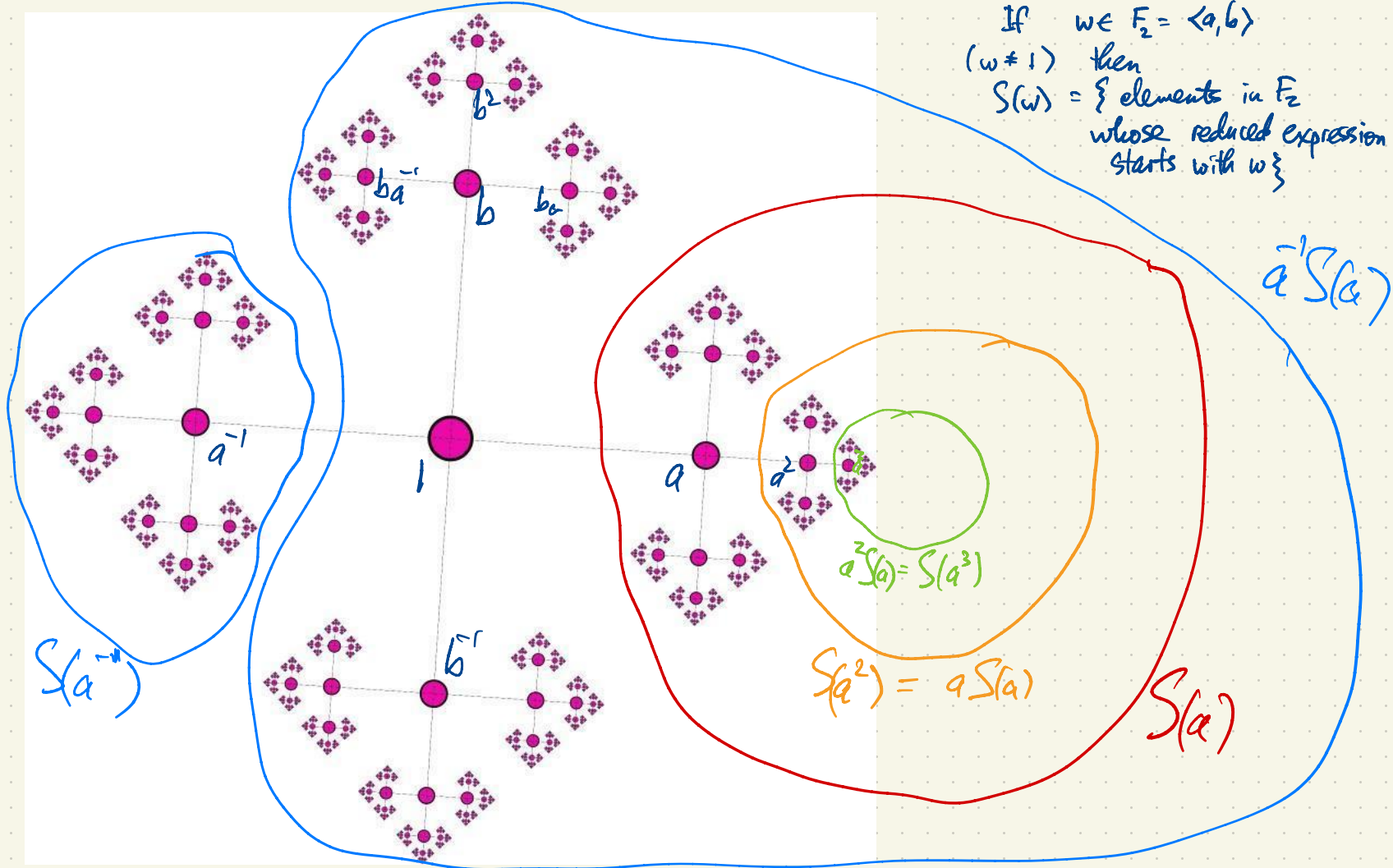
If $w \in F_2$, $w \neq 1$ then $S(w) \subset F_2$ is the subset consisting of words starting with w (they have w as a prefix). I'm only considering reduced words.


 $S(a^{-1})$


$$\begin{aligned} F_2 &= S(a^{-1}) \cup a^{-1}S(a) \\ &= S(b^{-1}) \cup b^{-1}S(b) \\ &= \{1\} \cup S(a) \cup S(b)S(a^{-1}) \cup S(b) \end{aligned}$$

This is almost a paradoxical decomposition of F_2 (unfortunately $\{1\}$ is "left over" ...)

If $w \in F_2 = \langle a, b \rangle$
 ($w \neq 1$) then
 $S(w) = \{ \text{elements in } F_2 \text{ whose reduced expression starts with } w \}$



An actual paradoxical
decomposition of F_2 without
emitting $\{1\}$:

$$F_2 = A \sqcup B \sqcup C \sqcup D$$

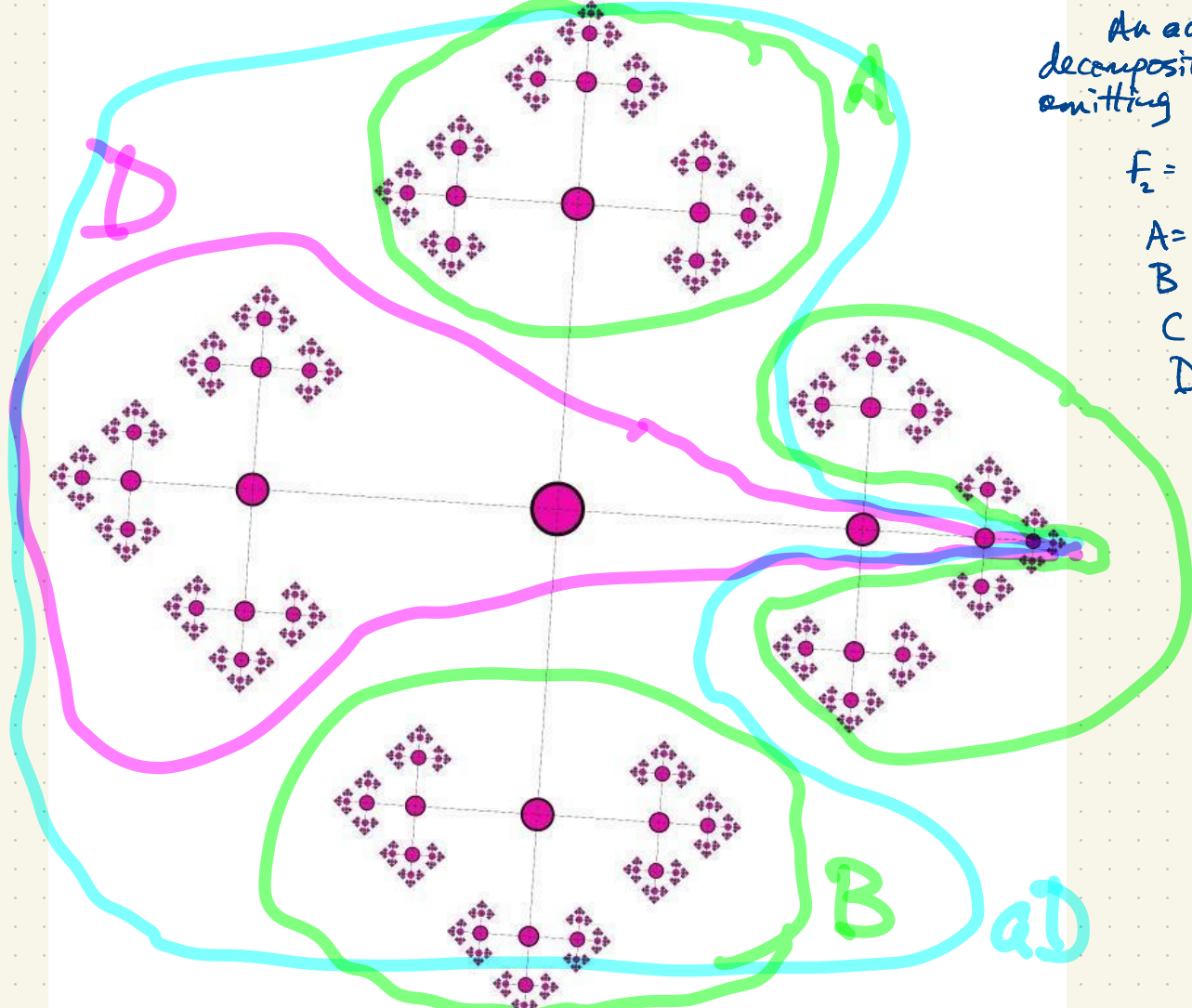
$$A = S(b)$$

$$B = S(b^{-1})$$

$$C = S(a) - \{a, a^2, a^3, a^4, \dots\}$$

$$D = S(a^{-1}) \cup \{1, a, a^2, a^3, a^4, \dots\}$$

$$\begin{aligned} F_2 &= \overline{A} \sqcup B \\ &= C \sqcup aD \end{aligned}$$



Banach-Tarski Theorem

Let $X = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$.

Then there exists a partition $X = X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4 \sqcup X_5$ such that

$$X = g_1 X_1 \sqcup g_2 X_2 = g_3 X_3 \sqcup g_4 X_4 \sqcup g_5 X_5 \quad \text{for some } g_1, \dots, g_5 \text{ direct isometries of } \mathbb{R}^3.$$

(orientation-preserving isometries)

An isometry is a transformation preserving distances.

Every isometry either preserves or reverses orientation.
e.g. rotations

The subsets $X_1, \dots, X_5$ are not all Lebesgue-measurable.

$$\text{Since } F_2 = g_1 A \sqcup g_2 B = g_3 C \sqcup g_4 D = A \sqcup B \sqcup C \sqcup D$$

and $SO_3(\mathbb{R})$ has a subgroup $\langle a, b \rangle \cong F_2$

the group $SO_3(\mathbb{R})$ also admits such a "paradoxical decomposition":

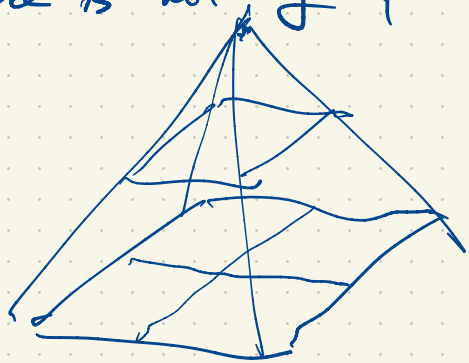
Let $T \subset SO_3(\mathbb{R})$ be a right transversal for $\langle a, b \rangle < SO_3(\mathbb{R}) = \bigsqcup_{\alpha} \langle a, b \rangle t_{\alpha}$
subset, not subgroup i.e. set of representatives of the right cosets

$$T = \{t_{\alpha} : \alpha\}$$

$$SO_3(\mathbb{R}) = \langle a, b \rangle T$$

$$SO_3(\mathbb{R}) = AT \sqcup BT \sqcup CT \sqcup DT = g_1 AT \sqcup g_2 BT = g_3 CT \sqcup g_4 DT$$

Note: A paradoxical composition requires a finite number of pieces;
there is nothing paradoxical about an infinite number of pieces.



Why can't a disk $X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ admit a "paradoxical decomposition" ?
If $X = X_1 \sqcup X_2 \sqcup \dots \sqcup X_5$ (or some other finite number of pieces)

with $X = g_1 X_1 \sqcup g_2 X_2 = g_3 X_3 \sqcup g_4 X_4 \sqcup g_5 X_5$, $g_1, \dots, g_5$ isometries,
what is the contradiction?

To obtain a contradiction we cannot invoke Lebesgue measure
(i.e. area) since the subsets X_i aren't necessarily Lebesgue measurable.
We use finitely additive measure.

Clara Loh, Geometric group theory

A finitely additive measure on a set X is a function $\mu: \mathcal{P}(X) \rightarrow [0, \infty]$ ($\mathcal{P}(X)$ = power set of X = $\{ \text{all subsets of } X \}$) such that

$$\mu(A_1 \sqcup A_2 \sqcup \dots \sqcup A_k) = \mu(A_1) + \dots + \mu(A_k) \quad (A \sqcup B = \text{disjoint union of } A \text{ and } B)$$

Now suppose G is a group acting on X i.e. group elements permute the points of X . We'll denote this as a left action: $g: X \rightarrow X$ is bijective mapping $x \mapsto g(x)$.

If $A \subseteq X$ then $g(A) = \{ g(a) : a \in A \}$. So g acts naturally on $\mathcal{P}(X)$.

Note: $(gh)(A) = g(h(A))$.

We often write $G \curvearrowright X$ to say G acts on X .

We say the f.a. measure μ is G -invariant if $\mu(gA) = \mu(A)$ for all $A \subseteq X$, $g \in G$.

eg. $X = \mathbb{R}^n$, $G = \{ \text{direct isometries of } \mathbb{R}^n \}$.

If X has a f.a. ^{G -invariant} measure μ on X and $E \subseteq X$ with $\mu(E) < \infty$ then E cannot have a paradoxical decomposition with respect to G .

Ex. $n=2$. $G = \{ \text{direct isometries of } \mathbb{R}^2 \}$.

$$E = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \}.$$

There is a f.a. G -invariant measure on $\mathbb{R}^2$ with $\mu(E) = 1$.
Consequently E has no "paradoxical decomposition" with respect to G .

There is no partition $E = E_1 \sqcup E_2 \sqcup E_3 \sqcup E_4 \sqcup E_5$ such that

$$E = g_1 E_1 \sqcup g_2 E_2 = g_3 E_3 \sqcup g_4 E_4 \sqcup g_5 E_5, \quad g_1, \dots, g_5 \in G.$$

Proof by contradiction: Assume such a decomposition exists. Then

$$\begin{aligned} 1 = \mu(E) &= \mu(E_1) + \dots + \mu(E_5) \\ &= \mu(g_1 E_1) + \mu(g_2 E_2) = \mu(g_3 E) + \mu(g_4 E_4) + \mu(g_5 E_5) \\ &= \mu(E_1) + \mu(E_2) = \mu(E_3) + \mu(E_4) + \mu(E_5) \end{aligned}$$

Contradiction.

In $n=3$ dimensions there is no f.a. measure invariant under direct isometries.

An action of G on X is amenable if X has a f.a. probability measure μ ^{invariant} under G . (Rather than $\mu: \mathcal{P}(X) \rightarrow [0, \infty]$ we have $\mu(X) = 1$, $\mu: \mathcal{P}(X) \rightarrow [0, 1]$.)

A group G is amenable if the left-action on itself is amenable.

Here G has a f.a.p. measure $\mu: \mathcal{P}(G) \rightarrow [0, 1]$, $\mu(G) = 1$, such that $\mu(gA) = \mu(A)$ for all $g \in G$, $A \subseteq G$.

All finite groups are amenable.

All abelian groups are amenable. (Not quite obvious!)

All solvable groups are amenable.

The group F_n ($n \geq 2$) is not amenable.

$$F_2 = A \sqcup B \sqcup C \sqcup D$$

$$= g_1 A \sqcup g_2 B = g_3 C \sqcup g_4 D$$

$$g_i \in F_2$$

$F_2 = \langle a, b \rangle$ has $\begin{cases} 1 & \text{if } n=0 \\ 1+4 \cdot 3^{n-1} & \text{if } n \geq 1 \end{cases}$ words of length at most n .

$$l(a^2b) = l(aab) = 3$$

$$l(a^{-1}b^3b^{-1}) = 6.$$

What is a f.a.p. measure on $\mathbb{Z}$, invariant (under translation)? or not?

Let $A \subseteq \mathbb{Z}$. We want a f.a.p. measure $\mu(A) \in [0,1]$, $\mu(\mathbb{Z}) = 1$?

$$A = 2\mathbb{Z} = \{\dots, -4, -2, 0, 2, 4, 6, \dots\}$$

$$B = 2\mathbb{Z} + 1 = \{\dots, -5, -3, -1, 1, 3, 5, \dots\}$$

$$\mathbb{Z} = A \sqcup B \Rightarrow 1 = \mu(\mathbb{Z}) = \mu(A) + \mu(B).$$

In order for μ to be translation-invariant, $\mu(A) = \mu(B) = \frac{1}{2}$.

For every $A \subseteq \mathbb{Z}$:

$$\mu\{\text{primes}\} = 0.$$

We probably want $\mu(A) = \lim_{n \rightarrow \infty} \frac{|A \cap [-n, n]|}{2n+1}$.

However this limit is not defined for arbitrary $A \subseteq \mathbb{Z}$.

Eg. $A = \{n \in \mathbb{Z} : |n| \text{ has first digit } 1 \text{ when written in decimal form}\}$.

$$\text{For } n = 100, \quad A \cap [-100, 100] = \{-100, -19, -18, \dots, -10, -1, 1, 10, 11, \dots, 19, 100\}, \quad |A \cap [-100, 100]| = 24$$
$$\frac{|A \cap [-100, 100]|}{2 \cdot 100 + 1} = \frac{24}{201} \approx 0.119$$

$$\text{For } n = 200, \quad A \cap [-200, 200] = \underbrace{\{\pm 1, \pm 10, \dots, \pm 19\}}_{20}, \underbrace{\{\pm 100, \dots, \pm 199\}}_{200}, \quad |A \cap [-200, 200]| = 222,$$
$$\frac{|A \cap [-200, 200]|}{2 \cdot 200 + 1} = \frac{222}{401} \approx 0.5536$$

Replace $\lim_{n \rightarrow \infty} \frac{|A \cap [-n, n]|}{2n+1}$ by $\text{glim}_{n \rightarrow \infty} \frac{|A \cap [-n, n]|}{2n+1}$

There is a "new improved limit" defined for every bounded sequence $a_0, a_1, a_2, \dots$ in $\mathbb{R}$ we can define $\text{glim}_{n \rightarrow \infty} a_n$ satisfying

- (i) If (a_n) converges then $\text{glim}_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_n$
- (ii) If $a_n \in [M_1, M_2]$ then $\text{glim}_{n \rightarrow \infty} a_n \in [M_1, M_2]$
- (iii) $\text{glim}_{n \rightarrow \infty} a_n$ is a cluster point of (a_n) (there is at least one cluster point of (a_n) in $[M_1, M_2]$ by Bolzano-Weierstrass)
- (iv) $\text{glim}_{n \rightarrow \infty} (a_n + b_n) = \text{glim}_{n \rightarrow \infty} a_n + \text{glim}_{n \rightarrow \infty} b_n$
 $\text{glim}_{n \rightarrow \infty} a_n b_n = (\text{glim}_{n \rightarrow \infty} a_n)(\text{glim}_{n \rightarrow \infty} b_n)$

Define $\mu(A) = \text{glim}_{n \rightarrow \infty} \frac{|A \cap [-n, n]|}{2n+1}$

$\mathbb{R}^3 = \{(a_0, a_1, a_2) : a_i \in \mathbb{R}\}$ is a vector space over $\mathbb{R}$; it is also a ring

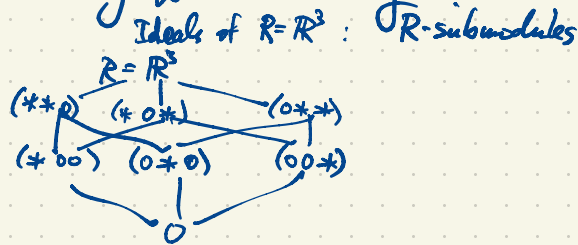
$(a_0, a_1, a_2)(b_0, b_1, b_2) = (a_0 b_0, a_1 b_1, a_2 b_2)$ commutative ring with identity

zero divisors: $(1, 0, 0)(0, 1, 0) = (0, 0, 0)$

There are infinitely many subspaces ($\mathbb{R}$ -submodules)



2-dim. subspaces
1-dim subspaces



The subspace of $\mathbb{R}^3$ spanned by $(3, 0, -5)$ i.e. $\{t(3, 0, -5) : t \in \mathbb{R}\}$ is not an ideal
 $(3t, 0, -5t)$

$$(a_0, a_1, a_2)(-3, 0, -5) = (*, 0, *) \quad (a_0, a_1, a_2) \in \mathbb{R}$$

$$R = \mathbb{R}^\infty = \{(a_0, a_1, a_2, a_3, \dots) : a_i \in \mathbb{R}\}$$

$$a+b = (a_0+b_0, a_1+b_1, \dots)$$

$$ab = (a_0b_0, a_1b_1, \dots)$$

commutative ring $R = \mathbb{R}^\infty$ with identity $1 = (1, 1, 1, \dots) \leftarrow r(1, 1, 1, \dots) = (r, r, r, \dots) = r$

$\lim a_n$ is defined on certain sequences forming a subspace of $\mathbb{R}^\infty$, where "lin" is a well-defined linear functional.

Extend this if possible to the subspace $\ell_\infty \subset \mathbb{R}^\infty$ consisting of bounded sequences.

This can be done using the Hahn-Banach Theorem.

Rather than use this, let's use an algebraic idea.

Start with the ideal $J \subset \mathbb{R}^\infty$ consisting of all sequences of finite support.

This is an ideal. (If $a, b \in J$ then $a+b \in J$. If $a \in J$ and $r \in R$, then $ra \in J$.)

$R \cdot (1, 0, 1, 0, 1, 0, \dots) = (*, 0, *, 0, *, 0, \dots)$ is an ideal not contained in J .

We want to extend J to a maximal ideal $M \subset R$ i.e. $J \subseteq M \subset R$

Such an extension exists by Zorn's lemma. So $R/M \cong$ field of hyper-real numbers ${}^*\mathbb{R}$

$${}^*\mathbb{R} \supset \mathbb{R}$$

In $R = \mathbb{R}^\infty$, we have zero divisors e.g. $(1, 0, 1, 0, 1, 0, \dots)(0, 1, 0, 1, 0, 1, \dots) = (0, 0, 0, 0, 0, 0, \dots)$

$$(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots) \in R, \quad (1, 2, 3, 4, 5, \dots) \in R$$

Given a bounded sequence $a = (a_0, a_1, a_2, \dots) \in \mathbb{R} = \mathbb{R}^\infty$ so $a_n \in [M_1, M_2]$ for all n .
 ${}^*\mathbb{R}$ is totally ordered. For $u, v \in {}^*\mathbb{R}$ either $u = v$ or $u < v$ or $u > v$.

eg. $u = (1, -1, 1, -1, 1, -1, \dots)$, $v = (0, 0, 0, 0, 0, 0, \dots)$

Which $r \in \mathbb{R}$ give $r = (r, r, r, \dots) < (a_0, a_1, a_2, \dots) \in \mathbb{R}$?

there is a real $r \in \mathbb{R}$ which is maximal s.t. $(r, r, r, \dots)_{\text{mod } M} \leq (a_0, a_1, a_2, \dots)_{\text{mod } M}$

This supremum gives $\lim_{n \rightarrow \infty} a_n = r$.

This gives a translation-invariant f.a.p. measure μ on $\mathbb{Z}$.

Same works over $\mathbb{R}$

and over any amenable group

$a = (a_0, a_1, a_2, \dots)_{\text{mod } M} < b = (b_0, b_1, b_2, \dots)_{\text{mod } M}$ iff $b - a = (b_0 - a_0, b_1 - a_1, b_2 - a_2, \dots) > 0$

If M is a Tarski group of exponent p , is M amenable?