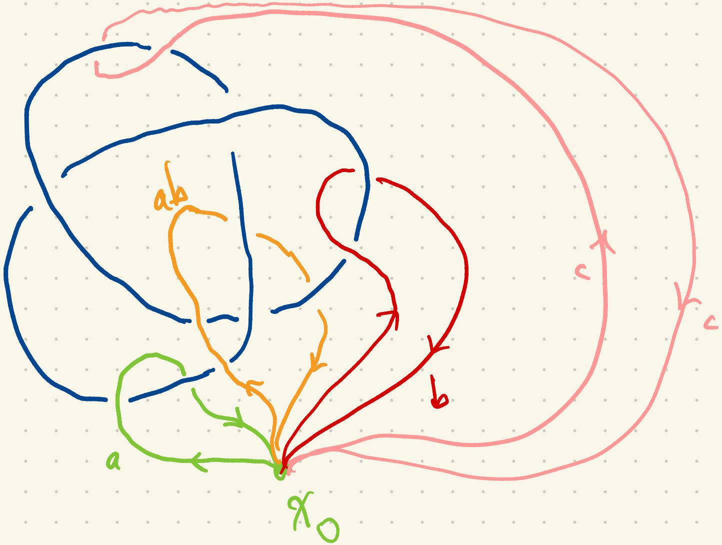
Group Theory

Book 3



$ab = bc = ca \Rightarrow c = b^{-1}ab = aba^{-1}$

$\pi_1(\mathbb{R}^3 - K) = \langle a, b, c : ab = bc = ca \rangle$

$= \langle x, y : x^3 = y^2 \rangle$

$\left.\begin{array}{l} x = ab = bc = ca \\ y = abc \end{array}\right\} \Rightarrow x^3 = ab \cdot ca \cdot bc = abc \cdot abc = y^2$

Observation: $\pi_1(\mathbb{R}^3 - K)$ has no (nontrivial) torsion elements (a "torsion" element in a group is an element of finite order)

If X is a path-connected subset of $\mathbb{R}^2$ then $\pi(X)$ has no (nontrivial) torsion elements. (known for a few decades, this considered "folklore"). Intuitive!

The real proj. plane $P^2\mathbb{R}$ has torsion (its fund. gp. $\pi_1(P^2\mathbb{R})$ has order 2).

$\leftarrow$ a subspace of $\mathbb{R}^4$, not embeddable in $\mathbb{R}^3$.

For (top. spaces) subsets $X \subset \mathbb{R}^4$, $\pi_1(X)$ can have nontrivial torsion elements.

What about $X \subset \mathbb{R}^3$? Can $\pi_1(X)$ have nontrivial torsion elements? Famous open problem.

We will show $SL_2(\mathbb{Z})$ has a subgroup isomorphic to F_2:

$$\left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right\rangle < SL_2(\mathbb{Z})$$

use the <u>Ping-Pong Lemma</u>:

One version is as follows.

Let G be a group acting on a set X. We can think of G as a subgroup of $\operatorname{Sym} X = \{\text{bijections } X \to X\}$

i.e. permutations of X. Here I'm using left action, so we compose "right-to-left"

More generally, to say G acts on X means: For every $g \in G$ we have a permutation of X, $x \mapsto gx$

such that for all $g, h \in G$, $g(hx) = (gh)x$ i.e. the map $G \to \operatorname{Sym} X$ is a group homomorphism.

If the map $G \to \operatorname{Sym} X$ is one-to-one, we say G acts <u>faithfully</u> on X.

(So G is identified with a subgroup of $\operatorname{Sym} X$.)

Eg. $SL_2(F)$ acts on P^1F as fractional linear transformations.

$$SL_2(F) \ni \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad P^1F = F \cup \{\infty\} \ni x \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}(x) = \frac{ax+b}{cx+d}$$

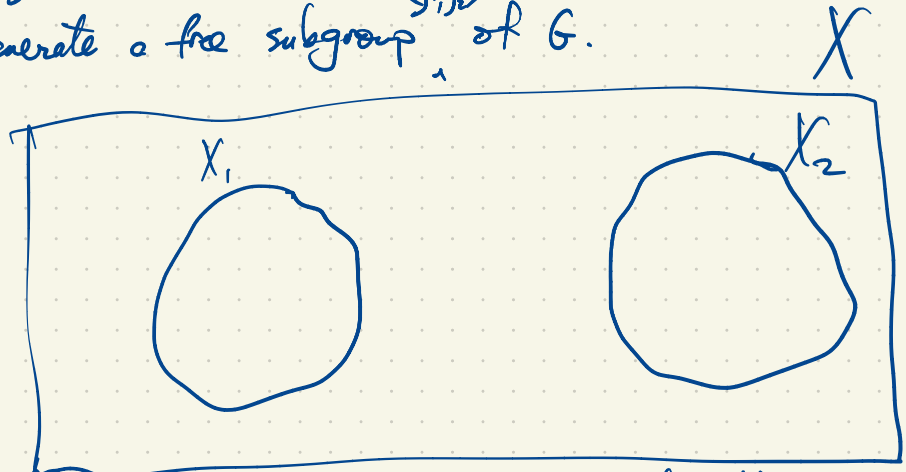
But this action is not faithful: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $-\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ give the same frac. lin. transf.

$PSL_2(F)$ acts faithfully on P^1F.

Given two subsets $X_1, X_2 \subset X$ which are disjoint nonempty and two elements $g_1, g_2 \in G$ such that

$$\left.\begin{aligned} g_1^k(X_2) &\subseteq X_1 \\ \text{and} \quad g_2^k(X_1) &\subseteq X_2 \end{aligned}\right\} \text{ for all integers } k \neq 0$$

then g_1, g_2 ~~generate~~ a free subgroup $\langle g_1, g_2 \rangle$ of G.



<u>Proof</u> A nontrivial word in two generators looks like

$w = g_1^{k_1} g_2^{l_1} g_1^{k_2} g_2^{l_2} \cdots g_1^{k_{r-1}} g_2^{l_{r-1}} g_1^{k_r}$ $\quad k_i, l_j$ nonzero integers up to conjugacy in the free group.

Then $w \neq 1$ since it maps $X \to X_1$.

Distinct words in the generators w_1, w_2 must give distinct permutations of X by considering $w = w_1^{-1} w_2$. $\square$

Application: $SL_2(\mathbb{Z})$ has $F_2 = \langle a,b \rangle$ (free group of rank 2) as a subgroup.

Try $u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. This doesn't work. These generate $\langle u,v \rangle = SL_2(\mathbb{Z})$

but this is not a free group since

$$uv^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad (uv^{-1})^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (uv^{-1})^6 = 1.$$

Next try $a = u^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $b = v^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. These generate $\langle a,b \rangle \cong F_2$.

To prove this, use the Ping-Pong Lemma.

$G = SL_2(\mathbb{Z})$ acts on $P^1\mathbb{Q} = \mathbb{Q} \cup \{\infty\}$ by fractional linear transformations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(x) = \frac{ax+b}{cx+d}$$

Let $X_1 = \{x \in P^1\mathbb{Q} : |x| < 1\}$, $X_2 = \{x \in P^1\mathbb{Q} : |x| > 1\}$. Note: $\infty \in X_2$.

$$\begin{tikzcd} X_1 \arrow[r, bend left, "a^*"] & X_2 \arrow[l, bend left, "b^*"] \end{tikzcd}$$

$$b^n = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix}, \quad a^n = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix} \quad n \neq 0$$

$$a^n(x) = \frac{1x + 2n}{0x+1} = x + 2n$$

If $x \in X_1$ then $|x_1| < 1$ ie. $-1 < x_1 < 1$ then $a^n(x) \in (2n-1, 2n+1)$ so $|a^n(x)| > 1$ ie. $a^n(x) \in X_2$. for $n \neq 0$.

If $x \in X_2$ then $|x| > 1$

$$b^n(x) = \frac{1x+0}{2nx+1} = \frac{1}{2n + \frac{1}{x}}, \quad \left|\frac{1}{x}\right| < 1, \quad \left|2n + \frac{1}{x}\right| > 1,$$

$$|b^n(x)| = \left|\frac{1}{2n+\frac{1}{x}}\right| < 1, \quad b^n(x) \in X_1. \qquad \square$$

In $G = SL_2(\mathbb{Z})$, $\langle a,b\rangle \cong F_2$.

Similarly in $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm I\}$, $\langle a,b\rangle \cong F_2$.

$$\langle a,b\rangle = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} : a_1, a_2, a_3, a_4 \in \mathbb{Z};\ a_1a_4 - a_2a_3 = 1;\ a_1, a_4 \text{ odd};\ a_2, a_3 \text{ even} \right\}.$$

In $SL_2(\mathbb{Z})$ elements have one of the forms

$$\begin{bmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{bmatrix}, \begin{bmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{bmatrix}, \begin{bmatrix} \text{odd} & \text{odd} \\ \text{even} & \text{odd} \end{bmatrix}, \begin{bmatrix} \text{odd} & \text{even} \\ \text{odd} & \text{odd} \end{bmatrix}, \begin{bmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{odd} \end{bmatrix}, \begin{bmatrix} \text{odd} & \text{odd} \\ \text{odd} & \text{even} \end{bmatrix},$$

$$\cancel{\begin{bmatrix} \text{even} & \text{odd} \\ \text{even} & \text{odd} \end{bmatrix}}, \ldots$$

ten choices of parity are excluded in $SL_2(\mathbb{Z})$.

$$|SL_2(\mathbb{F}_2)| = 2(2^2-1) = 6 \qquad SL_2(\mathbb{F}_2) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

Reducing matrix entries mod 2 gives an epimorphism (surjective homomorphism) $SL_2(\mathbb{Z}) \to SL_2(\mathbb{F}_2)$.

The kernel of this map is $\Gamma(2) = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in SL_2(\mathbb{Z}) : a_1, a_4 \text{ odd};\ a_2, a_3 \text{ even} \right\} = \langle a,b\rangle$
(a principal congruence subgroup)

$$[SL_2(\mathbb{Z}) : \Gamma(2)] = 6.$$

Tits Alternative: G linear group over F $\Rightarrow$ either

G virtually solvable

or G has F_2 as a subgroup (exclusive 'or').

conjectured originally by Bass & Serre

A linear group is a matrix group over a field F, in particular $GL_n(F)$ and its subgroups such as $SL_n(F)$, $O_n(F)$, $U_n(F)$, $Sp_n(F)$, ...

Tits proved that the alternative holds in GL_n.

G is solvable if it has a composition series with abelian factors i.e.

$$G_0 = 1 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_n = G \quad \text{with } G_k/G_{k-1} \text{ abelian.}$$

eg. S_3 is solvable: $1 \triangleleft A_3 \triangleleft S_3$

G is virtually solvable if it has a solvable subgroup of finite index.

Every finite group is virtually solvable by considering the trivial subgroup.

Eg. the linear group $O_2(\mathbb{R}) = \{\text{orthogonal } 2\times 2 \text{ real matrices}\} = \{A \in GL_2(\mathbb{R}) : A^TA = I\}$

If $A \in O_n(\mathbb{R})$ then $AA^T = A^TA = I$, so $(\det A)^2 = 1 \Rightarrow \det A = \pm 1$.

$SO_n(\mathbb{R}) = \{A \in O_n(\mathbb{R}) : \det A = 1\}$ = special orthogonal group = $\{\text{rotations fixing } 0 \in \mathbb{R}^n\}$

$[O_n(\mathbb{R}) : SO_n(\mathbb{R})] = 2$

$SO_2(\mathbb{R}) \cong S^1$ as Lie groups (abelian)

$\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a^2+b^2=1 \right\}$

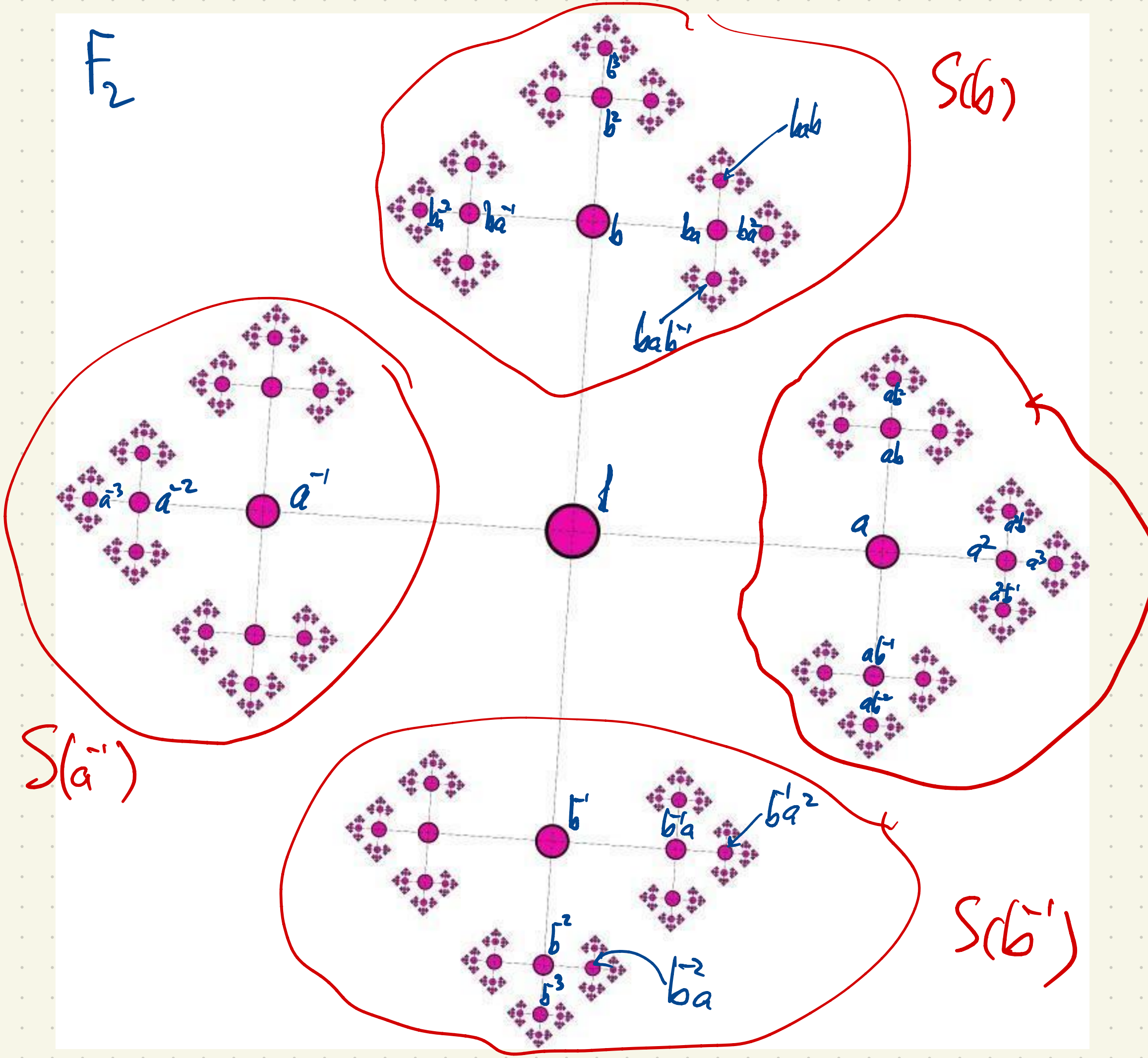
$O_2(\mathbb{R})$ is nonabelian but solvable. $\underbrace{1 \triangleleft SO_2(\mathbb{R})} \underbrace{\triangleleft O_2(\mathbb{R})}$

$O_2(\mathbb{R})$ cannot have a subgroup isomorphic to F_2.

$SO_3(\mathbb{R})$ has a subgroup $\cong F_2$. See handout on free groups.

F_2 has a "paradoxical decomposition" $F_2 = A \sqcup B \sqcup C \sqcup D$ (disjoint union ie. partition)

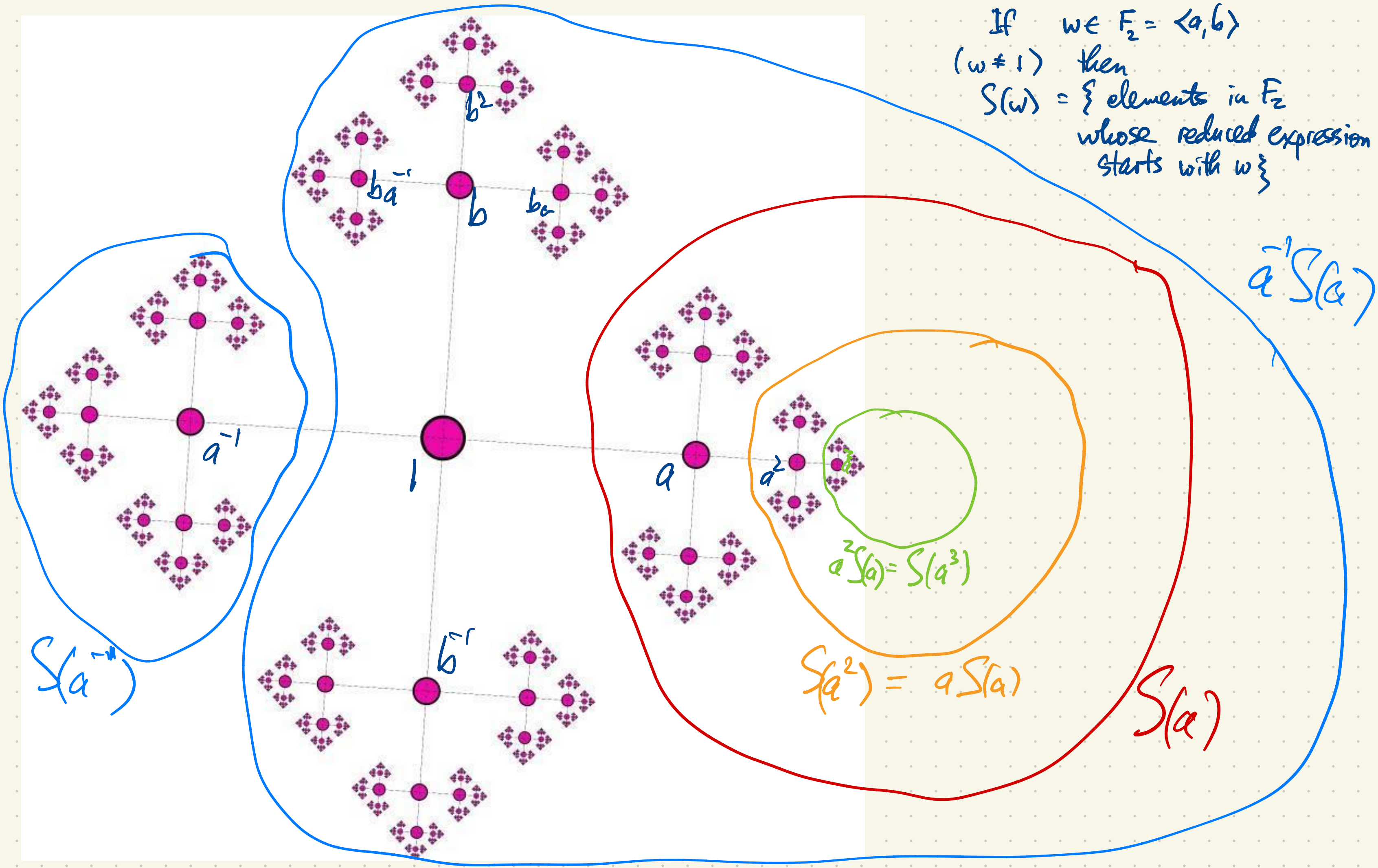
$F_2 = g_1 A \sqcup g_2 B = g_3 C \sqcup g_4 D \quad g_i \in F_2.$



If $w \in F_2$, $w \neq 1$ then $S(w) \subset F_2$ is the subset consisting of words starting with w (they have w as a prefix). I'm only considering reduced words.

$$F_2 = S(a^{-1}) \cup a^{-1}S(a)$$
$$= S(b^{-1}) \cup b^{-1}S(b)$$
$$= \{1\} \cup S(a) \cup S(b) \cup S(a^{-1}) \cup S(b^{-1})$$

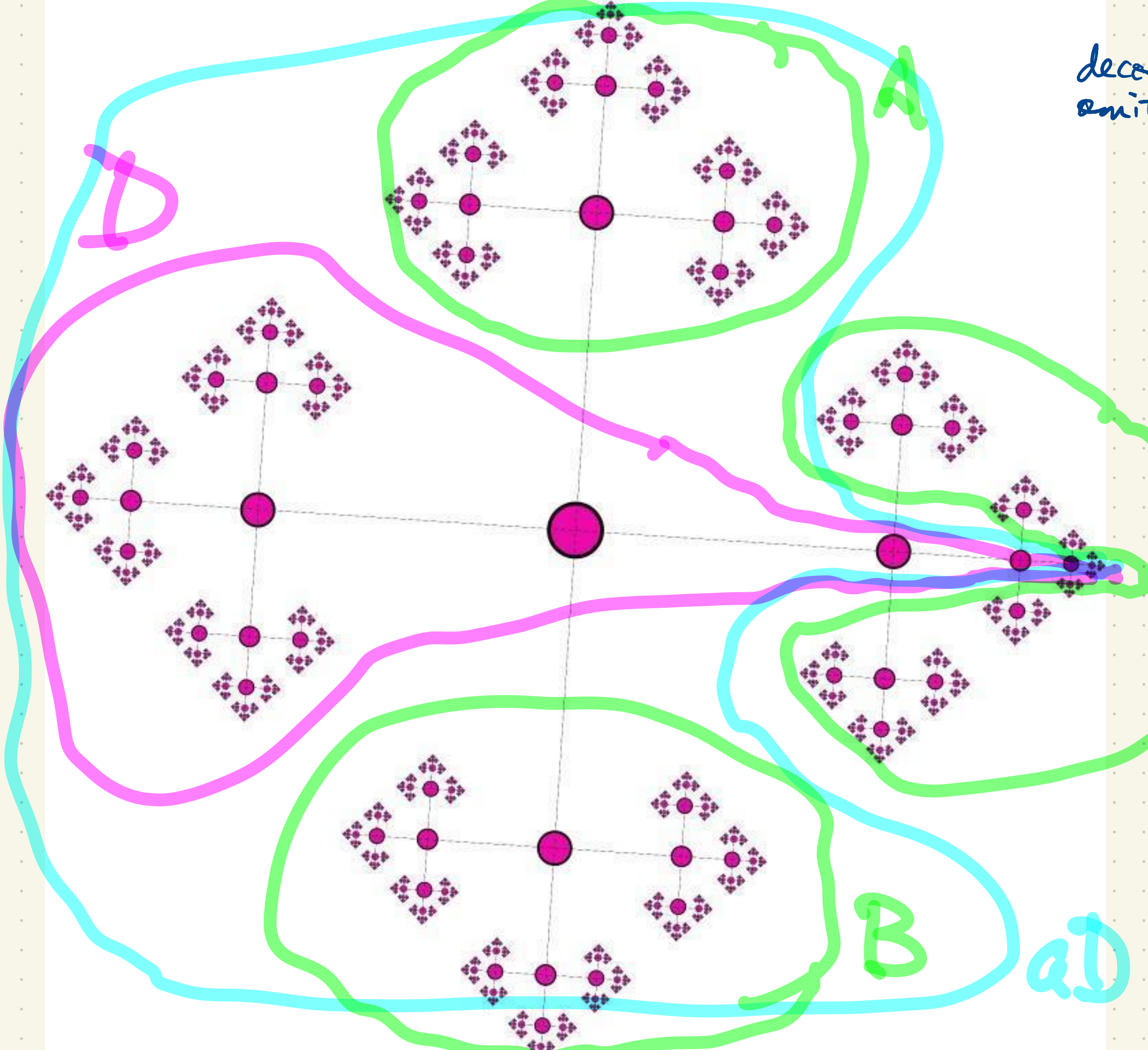
This is almost a paradoxical decomposition of F_2 (unfortunately $\{1\}$ is "left over"...)



If $w \in F_2 = \langle a, b \rangle$ $(w \neq 1)$ then

$S(w) = \{$ elements in F_2 whose reduced expression starts with $w \}$

An actual paradoxical decomposition of F_2 without omitting $\{1\}$:



$$F_2 = A \sqcup B \sqcup C \sqcup D$$

$$A = S(b)$$
$$B = S(b^{-1})$$
$$C = S(a) - \{a, a^2, a^3, a^4, \dots\}$$
$$D = S(a^{-1}) \cup \{1, a, a^2, a^3, a^4, \dots\}$$

$$F_2 = b^{-1}A \sqcup B$$
$$= C \sqcup aD$$

Banach-Tarski Theorem

Let $X = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1\}$.

Then there exists a partition $X = X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4 \sqcup X_5$ such that

$$X = g_1 X_1 \sqcup g_2 X_2 = g_3 X_3 \sqcup g_4 X_4 \sqcup g_5 X_5 \quad \text{for some } g_1, \dots, g_5 \text{ direct isometries of } \mathbb{R}^3.$$

(orientation-preserving isometries)

An isometry is a transformation preserving distances.

Every isometry either preserves or reverses orientation.

e.g. rotations

The subsets $X_1, \dots, X_5$ are not all Lebesgue-measurable.

Since $F_2 = g_1 A \sqcup g_2 B = g_3 C \sqcup g_4 D = A \sqcup B \sqcup C \sqcup D$

and $SO_3(\mathbb{R})$ has a subgroup $\langle a,b \rangle \cong F_2$

the group $SO_3(\mathbb{R})$ also admits such a "paradoxical decomposition":

Let $T \subset SO_3(\mathbb{R})$ be a right transversal for $\langle a,b \rangle < SO_3(\mathbb{R}) = \bigsqcup_\alpha \langle a,b \rangle t_\alpha$

subset, not subgroup — i.e. set of representatives of the right cosets

$T = \{t_\alpha : \alpha\}$

$SO_3(\mathbb{R}) = \langle a,b \rangle T$

$$SO_3(\mathbb{R}) = AT \sqcup BT \sqcup CT \sqcup DT = g_1 AT \sqcup g_2 BT = g_3 CT \sqcup g_4 DT$$

Note: A paradoxical composition requires a finite number of pieces; there is nothing paradoxical about an infinite number of pieces.

Why can't a disk $X=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 1\}$ admit a "paradoxical decomposition"?

If $X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_5$ (or some other finite number of pieces)

with $X = g_1X_1 \sqcup g_2X_2 = g_3X_3 \sqcup g_4X_4 \sqcup g_5X_5$, $g_1,\dots,g_5$ isometries, what is the contradiction?

To obtain a contradiction we cannot invoke Lebesgue measure (i.e. area) since the subsets X_i aren't necessarily Lebesgue measurable.

We use finitely additive measure.