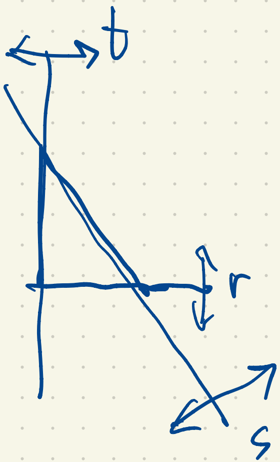
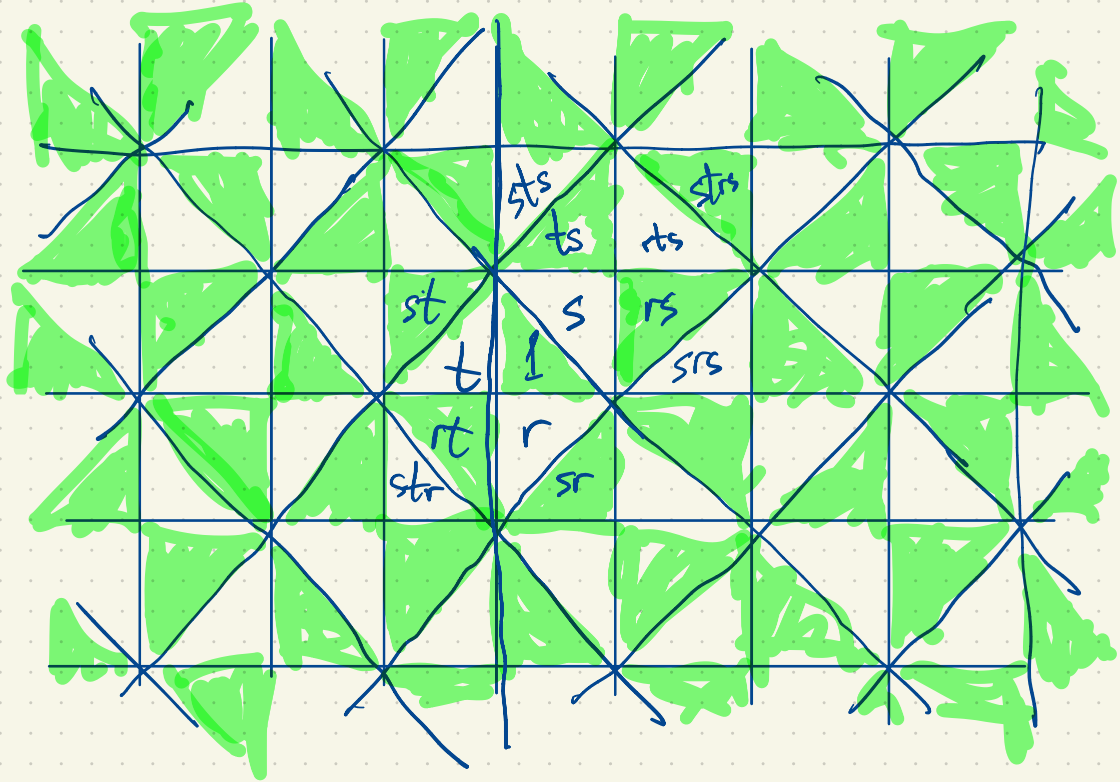
Group Theory

Book 2

reflections interchange
white triangles $\leftrightarrow$ green triangles

Elements of $G = G_{map}(4,4,2)$
white? green?





$G = G(4,4,2)$ labels the green triangles
W labels all triangles

More generally if $l, m, n \geq 2$ satisfying $\frac{1}{l}+\frac{1}{m}+\frac{1}{n} = 1$ then $G = G(l,m,n)$ is a group of isometries of the Euclidean plane generated by rotations of order l, m, n:

$$G = \langle a, b, c \mid a^l, b^m, c^n, abc \rangle$$

Moreover $[W:G]$ where $W = \langle r,s,t \mid r^2, s^2, t^2, (rs)^l, (st)^m, (tr)^n \rangle$.

If $\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}>1$, then $G=G(\ell,m,n)$ finite in place of a tiling of the Euclidean plane, we get a tiling of S^2 (Euclidean sphere).

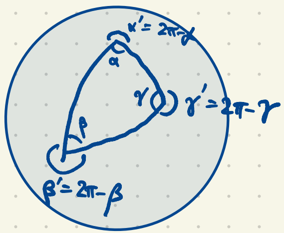
If $\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}<1$, then $G=G(\ell,m,n)$ infinite we get a tiling of the hyperbolic plane by congruent triangles.

A spherical example: $G=G(2,3,4)$, $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{6+4+3}{12}=\frac{13}{12}>1$

Eine Kleine spherical geometry

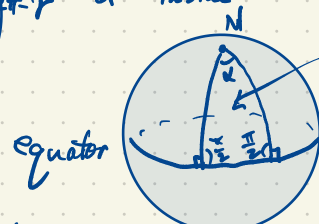
Let $S\subset\mathbb{R}^3$ be a unit sphere; its surface area is $4\pi r^2=4\pi$. "Lines" on S are geodesics (great circles).

Triangles in S have area = angular excess = $\alpha+\beta+\gamma-\pi>0$



$(\alpha+\beta+\gamma-\pi)+(\alpha'+\beta'+\gamma'-\pi)=6\pi-2\pi=4\pi$

angular excess of "inside" — angular excess "outside"



Area $=\alpha+\frac{\pi}{2}+\frac{\pi}{2}-\pi=\alpha$

$|G|=24$.

$G\cong S_4$

W consists of isometries of S preserving the tiling.

$[W:G]=2$, $G=G(2,3,4)$

$G=\langle a,b,c\mid a^2,b^3,c^4,abc\rangle$

$a=rs$, $b=st$, $c=tr$

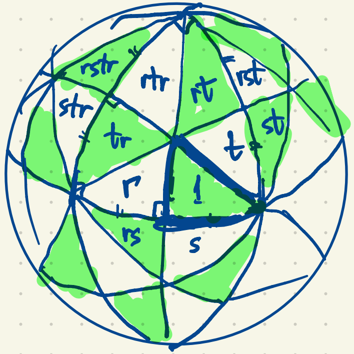
We subdivide S into 48 congruent spherical triangles each with angles $\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{4}$ and area $\frac{4\pi}{48}=\frac{\pi}{12}=\frac{\pi}{2}+\frac{\pi}{3}+\frac{\pi}{4}-\pi$

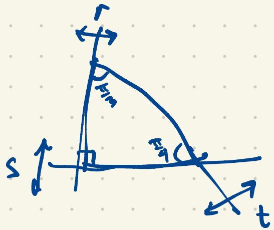
r,s,t: reflections of S in the "lines" bounding one triangle i.e. reflections in the corresponding planes through the origin

$W\cong C_2\times S_4$

$|W|=48$

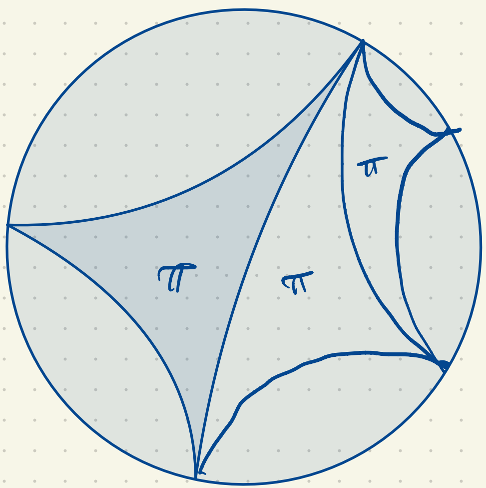
$W=W(\overset{3}{\bullet\!-\!\bullet}\overset{4}{=\!\bullet})=\langle r,s,t\mid r^2,s^2,t^2,(rs)^2,(rt)^3,(st)^4\rangle$, Starting Fri Oct 3 new room is BU24





Case $\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}<1$: $G(\ell,m,n)$ preserves a triangulation of the hyperbolic plane with triangles having angles $\frac{\pi}{\ell}, \frac{\pi}{m}, \frac{\pi}{n}$

In hyperbolic plane, a triangle (sides are lines = geodesics) having angular defect $\pi-(\alpha+\beta+\gamma)$ = area.

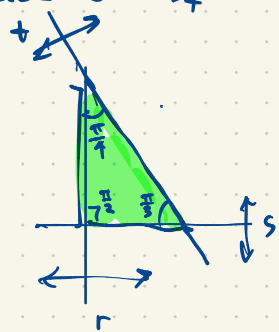


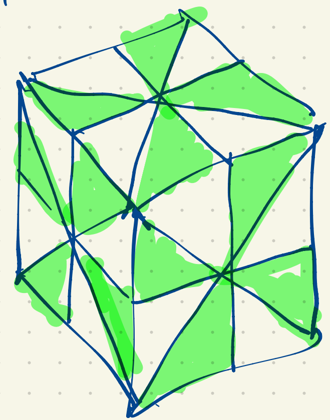
Today: $G = G(2,3,4) = \langle a,b,c : a^2, b^3, c^4, abc \rangle \cong S_4$

$W = W(B_2) \cong C_2 \times S_4$

$= \langle r,s,t \mid r^2, s^2, t^2, (rs)^2, (rt)^3, (st)^4 \rangle$

To recognize $G \cong S_4$ without computer: $G = \langle rs, st, tr$: rs, st, tr are rotations by angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$ about vertices of triangle shown$\rangle$





G = Group of rotational symmetries of a cube $\cong S_4$.

G permutes the four "body diagonals" of the cube in all $4! = 24$ ways.
(A "body diagonal" joins two opposite vertices.)

G permutes the 24 green triangles transitively (and regularly).

If $W = HK$ where H and K are normal subgroups with $H \cap K = 1$ (so H and K commute with each other i.e. $hk = kh$ for all $h \in H$, $k \in K$) then $W \cong H \times K = \{(h,k) : h \in H, k \in K\}$ (direct product)

In our case $|W| = 48$, $|G| = 24$, $W = G \cup Gr$, W has a subgroup $H = \langle h \rangle$ of order 2, $H \triangleleft W$, $h = -I$

G: preserve orientation; Gr: reverse orientation

$H = Z(W)$

Cube has $3 + 6 = 9$ planes of symmetry but altogether 24 orientation-reversing symmetries

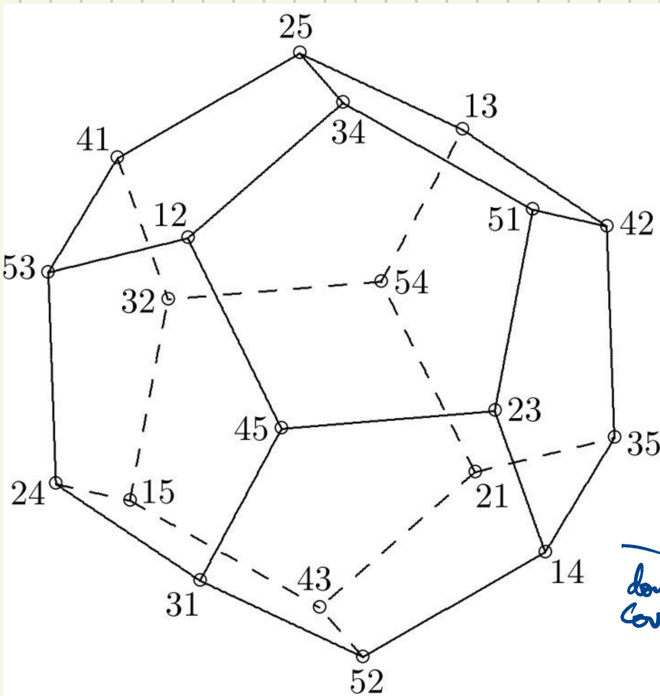
Similarly the regular dodecahedron (12 pentagonal faces) has rotational symmetry group G with $|G| = 60$, $G = A_5$ (Alt_5)

Elements of A_5 give rotational symmetries of the dodecahedron.

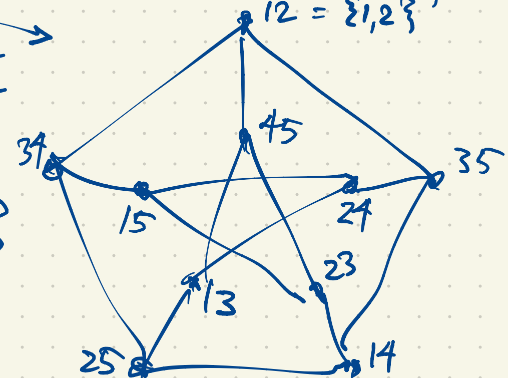
The full symmetry group of the regular dodecahedron has order 2×60 (60 orientation-preserving [rotations] and 60 orientation-reversing [reflections + other])

The full group of symmetries has order 120 and it has a subgroup isomorphic to A_5 but the group is not S_5. S_5 is not a group of isometries of $\mathbb{R}^3$.

Instead the full group of isometries of the regular dodecahedron is isomorphic to $C_2 \times A_5$.



double cover



This graph, the Petersen graph, has isomorphism group S_5

$$(i\ j) \mapsto \{i, j\}$$
$$(j\ i) \mapsto \{i, j\}$$

A presentation of a group G is an expression $G = \langle X : R \rangle$ where X is a set of symbols (letters) and $R \subset F(X)$ = free group on $X = \{x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} : x_j \in X,\ i_j \in \mathbb{Z}\}$

$x^j x^k = x^{j+k}$ $(j,k \in \mathbb{Z};\ x \in X)$

$x^0 = 1$ = identity.

X = set of "generators"

R = set of "relators" (words in the generators)

If X is finite then G is <u>finitely generated</u>.

If X and R are both finite then G is <u>finitely presented</u>.

Burnside groups are finitely generated (usually) but not finitely presented.

$G = F /$ subgroup of F generated by R and their conjugates

= "<u>largest</u>" homomorphic image of $F = F(X)$ having R in kernel

universal as we'll discuss later – see handout.

Every group has a presentation. Given G, for every $g \in G$, introduce a generator x_g. So $X = \{x_g : g \in G\}$.

For every pair $g, h \in G$ we want to force $x_g x_h = x_{gh}$ but this doesn't happen in $F = F(X)$ so introduce relators $x_g x_h x_{gh}^{-1} \in R$. $R = \{x_g x_h x_{gh}^{-1} : g, h \in G\}$. Then $F / \{\ldots R \ldots\} \cong G$.

If G is finitely generated then G is countable i.e. finite or countably infinite.

If $X = \{x_1, \ldots, x_n\}$ then $F = F(X) = \{$<u>products</u> of $x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\} = \bigcup_{\ell=0}^{\infty} S_\ell$ where

finite products

F is a countable union of finite sets, hence countable.

$S_\ell = \{x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_\ell}^{e_\ell} :$ $i_1, \ldots, i_\ell \in \{1, \ldots, n\}$, $e_j \in \{\pm 1\}\}$

$|S_\ell| \le 2^\ell$

If G is countably generated (X countable) then $F = F(X)$ is countable.

$S_\ell = \{$words of length $\ell\}$ is countable.

If $|X| = m$ and $|R| = n$, m, n pos. integers, what can we say about $|G|$ where $G = \langle X \mid R \rangle$?
If $n < m$ then G is infinite. That is (contrapositive form) in order for $|G| < \infty$, we need at least as many relators as generators.

eg. $m = 1$. $X = \{x\}$. If $R = \emptyset$ then $G = \langle x \mid \emptyset \rangle = \{\ldots, x^{-2}, x^{-1}, 1, x, x^2, \ldots\}$

If $R = \{1\}$ then $G = \langle x \mid 1 \rangle = F(x) = \{\ldots, x^{-2}, x^{-1}, 1, x, x^2, \ldots\}$

If $R = \{x^{15}\}$ then $G = \langle x \mid x^{15} \rangle = \{1, x, x^2, \ldots, x^{14}\} \cong C_{15}$

If $R = \{x^{15}, x^{40}\}$ then $G = \langle x \mid x^{15}, x^{40} \rangle = \langle x \mid x^5 \rangle = \{1, x, x^2, x^3, x^4\}$

$$x^5 = (x^{15})^3 (x^{40})^{-1}$$

If G is a group then a homomorphic image of G is the image of G under a homomorphism $G \xrightarrow{\phi} H$ (surjective, otherwise replace H by $\phi(G) \le H$.)

Note: $H \cong G/K$ where $K = \ker \phi \trianglelefteq G$. A homomorphic image of G is the same thing as a quotient group G/K, $K \trianglelefteq G$.

In particular the abelianization of G is the largest homo. image of G which is abelian.

A normal subgroup $K \trianglelefteq G$ yields an abelian quotient group G/K iff $K \ge G' =$ derived subgroup of G

i.e. $G' = \langle [g,h] : g, h \in G \rangle$, $[g,h] = g^{-1}h^{-1}gh$.

If $K \ge G'$ then in G/K, take any two elements $gK, hK \in G/K$, we have

$$hg[g,h] = hg\,g^{-1}h^{-1}gh = gh$$

$$(hK)(gK) = hgK = ghK = (gK)(hK) \quad \text{and conversely.}$$

eg. the abelianization of an abelian gp. is itself.

eg. the abelianization of S_n, $n \ge 2$ is C_2 (group of order 2).

The abelianization of $GL_n(F)$ $(n \geq 1)$ is $F^\times = \{$ nonzero elements of $F \}$

$$GL_n(F) \xrightarrow{\det} F^\times \qquad \ker(\det) = SL_n(F)$$

If $F = F(X)$ free group on m generators $X = \{x_1, \cdots, x_m\}$ e.g. $m=3$

The abelianization of F $w = x_1 x_2^3 x_1^{-4} x_3^2 x_1 x_2^{-1} \longmapsto (-2, 2, 2) \in \mathbb{Z}^3$

$F/F' \cong \mathbb{Z}^m$ where $F' = \{w \in F :$ every x_i has exponents in w adding to 0, $i = 1, 2, \cdots, m.\}$ (Multiplicatively, $x_1^{-2} x_2^2 x_3^2$.

(free abelian group on m generators)

$\mathbb{Z}^3 / \langle (1,2,3), (4,9,-1) \rangle$

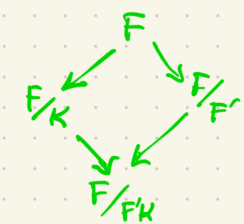
Now consider $G = \langle x_1, \cdots, x_m \mid r_1, \cdots, r_n \rangle$, $r_1, \cdots, r_n \in F = F(x_1, \cdots, x_m)$

$= F/K$, K is the normal subgp of F generated by $r_1, \cdots, r_n$ and their conjugates in F.

We want to show that if $n < m$ then $|G| = \infty$.

To prove this, first consider $F/F'K$ Here $F', K \trianglelefteq F$.

$F'K = \{ab : a \in F', b \in K\} \trianglelefteq F$.



$$F/F'K \cong \frac{F/F'}{F'K/F'}$$ (Third Isomorphism Theorem also sometimes called the Second Isomorphism theorem).

$F/F'K$ is a homomorphic image of $F/F' \cong \mathbb{Z}^m$.

$F/F'K \cong \mathbb{Z}^m /$ subgp. generated by n elements.

$F'K/F' \cong K/K \cap F'$ (Second Iso. Thm.)

$$F/F'K \cong \frac{F/K}{F'K/K}$$

$$\mathbb{Z}^3 / \langle (1,0,0), (0,2,0) \rangle \cong (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$$

$$\mathbb{Z}^3 / \langle (1,0,0), (0,2,0), (0,0,3) \rangle \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$$

Coxeter-Todd coset enumeration is the best algorithm we have for deciding certain "word problems" in group theory but it doesn't work in all cases.

Group order: Given a presentation $G = \langle X \mid R \rangle$ (X, R finite)

what is $|G|$? Is it finite?

Is G trivial?

Given two words in F, $= F(X)$ do they yield the same element of G?

The word problem for groups is undecidable.

Matrix mortality problem

You are given a positive integer n and a list of $n \times n$ integer matrices $A_1, \cdots, A_k$.

Is there a finite product of these A_i that equals the zero matrix? This is a decision problem.

For $n = 1$, this problem is decidable.

For $n \geq 3$, this problem is undecidable.

For $n = 2$, we don't know whether the problem is decidable.

Burnside Groups $B(m,n)$

The restricted Burnside group $B(m,n)$ is the universal (ie. largest) group G generated by m elements $r_1, r_2, \dots, r_m$ subject to the condition that $x^n = 1$ for all $x \in G$.

G is finitely generated but may be infinite and the number of defining relations $x^n = 1$ for all $x \in G$ is in general infinite.

For $n=2$: such a group is necessarily abelian.

$x^2 = 1 = y^2 = (xy)^2$ for all $x, y \in G$

$xyxy = 1$

$yx \cdot xyxy = yx$

$xy = yx$

William Burnside
(1852-1927)

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$$B(2,2) = \langle r, s \mid r^2, s^2, (rs)^2 \rangle = \{1, r, s, rs\}$$

is a Klein 4-group.

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is a Klein 4-group.

B(2,3) is a nonabelian group of order 27 isomorphic to a Sylow 3-subgroup of $GL_3(\mathbb{F}_3)$:

$$B(2,3) = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{F}_3 \right\}$$

generated by $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Burnside Groups B(m, n)

The restricted Burnside group $B(m,n)$ is the universal (ie. largest) group G generated by m elements $r_1, r_2, \cdots, r_m$ subject to the condition that $x^n = 1$ for all $x \in G$. (m is the rank, n is the exponent).

n	$\lvert B(2,n) \rvert$
2	4
3	27

Burnside Groups $B(m,n)$

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n	$\lvert B(2,n) \rvert$
2	4
3	27
4	4096

Marshall Hall, Jr
(1920-1990)

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n	$\lvert B(2,n) \rvert$	
2	4	
3	27	
4	4096	
5	?	finite or infinite?

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n	$\lvert B(2,n) \rvert$
2	4
3	27
4	4096
5	? finite or infinite?
6	$2^{28} 3^{25} =$ 227,442,304,239,437,611,008 (M. Hall, 1957)

Marshall Hall, Jr
(1920-1990)

	a		a		a		b		b		b		a		b		a		b		a		b		a		a		b		a		a		b		a		a		b	
1		1		1		1		2		3		1		1		2						3		1		1		1		2										3		1

	a
1	1
2	
3	
4	
5	
6	
7	

	b
1	2
2	3
3	1
4	
5	
6	
7	

$B(2,3)$ is the largest ("universal") group generated by two generators a, b such that $x^3 = 1$ for all x.

In particular $a^3 = 1 = b^3 = (ab)^3 = (a^2b)^3 = \cdots$

We show that $G = \langle a, b \mid a^3, b^3, (ab)^3, (a^2b)^3 \rangle$ is the example of nonabelian gp of order 27 above with exponent 3.

Use coset enumeration on $H = \langle a \rangle$ of order ≤ 3 to show $[G:H] \leq 9$. (so $|G| \leq 27$).

Together with our example of order 27 attaining the upper bound, this gives equality $|G| = 27$.

	a	a	a	b	b	b	a	b	a	b	a	b	a	a	b	a	a	b	a	a	b
1	1	1	1	2	3	1	1	2	4	5	3	1	1	1	2	4	6	7	5	3	1

a	
1	1
2	4
3	
4	6
5	3
6	
7	5

b	
1	2
2	3
3	1
4	5
5	
6	7
7	

	a	a	a	b	b	b	a	b	a	b	a	b	a	a	b	a	a	b	a	a	b
1	1	1	1	2	3	1	1	2	4	5	3	1	1	1	2	4	6	7	5	3	1

a	
1	1
2	4
3	
4	6
5	3
6	
7	5

b	
1	2
2	3
3	1
4	5
5	
6	7
7	

	a	a	a	b	b	b	a	b	a	b	a	b	a	a	b	a	a	b	a	a	b
1	1	1	1	2	3	1	1	2	4	5	3	1	1	1	2	4	6	7	5	3	1
2	4	6	2	3	1	2	4	5	3	1	1	2	4	6	7	5	3	1	1	1	2

a	
1	1
2	4
3	
4	6
5	3
6	2
7	5

b	
1	2
2	3
3	1
4	5
5	
6	7
7	

	a	a	a	b	b	b	a	b	a	b	a	b	a	a	b	a	a	b	a	a	b
1	1	1	1	2	3	1	1	2	4	5	3	1	1	1	2	4	6	7	5	3	1
2	4	6	2	3	1	2	4	5	3	1	1	2	4	6	7	5	3	1	1	1	2

a	
1	1
2	4
3	
4	6
5	3
6	2
7	5

b	
1	2
2	3
3	1
4	5
5	
6	7
7	

	a	a	a	b	b	b	a	b	a	b	a	b	a	a	b	a	a	b	a	a	b
1	1	1	1	2	3	1	1	2	4	5	3	1	1	1	2	4	6	7	5	3	1
2	4	6	2	3	1	2	4	5	3	1	1	2	4	6	7	5	3	1	1	1	2
3	7	5	3	1	2	3	7	8	9	6	2	3	7	5	10	11	12	4	6	2	3

a	
1	1
2	4
3	7
4	6
5	3
6	2
7	5
8	9
9	
10	11
11	12
12	

b	
1	2
2	3
3	1
4	5
5	10
6	7
7	8
8	
9	6
10	
11	
12	4

	a	a	a	b	b	b	a	b	a	b	a	b	a	a	b	a	a	b	a	a	b
1	1	1	1	2	3	1	1	2	4	5	3	1	1	1	2	4	6	7	5	3	1
2	4	6	2	3	1	2	4	5	3	1	1	2	4	6	7	5	3	1	1	1	2
3	7	5	3	1	2	3	7	8	9	6	2	3	7	5	10	10	10	4	6	2	3
4	6	2	4	5	10	4	6	7	5	10	10	4	6	2	3	7	5	10	10	10	4

a	
1	1
2	4
3	7
4	6
5	3
6	2
7	5
8	9
9	
10	10

b	
1	2
2	3
3	1
4	5
5	10
6	7
7	8
8	
9	6
10	4

10=11=12

	a	a	a	b	b	b	a	b	a	b	a	b	a	a	b	a	a	b	a	a	b
1	1	1	1	2	3	1	1	2	4	5	3	1	1	1	2	4	6	7	5	3	1
2	4	6	2	3	1	2	4	5	3	1	1	2	4	6	7	5	3	1	1	1	2
3	7	5	3	1	2	3	7	8	9	6	2	3	7	5	10	11	12	4	6	2	3
4	6	2	4	5	10	4						4									4

a	
1	1
2	4
3	7
4	6
5	3
6	2
7	5
8	9
9	
10	11
11	12
12	

b	
1	2
2	3
3	1
4	5
5	10
6	7
7	8
8	
9	6
10	4
11	
12	4

	a	a	a	b	b	b	a	b	a	b	a	b	a	a	b	a	a	b	a	a	b
1	1	1	1	2	3	1	1	2	4	5	3	1	1	1	2	4	6	7	5	3	1
2	4	6	2	3	1	2	4	5	3	1	1	2	4	6	7	5	3	1	1	1	2
3	7	5	3	1	2	3	7	8	9	6	2	3	7	5	10	10	10	4	6	2	3
4	6	2	4	5	10	4	6	7	5	10	10	4	6	2	3	7	5	10	10	10	4

a	
1	1
2	4
3	7
4	6
5	3
6	2
7	5
8	9
9	
10	10

b	
1	2
2	3
3	1
4	5
5	10
6	7
7	8
8	
9	6
10	4

10=11=12

	a	a	a	b	b	b	a	b	a	b	a	b	a	a	b	a	a	b	a	a	b
1	1	1	1	2	3	1	1	2	4	5	3	1	1	1	2	4	6	7	5	3	1
2	4	6	2	3	1	2	4	5	3	1	1	2	4	6	7	5	3	1	1	1	2
3	7	5	3	1	2	3	7	8	8	6	2	3	7	5	10	10	10	4	6	2	3
4	6	2	4	5	10	4	6	7	5	10	10	4	6	2	3	10	5	10	10	10	4
5	3	7	5	10	4	5	3	1	1	2	4	5	3	7	8	10	8	6	2	4	5

a	
1	1
2	4
3	7
4	6
5	3
6	2
7	5
8	8
10	10

b	
1	2
2	3
3	1
4	5
5	10
6	7
7	8
8	6
10	4

8=9

	a	a	a	b	b	b	a	b	a	b	a	b	a	a	b	a	a	b	a	a	b
1	1	1	1	2	3	1	1	2	4	5	3	1	1	1	2	4	6	7	5	3	1
2	4	6	2	3	1	2	4	5	3	1	1	2	4	6	7	5	3	1	1	1	2
3	7	5	3	1	2	3	7	8	9	6	2	3	7	5	10	10	10	4	6	2	3
4	6	2	4	5	10	4	6	7	5	10	10	4	6	2	3	7	5	10	10	10	4

a	
1	1
2	4
3	7
4	6
5	3
6	2
7	5
8	9
9	
10	10

b	
1	2
2	3
3	1
4	5
5	10
6	7
7	8
8	
9	6
10	4

10=11=12

	a	a	a	b	b	b	a	b	a	b	a	b	a	a	b	a	a	b	a	a	b
1	1	1	1	2	3	1	1	2	4	5	3	1	1	1	2	4	6	7	5	3	1
2	4	6	2	3	1	2	4	5	3	1	1	2	4	6	7	5	3	1	1	1	2
3	7	5	3	1	2	3	7	8	8	6	2	3	7	5	10	10	10	4	6	2	3
4	6	2	4	5	10	4	6	7	5	10	10	4	6	2	3	10	5	10	10	10	4
5	3	7	5	10	4	5	3	1	1	2	4	5	3	7	8	10	8	6	2	4	5

a	
1	1
2	4
3	7
4	6
5	3
6	2
7	5
8	8
10	10

b	
1	2
2	3
3	1
4	5
5	10
6	7
7	8
8	6
10	4

8=9

	a	a	a	b	b	b	a	b	a	b	a	b	a	a	b	a	a	b	a	a	b
1	1	1	1	2	3	1	1	2	4	5	3	1	1	1	2	4	6	7	5	3	1
2	4	6	2	3	1	2	4	5	3	1	1	2	4	6	7	5	3	1	1	1	2
3	7	5	3	1	2	3	7	8	8	6	2	3	7	5	10	10	10	4	6	2	3
4	6	2	4	5	10	4	6	7	5	10	10	4	6	2	3	10	5	10	10	10	4
5	3	7	5	10	4	5	3	1	1	2	4	5	3	7	8	10	8	6	2	4	5
6	2	4	6	7	8	6	2	3	7	8	8	6	2	4	5	3	7	8	8	8	6
7	5	3	7	8	6	7	5	10	10	4	6	7	5	3	1	1	1	2	4	6	7
8	8	8	8	6	7	8	8	6	2	3	7	8	8	8	6	2	4	5	3	7	8
10	10	10	10	4	5	10	10	4	6	7	5	10	10	10	4	6	2	3	7	5	10

a	
1	1
2	4
3	7
4	6
5	3
6	2
7	5
8	8
10	10

b	
1	2
2	3
3	1
4	5
5	10
6	7
7	8
8	6
10	4

	a	a	a	b	b	b	a	b	a	b	a	b	a	a	b	a	a	b	a	a	b
1	1	1	1	2	3	1	1	2	4	5	3	1	1	1	2	4	6	7	5	3	1
2	4	6	2	3	1	2	4	5	3	1	1	2	4	6	7	5	3	1	1	1	2
3	7	5	3	1	2	3	7	8	9	6	2	3	7	5	10	10	10	4	6	2	3
4	6	2	4	5	10	4	6	7	5	10	10	4	6	2	3	7	5	10	10	10	4

a	
1	1
2	4
3	7
4	6
5	3
6	2
7	5
8	9
9	
10	10

b	
1	2
2	3
3	1
4	5
5	10
6	7
7	8
8	
9	6
10	4

10=11=12

	a	a	a	b	b	b	a	b	a	b	a	b	a	a	b	a	a	b	a	a	b
1	1	1	1	2	3	1	1	2	4	5	3	1	1	1	2	4	6	7	5	3	1
2	4	6	2	3	1	2	4	5	3	1	1	2	4	6	7	5	3	1	1	1	2
3	7	5	3	1	2	3	7	8	8	6	2	3	7	5	10	10	10	4	6	2	3
4	6	2	4	5	10	4	6	7	5	10	10	4	6	2	3	10	5	10	10	10	4
5	3	7	5	10	4	5	3	1	1	2	4	5	3	7	8	10	8	6	2	4	5

a	
1	1
2	4
3	7
4	6
5	3
6	2
7	5
8	8
10	10

b	
1	2
2	3
3	1
4	5
5	10
6	7
7	8
8	6
10	4

8=9

	a	a	a	b	b	b	a	b	a	b	a	b	a	a	b	a	a	b	a	a	b
1	1	1	1	2	3	1	1	2	4	5	3	1	1	1	2	4	6	7	5	3	1
2	4	6	2	3	1	2	4	5	3	1	1	2	4	6	7	5	3	1	1	1	2
3	7	5	3	1	2	3	7	8	8	6	2	3	7	5	10	10	10	4	6	2	3
4	6	2	4	5	10	4	6	7	5	10	10	4	6	2	3	10	5	10	10	10	4
5	3	7	5	10	4	5	3	1	1	2	4	5	3	7	8	10	8	6	2	4	5
6	2	4	6	7	8	6	2	3	7	8	8	6	2	4	5	3	7	8	8	8	6
7	5	3	7	8	6	7	5	10	10	4	6	7	5	3	1	1	1	2	4	6	7
8	8	8	8	6	7	8	8	6	2	3	7	8	8	8	6	2	4	5	3	7	8
10	10	10	10	4	5	10	10	4	6	7	5	10	10	10	4	6	2	3	7	5	10

a	
1	1
2	4
3	7
4	6
5	3
6	2
7	5
8	8
10	10

b	
1	2
2	3
3	1
4	5
5	10
6	7
7	8
8	6
10	4

a=(2,4,6)(3,7,5)
b=(1,2,3)(4,5,10)(6,7,8)

	a	a	a	b	b	b	a	b	a	b	a	b	a	a	b	a	a	b	a	a	b
1	1	1	1	2	3	1	1	2	4	5	3	1	1	1	2	4	6	7	5	3	1
2	4	6	2	3	1	2	4	5	3	1	1	2	4	6	7	5	3	1	1	1	2
3	7	5	3	1	2	3	7	8	9	6	2	3	7	5	10	10	10	4	6	2	3
4	6	2	4	5	10	4	6	7	5	10	10	4	6	2	3	7	5	10	10	10	4

a	
1	1
2	4
3	7
4	6
5	3
6	2
7	5
8	9
9	
10	10

b	
1	2
2	3
3	1
4	5
5	10
6	7
7	8
8	
9	6
10	4

10=11=12

	a	a	a	b	b	b	a	b	a	b	a	b	a	a	b	a	a	b	a	a	b
1	1	1	1	2	3	1	1	2	4	5	3	1	1	1	2	4	6	7	5	3	1
2	4	6	2	3	1	2	4	5	3	1	1	2	4	6	7	5	3	1	1	1	2
3	7	5	3	1	2	3	7	8	8	6	2	3	7	5	10	10	10	4	6	2	3
4	6	2	4	5	10	4	6	7	5	10	10	4	6	2	3	10	5	10	10	10	4
5	3	7	5	10	4	5	3	1	1	2	4	5	3	7	8	10	8	6	2	4	5

a	
1	1
2	4
3	7
4	6
5	3
6	2
7	5
8	8
10	10

b	
1	2
2	3
3	1
4	5
5	10
6	7
7	8
8	6
10	4

8=9

	a	a	a	b	b	b	a	b	a	b	a	b	a	a	b	a	a	b	a	a	b
1	1	1	1	2	3	1	1	2	4	5	3	1	1	1	2	4	6	7	5	3	1
2	4	6	2	3	1	2	4	5	3	1	1	2	4	6	7	5	3	1	1	1	2
3	7	5	3	1	2	3	7	8	8	6	2	3	7	5	10	10	10	4	6	2	3
4	6	2	4	5	10	4	6	7	5	10	10	4	6	2	3	10	5	10	10	10	4
5	3	7	5	10	4	5	3	1	1	2	4	5	3	7	8	10	8	6	2	4	5
6	2	4	6	7	8	6	2	3	7	8	8	6	2	4	5	3	7	8	8	8	6
7	5	3	7	8	6	7	5	10	10	4	6	7	5	3	1	1	1	2	4	6	7
8	8	8	8	6	7	8	8	6	2	3	7	8	8	8	6	2	4	5	3	7	8
10	10	10	10	4	5	10	10	4	6	7	5	10	10	10	4	6	2	3	7	5	10

a	
1	1
2	4
3	7
4	6
5	3
6	2
7	5
8	8
10	10

b	
1	2
2	3
3	1
4	5
5	10
6	7
7	8
8	6
10	4

a=(2,4,6)(3,7,5)
b=(1,2,3)(4,5,10)(6,7,8)

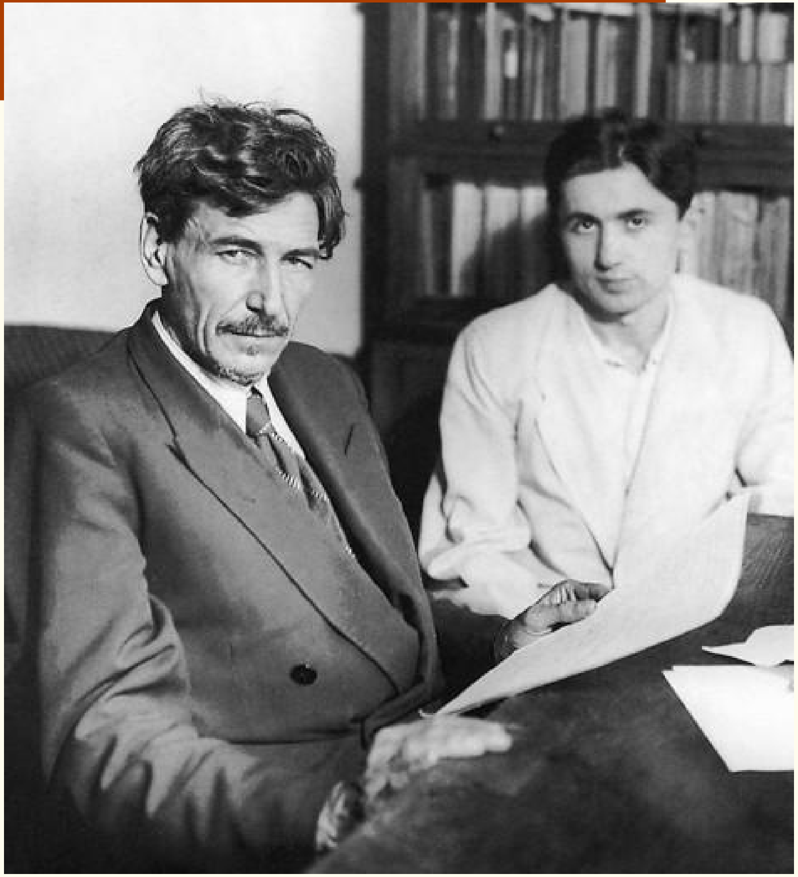
After renaming 10 as 9, we obtain generators: a=(2,4,6)(3,7,5)
b=(1,2,3)(4,5,9)(6,7,8)

Theorem (Novikov & Adian, 1968).

For all $n > 4381$ and $m \geq 2$, $|B(m,n)| = \infty$.

Pyotr Novikov
(1901-1975)

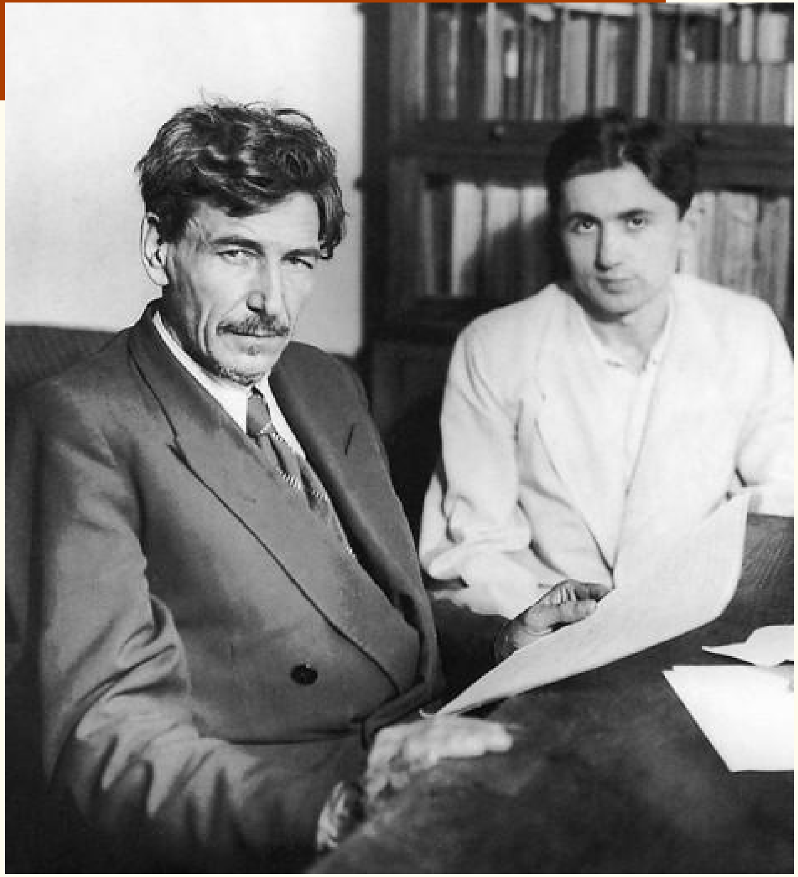
Sergei Adian
(1931-2020)



Theorem (Novikov & Adian, 1968).

For all $n > 4381$ and $m \geq 2$, $|B(m,n)| = \infty$.

(That is, for every exponent $n > 4381$, there exist infinite groups of exponent n which are finitely generated.)



Pyotr Novikov (1901-1975)

Sergei Adian (1931-2020)

Theorem (Zelmanov, 1989). Let $m, n \geq 1$. Among all **finite** groups of rank m and exponent n, there is a largest one.

Efim Zelmanov
(1955-)

Theorem (Zelmanov, 1989). Let $m, n \geq 1$. Among all *finite* groups of rank m and exponent n, there is a largest one. (So there are only finitely many groups of rank m and exponent n.)

Efim Zelmanov
(1955-   )

Theorem (Zelmanov, 1989). Let $m, n \geq 1$. Among all finite groups of rank m and exponent n, there is a largest one. (So there are only finitely many groups of rank m and exponent n.)

It was for this result that he received the Fields Medal (1994).

Efim Zelmanov
(1955-   )

__Theorem__ (Ol'shanskii, 1979) For every prime $p > 10^{75}$, there exists a __Tarski Monster__ G of exponent p:

- $|G| = \infty$
- Every nonidentity element $x \in G$ has order p.

Alexander Olshanskii (1946-)