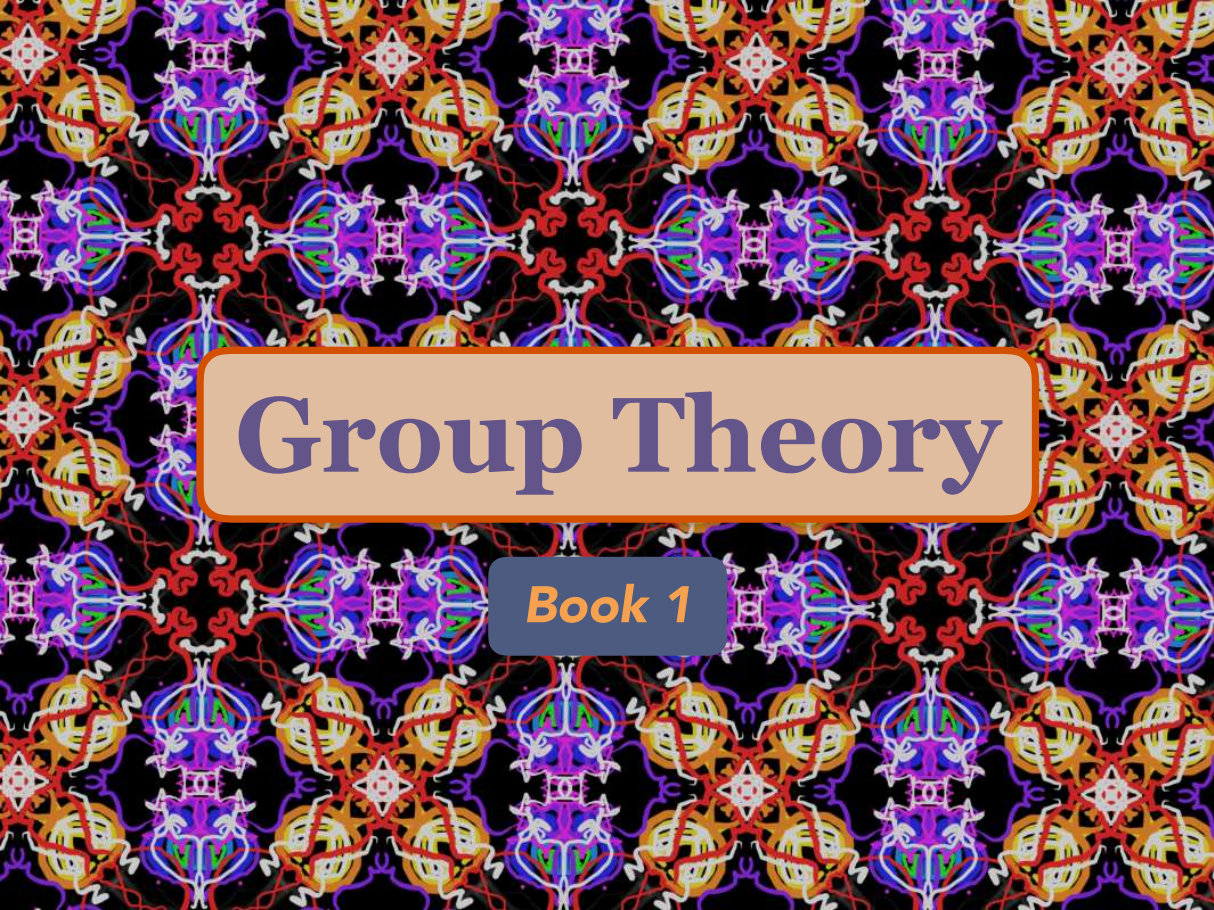




Group Theory

Book 1

Finite groups (up to isomorphism)

n	1	2	3	4	5	6	7	8	9	10	11	...	59	60	61	62	63	64	65	...
no. of groups of order n	1	1	1	2	1	2	1	5	2	2	1			1	13	1	2	1	247	1

$C_2 \times C_2 \times C_2 \times C_2 \cong C_4$
 $C_2 \times C_2 \times C_2 \cong C_6$
 $S_3 \cong D_3$
 $C_2 \times C_2 \times C_2$ elementary abelian
 $C_2 \times C_4$
 C_8
 $C_2 \times C_3$ elementary abelian

Q Quaternion
 D_4 Dihedral

S_n = symmetric group of degree n , $|S_n| = n!$
 A_n = alternating group of degree n , order, $|A_n| = \frac{1}{2}n!$ ($n \geq 2$)
 A_n is simple for $n \geq 5$
 $|S_4| = 24$ solvable

$G_1 \cong 1 \triangleleft G_2 \triangleleft G_3 \triangleleft G_4 = A_4 \triangleleft G = S_4$
 $\langle (12)(34) \rangle$ order 2
 $\langle (12)(34), (13)(24) \rangle$ order 4

Composition series with composition factors of prime order: $|G_2/G_1| = 2$, $|G_3/G_2| = 2$,
 $|G_4/G_3| = 3$, $|G/G_4| = 2$

G is solvable if all its composition factors are cyclic of prime order.

Jordan-Hölder Theorem: Every finite group has a composition series with its factors being simple groups.

G is simple if its only composition series is $1 \triangleleft G$ (the only normal subgroups are 1 and G).
 eg. cyclic groups of prime order are simple.
 A_n is simple for $n \geq 5$.

$|S_5| = 5! = 120$
 S_5 has composition series
 $1 \triangleleft A_5 \triangleleft S_5$
 Simple groups $\begin{cases} \text{cyclic of prime order} \\ \text{nonabelian simple groups of order} \end{cases}$
 $60, 168, 360, 504, 660, 1092, 2496, 2520, 3420, 1080, \dots$
 $A_5 \cong \text{symplectic}$
 $SL_3(2) \cong PSL_2(7)$
 $SL_2(4) \cong PSL_2(5) \cong A_5$

Classical groups of Lie type are analogous to Lie groups
 We use finite fields: Every finite field has prime power order $q = p^e$, p prime, $e \geq 1$.

including $\mathbb{F}_p = \{0, 1, 2, \dots, p-1\}$; $\mathbb{F}_4 = \{0, 1, \alpha, \beta\}$

+	0	1	α	β
0	0	1	α	β
1	1	0	β	α
α	α	β	0	1
β	β	α	1	0

·	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	β	1
β	0	β	1	α

$GL_n(F)$ = group of all invertible $n \times n$ matrices over F .

$$\alpha^2 = 1 + \alpha = \beta$$

$GL(F)$ is the general linear group of degree n over F .

$GL_2(\mathbb{R})$ is a Lie group

$$GL_2(\mathbb{F}_2) = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{order 2}}, \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{\text{order 6}} \right\} \cong S_3$$

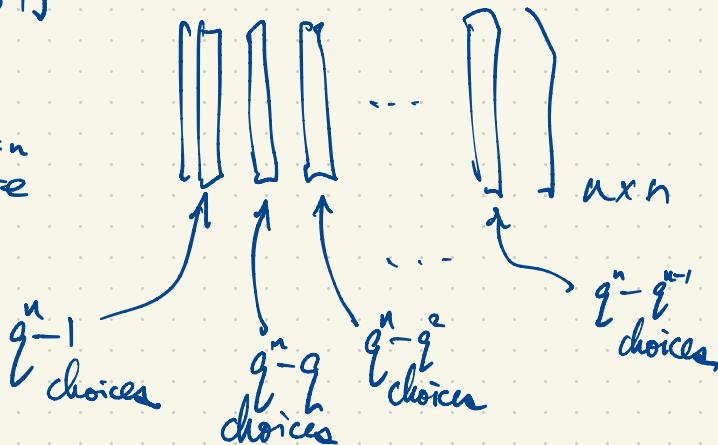
$GL_2(2)$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|GL_n(\mathbb{F}_q)| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$$

There are q^{n^2} matrices of size $n \times n$ over $\mathbb{F}_q$ but most of them are not invertible.

$$\begin{aligned} |GL_n(\mathbb{F}_q)| &= q^{\frac{n(n-1)}{2}} (q^{n-1} - 1)(q^{n-2} - 1) \cdots (q - 1) \\ &= q^{\frac{n(n-1)}{2}} \prod_{j=1}^n (q^j - 1) \end{aligned}$$



A Sylow p -subgroup

$$\begin{bmatrix} 1 & * & \cdots & * \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

of order $q^{\frac{n(n-1)}{2}}$

$$|GL_2(2)| = (2^2 - 1)(2^2 - 2) = 6$$

$$|GL_3(2)| = (2^3 - 1)(2^3 - 2)(2^3 - 2^2) = 7 \cdot 6 \cdot 4 = 168$$

$GL_3(2)$ is the second-smallest nonabelian simple group

$$|GL_4(2)| = (2^4-1)(2^4-2)(2^4-2^2)(2^4-2^3) = 15 \cdot 14 \cdot 12 \cdot 8 = 20160$$

$$|A_8| = \frac{8!}{2} = \frac{40320}{2} = 20160$$

There are two simple groups of order 20160 : $A_8 \cong GL_4(2)$, $PSL_3(4)$

$$|GL_3(4)| = (4^3-1)(4^3-4)(4^3-4^2) = 63 \cdot 60 \cdot 48 = 181440 = 9 \cdot 20160$$

$GL_3(4)$ is not simple.

$GL_n(F)$ has a normal subgroup $SL_n(F) = \text{special linear group of degree } n \text{ over } F$
 $= \{ A \in GL_n(F) : \det A = 1 \}$

$$|SL_n(F_q)| = \frac{|GL_n(F_q)|}{q-1}$$

there is a surjective homomorphism $GL_n(F_q) \rightarrow F_q^* = \{\text{nonzero field elements}\}$
 $A \mapsto \det A$

First isomorphism theorem: $GL_n(F_q) / SL_n(F_q) \cong F_q^*$

If $q=2$ then $SL_n(2) = GL_n(2)$.

Smaller example: $|A_5| = \frac{1}{2} 5! = 60$ Smallest nonabelian simple group

$$SL_2(\mathbb{F}_5) = \frac{1}{2} \cdot 180 = 60$$

$$|GL_2(\mathbb{F}_5)| = (5^2 - 1)(5^2 - 5) = 15 \times 12 = 180$$

$$\mathbb{F}_5 = \{0, 1, \alpha, \beta\}$$

+	0	1	α	β
0	0	1	α	β
1	1	0	β	α
α	α	β	0	1
β	β	α	1	0

·	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	β	1
β	0	β	1	α

There is only one simple group of order 60;

$$SL_2(4) \cong A_5$$

What five points is $SL_2(4)$ permuting (all 60 even permutations of the five points)?

We'll answer this soon but first use group presentations:

$A_5 \cong \langle a, b \mid a^5 = b^3 = (ab)^2 = 1 \rangle$ = the group generated by a, b subject to three relations
 $a^5 = 1, \quad b^3 = 1, \quad (ab)^2 = 1.$

What does this mean?

In A_5 , let $a = (12345)$, $b = (142)$, so $ab = (12345)(142) = (23)(45)$

"left-to-right" composition: see handout

~~$$\begin{aligned}
 (142)(1) &= 4 \\
 (142)(2) &= 1 \\
 (142)(3) &= 3 \\
 &\dots
 \end{aligned}$$~~

$$(142): \begin{array}{l} 1 \mapsto 4 \\ 2 \mapsto 1 \\ 3 \mapsto 3 \\ \dots \end{array}$$

$$\begin{bmatrix} \beta & \alpha \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} \beta & \alpha \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$1(12345)(142)$$

$$= 2(142) = 1$$

$SL_2(4)$ is generated by $a = \begin{bmatrix} \alpha & 1 \\ 1 & \beta \end{bmatrix}$, $b = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$

$$ab = \begin{bmatrix} \alpha & 1 \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$$

$$(ab)^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$1(12345)(142)$$

$$\begin{aligned}
 1^{(142)} &= 4 \\
 3^{(142)} &= 3 \\
 &\dots
 \end{aligned}$$

$$\begin{aligned}
 &= \left[1^{(12345)} \right] (142) \\
 &= 2^{(142)} = 1
 \end{aligned}$$

$$|SL_2(5)| = |SL_2(\mathbb{F}_5)| = \frac{1}{4} |GL_2(5)| = \frac{1}{4} \cdot 480 = 120$$

$$|GL_2(5)| = (5^2 - 1)(5^2 - 5) = 24 \cdot 20 = 480$$

$$|PSL_2(5)| = |SL_2(5)| / \langle -I \rangle = \frac{120}{2} = 60.$$

$$-I = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Z(SL_2(5)) = \langle -I \rangle = \{I, -I\}$$

center

$$Z(GL_2(5)) = \{\lambda I : \lambda \neq 0\}$$

$$\det(\lambda I) = \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = \lambda^2$$

$PSL_2(5)$ consists of all 2×2 matrices $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$, $x, y, z, w \in \mathbb{F}_5$ with $xw - yz = 1$
where we identify $\pm \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

$$a = \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad a^2 = \pm \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \dots, \quad a^5 = \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$b = \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad b^2 = \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b^3 = \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$ab = \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \pm \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad (ab)^2 = \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$PSL_2(\mathbb{F}_5)$ can be identified with the 60 fractional linear transformations permuting $\mathbb{F}_5 \cup \{\infty\} = \{0, 1, 2, 3, 4, \infty\}$

$A_5 \cong SL_2(4) \cong PSL_2(5)$ of order 60; all are defined by the presentation $\langle a, b \mid a^5 = b^3 = (ab)^2 = 1 \rangle$

Composition series for $SL_2(5)$:

$$\begin{array}{ccccc} \langle I \rangle & \triangleleft & \langle -I \rangle & \triangleleft & SL_2(5) \\ \text{order } 1 & & \text{order } 2 & & \text{order } 120 \end{array}$$

Why is $SL_2(\mathbb{F}_q) \cong A_5$? One way is to see that $SL_2(\mathbb{F}_q)$ permutes $\mathbb{F}_q \cup \{\infty\} = \{0, 1, \alpha, \beta, \infty\}$ as fractional linear transformations

$$f_A(z) = \frac{az+b}{cz+d}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$B = \begin{bmatrix} e & g \\ h & k \end{bmatrix}, \quad f_B(z) = \frac{ez+g}{gz+k}, \quad f_B(f_A(z)) = (f_B \circ f_A)(z) = \frac{e \frac{az+b}{cz+d} + g}{h \frac{az+b}{cz+d} + k} \cdot \frac{cz+d}{cz+d} = \frac{e(az+b) + g(cz+d)}{h(az+b) + k(cz+d)}$$

$$= \frac{(ae+cg)z + (be+dg)}{(ah+ck)z + (bh+dk)} = f_C(z), \quad C = \begin{bmatrix} ae+cg & be+dg \\ ah+ck & bh+dk \end{bmatrix} \quad f_b = (0, \beta)(\alpha, \infty)$$

$$BA = \begin{bmatrix} e & g \\ h & k \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ae+cg & be+dg \\ ah+ck & bh+dk \end{bmatrix}$$

$$f_B \circ f_A = f_{BA}$$

$$\{a\beta + b\beta : a, b \in \mathbb{F}_2\} = \{0, 1, \beta, \beta\beta\}$$

$\mathbb{F}_4 \supset \mathbb{F}_2$ is a 2-dim vector space with basis $\{1, \alpha\}$.

$$\mathbb{F}_4 = \{0, 1, \alpha, \beta\}$$

+	0	1	α	β
0	0	1	α	β
1	1	0	β	α
α	α	β	0	1
β	β	α	1	0

·	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	β	1
β	0	β	1	α

The map $GL_2(F) \rightarrow \{\text{fractional linear transformations}\}$

$$A \mapsto f_A \quad \text{where } f_A(z) = \frac{az+b}{cz+d}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a homomorphism. This homomorphism is onto but in general not one-to-one:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mapsto f_I, \quad f_I(z) = \frac{1z+0}{0z+1} = z$$

$$\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad f_{\lambda I}(z) = \frac{\lambda z+0}{0z+\lambda} = z$$

The kernel of our homomorphism is $Z(GL_2(F)) = \{\lambda I : \lambda \in F, \lambda \neq 0\}$.

$$Z(SL_2(\mathbb{F}_q)) = \{\lambda I \in SL_2(\mathbb{F}_q) : \det(\lambda I) = 1\} = \{I\}$$

$$\det \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \lambda^2 = 1 \iff \lambda = 1.$$

Frac. lin. transfr. over F are permutations of $F \cup \{\infty\}$

$$\text{eg. } a = \begin{bmatrix} \alpha & 1 \\ 1 & \beta \end{bmatrix}, \quad b = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad ab = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$$

$$a^{-1} = b^{-1} = (ab)^{-1} = I$$

$$f_a(z) = \frac{z+1}{\alpha z + \beta}$$

$$f_b(z) = \frac{\alpha z + 0}{0z + \beta} = \beta z$$

$$f_{ab}(z) = \frac{\alpha z + \beta}{\beta z + \alpha}$$

$$f_a = (0, \alpha, \infty, \beta, 1)$$

$$f_b = (1, \beta, \alpha)$$

$$f_{ab} = (0, \alpha)(\beta, \infty) = f_a \circ f_b$$

$$|A_8| = \frac{1}{2} 8! = \frac{1}{2} \times 40320 = 20160$$

$$|GL_4(\mathbb{F}_2)| = (2^4-1)(2^4-2)(2^4-2^2)(2^4-2^3) = 15 \times 14 \times 12 \times 8 = 20160$$

$$GL_4(2) = SL_4(2) = PGL_4(2) = PSL_4(2)$$

$$|GL_3(4)| = (4^3-1)(4^3-4)(4^3-4^2) = 63 \cdot 60 \cdot 48 = 181,440$$

$$|SL_3(4)| = 60,480$$

$$Z(SL_3(4)) = \{cI : c \neq 0\} = \{I, \alpha I, \beta I\}$$

$$\uparrow$$

$$c \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}, \quad c = 1, \alpha, \beta \quad \text{in } \mathbb{F}_4 = \{0, 1, \alpha, \beta\}$$

$$\det(cI) = c^3 = 1$$

$PSL_3(4) = SL_3(4)/Z(SL_3(4))$. This group consists of invertible 3×3 matrices over $\mathbb{F}_4 = \{0, 1, \alpha, \beta\}$ of determinant 1 in which we identify scalar multiples i.e. $A, \alpha A, \beta A$ give the same group element.

$$|PSL_3(4)| = \frac{60,480}{3} = 20,160.$$

Easyish fact: $A_8, GL_4(2)$ have elements of order 15; $PSL_3(4)$ does not. $PSL_3(4) \not\cong A_8$ or $GL_4(2)$.

A harder: $A_8 \cong GL_4(2)$.

$(1\ 2\ 3\ 4\ 5)(6\ 7\ 8) \in A_8$ of order 15. Equivalently, $(1\ 2\ 3\ 4\ 5) \in A_5$ of order 5; $(6\ 7\ 8) \in A_3$ of order 3; and they commute.

Why does $GL_4(2)$ have an element of order 15? Consider $\mathbb{F}_{16} > \mathbb{F}_2$, an extension of degree 4. $\mathbb{F}_{16}^\times$ is a multiplicative cyclic group of order 15. $\mathbb{F}_{16}^\times = \{1, \omega, \omega^2, \omega^3, \dots, \omega^{14}\}$, $\omega^{15} = 1$.

$$\mathbb{F}_{16} = \{0, 1, \omega, \omega^2, \dots, \omega^{14}\}$$

$$\begin{aligned} \mathbb{F}_{16} &\rightarrow \mathbb{F}_{16} \\ x &\mapsto \omega x \end{aligned}$$

$$\begin{aligned} f(x) &= x^4 + x + 1 \in \mathbb{F}_2[x] \\ \mathbb{F}_{16} &= \mathbb{F}_2[\omega] \text{ where } \omega \text{ is a root of } f(x) \\ &= \{a + b\omega + c\omega^2 + d\omega^3 : a, b, c, d \in \mathbb{F}_2\} \\ \omega^4 &= \omega + 1 \end{aligned}$$

Equivalently, take $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ (companion matrix for $f(x)$)

$$\text{so } A^4 = A + I$$

$$\mathbb{F}_2[A] = \{aI + bA + cA^2 + dA^3 : a, b, c, d \in \mathbb{F}_2\}$$

$$\subset \{4 \times 4 \text{ matrices over } \mathbb{F}_2\}$$

$$\{0, I, A, A^2, \dots, A^{14}\}, \quad A^{15} = I.$$

$$A \in GL_4(2) \text{ has order 15.}$$

Claim: $\text{PSL}_3(4)$ has no element of order 15. (So it cannot be isomorphic to A_5 or $G_2(2)$.)

Proof: Suppose $A \in \text{PSL}_3(4)$ has order 15, so A^3 has order 5 and A^5 has order 3; here A^3 and A^5 commute.

By Sylow theory, any two subgroups of order 5 are conjugate in $G = \text{PSL}_3(4)$. $\langle A^3 \rangle$ is a Sylow 5-subgroup. WLOG, $A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}$. Next use a general fact from linear algebra

If A, B are $n \times n$ matrices which commute ($AB=BA$) then every eigenspace for A is an invariant subspace for B i.e. if $Av = \lambda v$ then $A(Bv) = \lambda(Bv)$. Proof: $A(Bv) = B(Av) = B(\lambda v) = \lambda(Bv)$.

$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}$ has $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ as an eigenvector with eigenvalue 1, and A^5 commutes with A^3 so $\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rangle$ is invariant under A^5 . ($\begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix}$ doesn't have eigenvalue 1, so the 1-eigenspace of A^3 is $\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rangle$.)

so $A^5 = \begin{bmatrix} c & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$ $c \neq 0 \Rightarrow c \in \{1, \alpha, \beta\}$, WLOG $c=1$ so $A^5 = \begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$ has order 3 in G

$A^5 = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & c \end{bmatrix}$ where $c \in \text{SL}_2(4)$, $c^3 = I$. $|\text{SL}_2(4)| = 60 = 2^2 \cdot 3 \cdot 5$. c^3 is conjugate (in $\text{SL}_2(4)$) to $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ and c^3 commutes with $\begin{bmatrix} 1 & * \\ \alpha & \beta \end{bmatrix}$. However, $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ has two ^{distinct} eigenspaces $\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$, $\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle$ and so c^3 has two distinct eigenspaces and these are not invariant under $\begin{bmatrix} 1 & * \\ \alpha & \beta \end{bmatrix}$ which has no eigenvectors; its char. poly. is irreducible over $\mathbb{F}_4$. Contradiction!

More examples of group presentations:

The dihedral group of order $2n$ has presentation $\langle a, b : a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle$

The cyclic group of order n is $\langle a : a^n = 1 \rangle$

The symmetric group S_n of degree $n \geq 3$ is generated by $(1, 2), (1, 2, 3, \dots, n)$ but not in a convenient way since their relations are very complicated.

We might instead use all $\binom{n}{2}$ transpositions (i, j) , $1 \leq i < j \leq n$ or simply the $n-1$ transpositions $(i, i+1)$, $1 \leq i \leq n-1$

S_n has presentation $\langle r_1, \dots, r_{n-1} : r_i^2 = r_1^2 = \dots = r_{n-1}^2 = 1, (r_i r_{i+1})^3 = 1 \text{ for } i=1, 2, \dots, n-2; (r_i r_j)^2 = 1 \text{ for } 1 \leq i < j \leq n-1, |i-j| \geq 2 \rangle$

$= \langle r_1, \dots, r_{n-1} : (r_i r_j)^2 = 1 \text{ if } i=j \text{ or } |i-j| \geq 2; (r_i r_j)^3 = 1 \text{ if } |i-j|=1 \rangle$

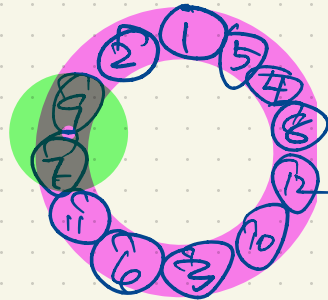
S_n is an example of a Coxeter group $\langle r_1, \dots, r_{n-1} : r_i r_j = 1 \text{ if } i=j \rangle$

i.e. a group given by a presentation $\langle r_1, \dots, r_{n-1} : (r_i r_j)^{m_{ij}} = 1 \rangle$,

$A_n < S_n$, $|A_n| = \frac{1}{2} n!$, A_n is generated by

$$\begin{aligned} g_1 &= (12)(23) = (1, 3, 2) \\ g_2 &= (12)(34) \\ g_3 &= (12)(45) \\ &\vdots \\ g_{n-2} &= (12)(n-1, n) \end{aligned}$$

$$\underbrace{bab^{-1} = a^{-1}}_{\text{relation}} \longleftrightarrow \underbrace{abab^{-1} = 1}_{\text{relator}}$$



$(3, 4)(7, 8)$ has order 2

$(3, 4)(4, 5) = (3, 5, 4)$ has order 3

$(3, 4)(3, 4) = 1$...

$$m_{ii} = 1, \quad m_{ij} = m_{ji} \geq 1$$

$$g_i^2 = 1$$

$$g_i^2 = g_j^2 = \dots = g_{n-1}^2 = 1$$

$$(g_i g_j)^2 = 1 \quad \text{if } |i-j| \geq 2$$

$$(g_i g_j)^3 = 1 \quad \text{if } |i-j| = 1$$

$$A_8 \cong GL_4(2)$$

2.5 Satz. Sei $V = V(4, 2)$ der Vektorraum der Dimension 4 über dem Körper $GF(2)$ und $GL(4, 2)$ die Gruppe aller linearen Abbildungen von V auf sich. Dann gilt $GL(4, 2) \cong \mathfrak{A}_8$.

Beweis. Die Matrizen

$$G_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad G_2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$G_3 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad G_4 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$G_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad G_6 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

liegen in $GL(4, 2)$. Triviale, aber etwas langwierige Rechnungen zeigen, daß sie die in I, 19.8 (S. 138) angegebenen definierenden Relationen

$$G_1^3 = G_i^3 = E \quad (2 \leq i \leq 6), \quad (G_i G_{i+1})^3 = E \quad (1 \leq i \leq 5),$$

$$(G_i G_j)^2 = E \quad (i + 1 < j)$$

von $\mathfrak{A}_8$ erfüllen. Also ist $\mathfrak{G} = \langle G_1, \dots, G_6 \rangle$ ein epimorphes Bild von $\mathfrak{A}_8$. Da $\mathfrak{A}_8$ nach 2.4 einfach ist, folgt $\mathfrak{G} \cong \mathfrak{A}_8$. Eine einfache Rechnung zeigt (siehe auch II, 6.2) $|GL(4, 2)| = |\mathfrak{A}_8|$. Wegen $\mathfrak{G} \leq GL(4, 2)$ folgt jetzt $\mathfrak{G} = GL(4, 2) \cong \mathfrak{A}_8$. q.e.d.

B. Huppert,
Endliche Gruppen I
1967.

With no relations in a group presentations, we have a free group.

$$F_n = \langle r_1, \dots, r_n \rangle = \text{free group on } n \text{ generators} = \{ \text{all (finite) products of } r_1, \dots, r_n, r_1^{-1}, r_2^{-1}, \dots, r_n^{-1} \}$$

no relations

$$r_i^k r_i^l = r_i^{k+l}, \quad r_i^0 = 1.$$

$$F_1 = \langle r \rangle = \{ \dots, r^{-3}, r^{-2}, r^{-1}, 1, r, r^2, r^3, \dots \} = \text{infinite cyclic group} \cong \mathbb{Z}$$

Informally, $F_1 = \mathbb{Z}$.

$$\mathbb{Z} \xrightarrow{\cong} F_1$$

$$k \mapsto r^k$$

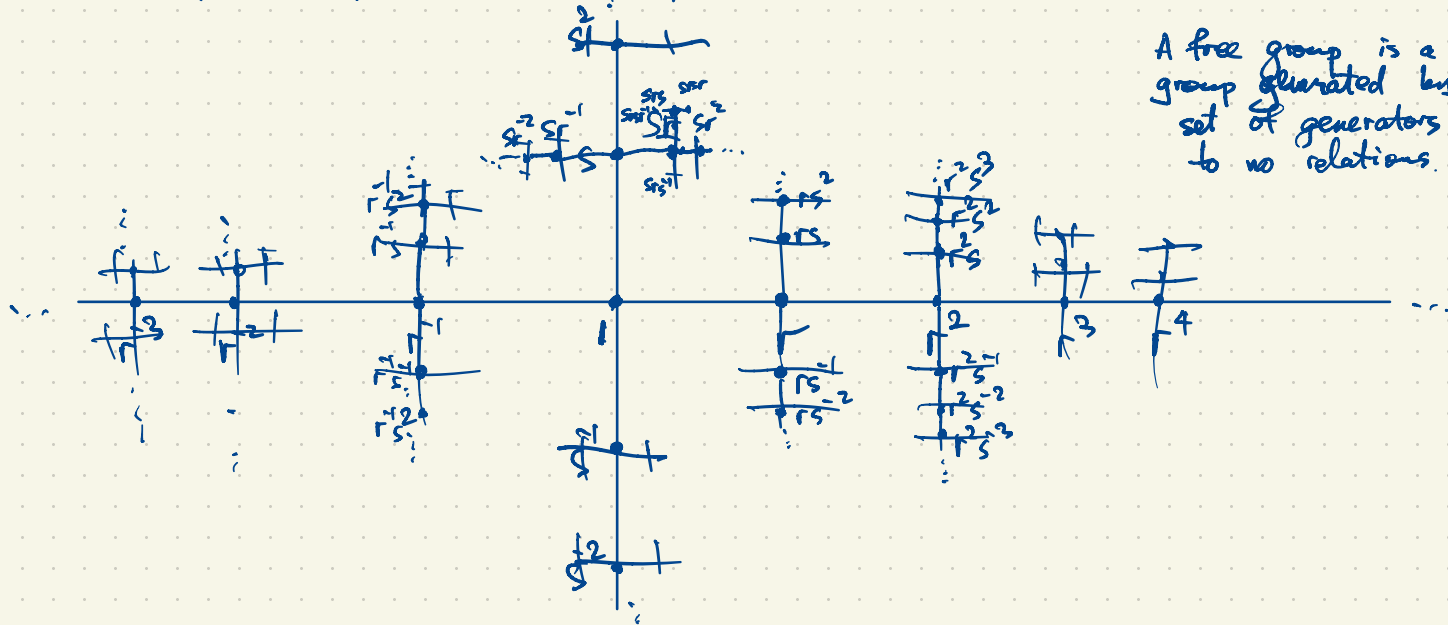
$$d(r_i, r_j) = |i-j|$$

$$\overline{r^{-3} r^{-2} r^{-1} 1 r r^2 r^3} \dots$$

$$F_0 = \{1\} \text{ trivial group}$$

$$F_2 = \langle r, s \rangle = \{ \text{all products of powers of } r\text{'s and } s\text{'s} \} = \{ 1, r, r^{-1}, s, s^{-1}, rs, rs^{-1}, \dots, r^3 s^{-1} r^2 s^5 r, \dots \}$$

A free group is a group generated by a set of generators subject to no relations.



A group presentation expresses a group G as $G \cong \langle X | R \rangle$

$\uparrow$ set of generators i.e. literal symbols
 $\uparrow$ relations: words on the symbols

words: strings of letters

Every relation can be rewritten as $w=1$ where the relator w is a word i.e. $w \in \langle X \rangle$

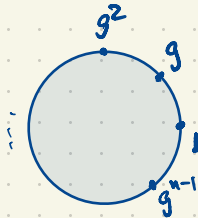
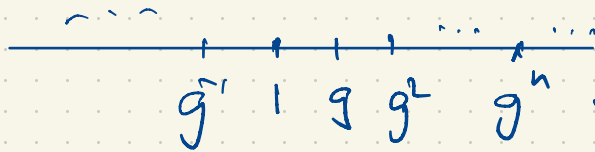
$G = \langle X | R \rangle$ is a homomorphic image of the free group $\langle X \rangle$;

$\langle X \rangle \longrightarrow \langle X | R \rangle = \langle X \rangle / K$ where $K \trianglelefteq \langle X \rangle$ is the normal subgroup generated by elements of R and their conjugates in $\langle X \rangle$.

eg. $C_n = \langle g | g^n = 1 \rangle = \langle g | g^n \rangle = \langle g \rangle / \langle g^n \rangle = \{ \dots, g^3, g^2, g, 1, g, g^2, g^3, \dots \} / \{ \dots, g^{2n}, g^n, 1, g, g^{2n}, \dots \}$

$\uparrow$ one generator $\uparrow$ one relator

$= \{ 1, g, g^2, \dots, g^{n-1} \}$



$$\langle r, s \mid rs = sr \rangle = \langle r, s \mid r^i s^j rs = 1 \rangle = \langle r, s \mid r^i s^j rs \rangle = \{ r^i s^j \mid i, j \in \mathbb{Z} \} \cong \mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$$

free abelian group on two generators

$$S_3 = \langle r, s \mid r^2 = s^2 = (rs)^3 = 1 \rangle = \langle r, s \mid r^2, s^2, (rs)^3 \rangle$$

If we have a group G generated by two elements r, s satisfying $r^2 = s^2 = (rs)^3 = 1$, then we show $|G| = 1, 2$ or 6 and G is $1, C_2$ or S_3 respectively.

Elements of G are products of powers of r and s . Since $r^{-1} = r$ and $s^{-1} = s$, elements of G must have the form $1, r, s, rs, sr, rsr, srs, rsrs, srsr, \dots$

$$\text{But } rsrsrs = 1 \Rightarrow rsrsrs \cdot srs = 1 \cdot srs \Rightarrow rsr = srs \Rightarrow rsrs = srsr = sr$$

$$\text{so } G = \{1, r, s, rs, sr, rsr\} \quad (\text{at most six distinct elements})$$

If these six elements are distinct then $G \cong S_3$; an isomorphism $G \rightarrow S_3$ is given by $r \mapsto (12), s \mapsto (23), (rs = (132) \text{ etc.})$

These three possibilities for G are all realizable and no other possibilities occur.

All possibilities arise as S_3/K where $K \trianglelefteq S_3$. ($K = 1, A_3 = \langle rs \rangle = \langle (123) \rangle$, or S_3).

Next consider

$$G = \langle r, s \mid r^3, s^3, (rs)^2 \rangle$$

A group G generated by two elements r, s satisfying $r^3 = s^3 = (rs)^2 = 1$ has

$$|G| = 1, 3 \text{ or } 12 \quad (1, C_3, \text{ or } A_4)$$

Coxeter, Todd (1936) described an algorithm for determining G from its presentation.

We will deduce $|G| \leq 12$ by showing G has a subgroup H with $|H| \leq 3$ and $[G:H] \leq 4$.

(So $|G| = [G:H]|H| \leq 4 \cdot 3 = 12$.)

(Lagrange's Theorem)

Recall: Right cosets of $H \leq G$ have the form $Hg = \{hg : h \in H\}$, $g \in G$; $|Hg| = |H|$
 G permutes the right cosets of H by $Hx \mapsto Hxg$.

Let's take $H = \langle r \rangle$ so $|H| = 1$ or 3 . Right cosets of H in G :

$$(1) \bullet H = H1 = Hr$$

$$(2) \bullet Hs$$

$$(3) \bullet Hs^2 = Hsr$$

$$(4) \bullet Hs^2r = Hsr^2$$

$$rsrs = 1 \Rightarrow rsrs \cdot s^2r^2 = 1 \cdot s^2r^2 \Rightarrow rs = s^2r^2$$

$$sr = r^2s^2$$

$$Hsr = Hr^2s^2 = Hs^2$$

$$(Hs^2r)s = Hs^2 \cdot s^2r^2 = Hsr^2 = Hs^2r$$

$$Hsr^2 = Hsr \cdot r = Hr^2s^2r = Hs^2r$$

A_4 has normal subgroups 1 , $\langle (12)(34), (13)(24) \rangle$, A_4

These must be the only right cosets of H (not necessarily distinct)

Right multiplication by s permutes the right cosets of H as $(123)(4) = (123)$

$$(1)(234)$$

Assuming we actually have 4 distinct right cosets of H in G then $[G:H] = 4$, $|H| = 3$ and $|G| = 12$.
 And we have an isomorphism $G \rightarrow A_4$, $r \mapsto (234)$, $s \mapsto (123)$. So $rs \mapsto (234)(123) = (12)(34)$

$$G = \langle r, s \mid r^3, s^3, (rs)^2 \rangle, \quad H = \langle r \rangle$$

$$Hr = H$$

	r	r	r	s	s	s	r	s	r	s
1	1	1	1	2	3	1	1	2	3	1
2	3	4	2	3	1	2	3	1	1	2
3	4	2	3	1	2	3	4	4	2	3
4	2	3	1	4	4	4	2	3	4	4

This is the Coxeter-Todd coset enumeration algorithm.

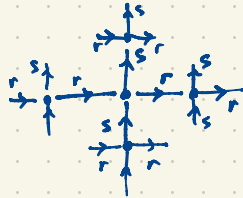
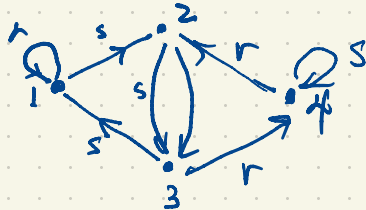
The largest (i.e. most general) group generated by two elements r, s satisfying $r^3 = s^3 = (rs)^2 = 1$ has

$$[G:H] = 4, \quad |H| = 3, \quad |G| = 12.$$

Such a group is represented as a permutation group $\langle (234), (123) \rangle = A_4$.

$$\langle r, s \mid r^3, s^3, (rs)^2 \rangle \cong A_4 = F_2 / \text{normal subgp. gen. by } r^3, s^3, (rs)^2 \text{ and their conjugates}$$

The Schreier graph of $G = \langle r, s \mid r^3, s^3, (rs)^2 \rangle$ on the cosets of $H = \langle r \rangle$



This tree is the Cayley graph of $F_2 = \langle r, s \rangle$ (free group on two generators)

vertex 1 represents $H = H1$
 " 2 " Hs
 " 3 " Hs^2

$S_3 \cong \langle r, s \mid r^2, s^2, (rs)^3 \rangle$ on cosets of $H = \langle r \rangle$
 Coxeter diagram: $\overset{r}{\circ} \text{---} \overset{s}{\circ}$
 When generators of G have order 2, write --- in place of $\curvearrowright$ and $\circ$ instead of $\bigcirc$.

Schreier graph



$$(Hsr)s = Hsr = Hsr$$

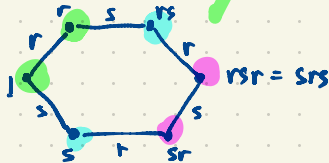
$$rsrsrs = 1$$

$$rsrsrs \cdot srs = 1 \cdot srs$$

$$rsr = srs$$

branched cover

S_3 on cosets of $H = \{1\}$: Schreier graph reduces to a Cayley graph: vertices of graph are elements of G (same as cosets of $\{1\}$).



Cayley graph of S_3 : homeomorphic to $S^1 = \bigcirc$

(12) (23) (34)

$$(rt)^2 = rtot = 1$$

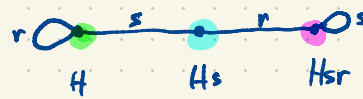
$$rt \cdot tr = 1 \cdot tr$$

A Coxeter group is a group having a presentation using generators and relators $(rs)^{m_{rs}}, (rt)^{m_{rt}}, (st)^{m_{st}}, \dots$
 (all exponents are positive integers)

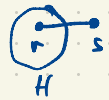
$\langle r, s \mid r^2, s^2, (rs)^2 \rangle$ has Coxeter diagram: $\overset{r}{\circ} \text{---} \overset{s}{\circ}$

This Coxeter group is a Klein 4-group $C_2 \times C_2$.

S_3 has Schreier graph on subgroup $H = \langle r \rangle$.



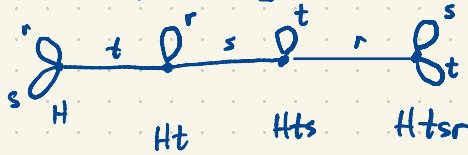
Coxeter diagram



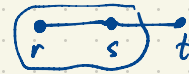
$$S_4 = \langle r, s, t : r^2, s^2, t^2, (rs)^2, (rt)^2, (st)^2 \rangle$$

Find Schreier graph on

$$H = \langle r, s \rangle \cong S_2$$



Coxeter diagram
 $rt = tr$



$$sts = tst$$

$$Hr = H$$

$$Hs = H$$

$$(Ht)r = Htr = Ht$$

$$(Hts)t = Hsts = Hts$$

$$(Htsr)t = Htstr = Htsr = Htsr$$

$$|S_4| = 4 \times 6 = 24$$

S_n has Coxeter diagram $A_{n-1} = 1 - 2 - 3 \dots n-1$

$S_n = W(A_{n-1}) =$ Weyl group
(Coxeter group)
of type A_{n-1}

$$|S_n| = n! = n |S_{n-1}|$$