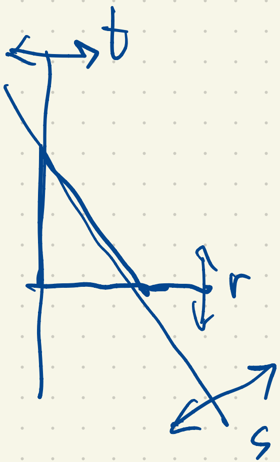
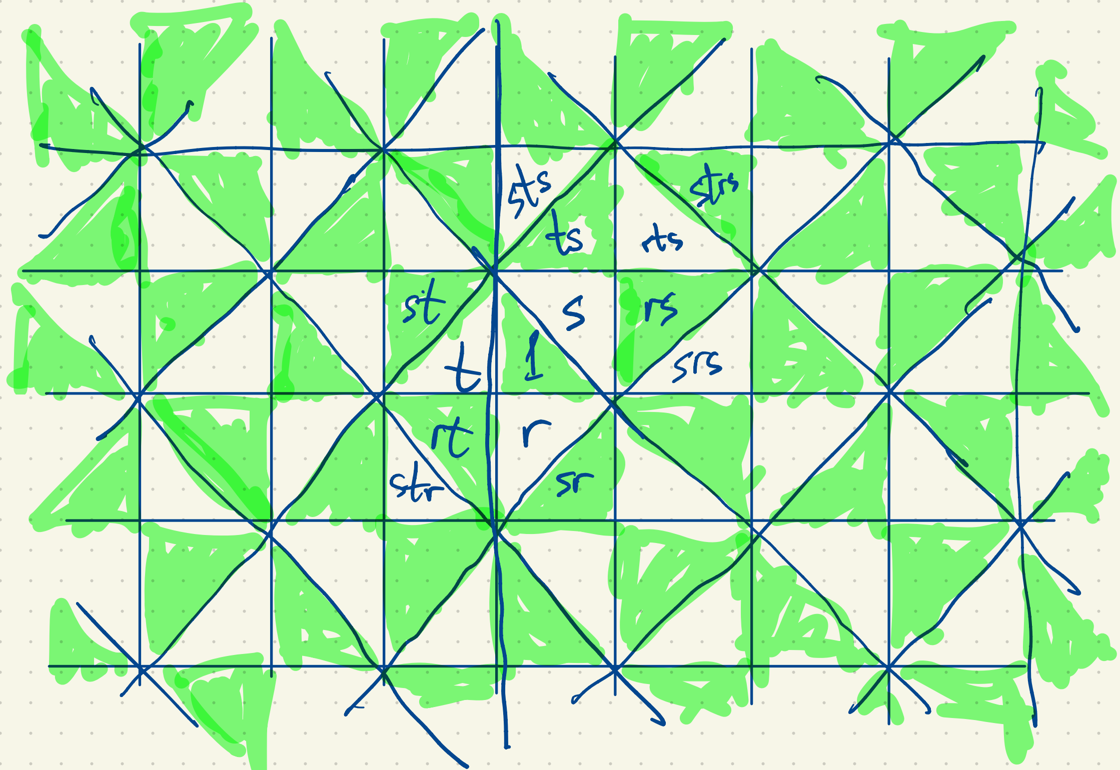
Group Theory

Book 2

reflections interchange white triangles $\longleftrightarrow$ green triangles

Elements of $G = G(4,4,2)$ map white? green?





$G = G(4,4,2)$ labels the green triangles
W labels all triangles

More generally if $l, m, n \geq 2$ satisfying $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1$ then $G = G(l,m,n)$ is a group of isometries of the Euclidean plane generated by rotations of order l, m, n:

$$G = \langle a, b, c \mid a^l, b^m, c^n, abc \rangle$$

Moreover $[W:G]$ where $W = \langle r,s,t \mid r^2, s^2, t^2, (rs)^l, (st)^m, (tr)^n \rangle$.

If $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}>1$, then $G=G(l,m,n)$ finite in place of a tiling of the Euclidean plane, we get a tiling of S^2 (Euclidean sphere).

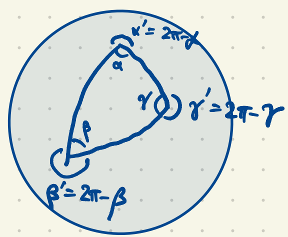
If $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}<1$, then $G=G(l,m,n)$ infinite we get a tiling of the hyperbolic plane by congruent triangles.

A spherical example: $G=G(2,3,4)$, $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{6+4+3}{12}=\frac{13}{12}>1$

Eine Kleine spherical geometry

Let $S\subset\mathbb{R}^3$ be a unit sphere; its surface area is $4\pi r^2=4\pi$. "Lines" on S are geodesics (great circles).

Triangles in S have area = angular excess = $\alpha+\beta+\gamma-\pi>0$



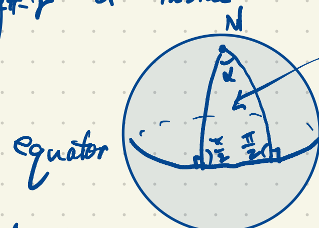
$$\underbrace{(\alpha+\beta+\gamma-\pi)}_{\text{angular excess of "inside"}}+\underbrace{(\alpha'+\beta'+\gamma'-\pi)}_{\text{angular excess "outside"}}=6\pi-2\pi=4\pi$$

W consists of isometries of S preserving the tiling:

$[W:G]=2$, $G=G(2,3,4)$

$G=\langle a,b,c \mid a^2,b^3,c^4,abc\rangle$

$a = rs$, $b = st$, $c = tr$



Area $=\alpha+\frac{\pi}{2}+\frac{\pi}{2}-\pi=\alpha$

$|G|=24$.

$G\cong S_4$

We subdivide S into 48 congruent spherical triangles each with angles $\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{4}$ and area $\frac{4\pi}{48}=\frac{\pi}{12}=\frac{\pi}{2}+\frac{\pi}{3}+\frac{\pi}{4}-\pi$

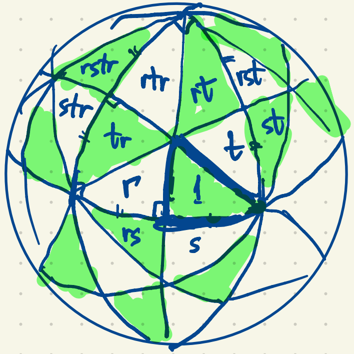
r,s,t: reflections of S in the "lines" bounding one triangle i.e. reflections in the corresponding planes through the origin

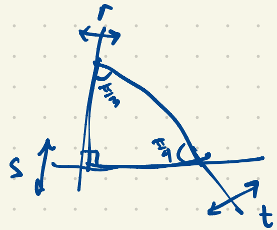
$W=W(\bullet\overset{3}{\text{---}}\bullet\overset{4}{=\!=}\bullet)=\langle r,s,t \mid r^2,s^2,t^2,(rs)^2,(rt)^3,(st)^4\rangle$,

$W\cong C_2\times S_4$

$|W|=48$

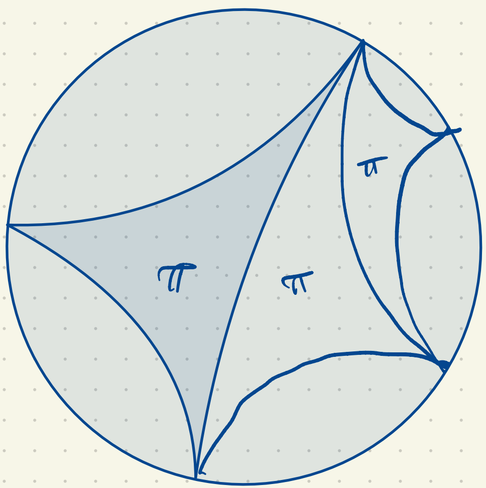
Starting Fri Oct 3 new room is BU24





Case $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}<1$: $G(l,m,n)$ preserves a triangulation of the hyperbolic plane with triangles having angles $\frac{\pi}{l}, \frac{\pi}{m}, \frac{\pi}{n}$

In hyperbolic plane, a triangle (sides are lines = geodesics) having angular defect $\pi-(\alpha+\beta+\gamma)$ = area.

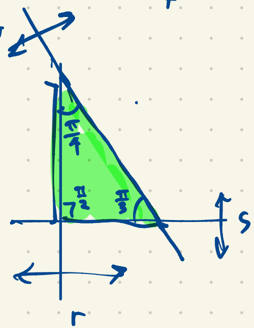


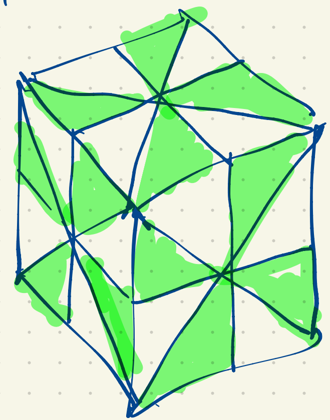
Today: $G = G(2,3,4) = \langle a,b,c : a^2, b^3, c^4, abc \rangle \cong S_4$

$W = W(\bullet\!\!-\!\!-\!\!\bullet\!\!=\!\!=\!\!\bullet) \cong C_2 \times S_4$
$\quad\quad B_2$

$= \langle r,s,t \mid r^2, s^2, t^2, (rs)^2, (rt)^3, (st)^4 \rangle$

To recognize $G \cong S_4$ without computer: $G = \langle rs, st, tr$: rs, st, tr are rotations by angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$ about vertices of triangle shown$\rangle$





G = Group of rotational symmetries of a cube $\cong S_4$.

G permutes the four "body diagonals" of the cube in all $4! = 24$ ways.
(A "body diagonal" joins two opposite vertices.)

G permutes the 24 green triangles transitively (and regularly).

If $W = HK$ where H and K are normal subgroups with $H \cap K = 1$ (so H and K commute with each other i.e. $hk = kh$ for all $h \in H$, $k \in K$) then $W \cong H \times K = \{(h,k) : h \in H, k \in K\}$ (direct product)

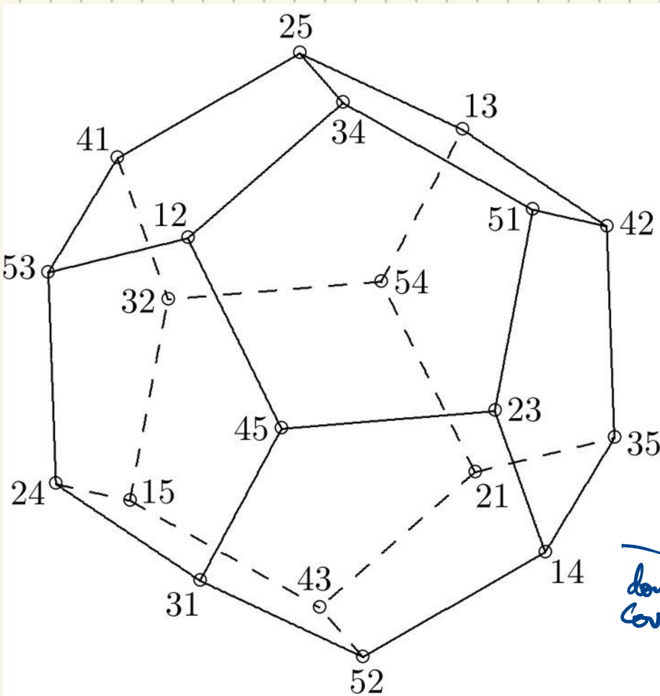
In our case $|W| = 48$, $|G| = 24$, $W = G \cup Gr$, W has a subgroup $H = \langle h \rangle$ of order 2, $H \triangleleft W$, $h = -I$

preserve orientation; reverse orientation

$H = Z(W)$

Cube has $3 + 6 = 9$ planes of symmetry but altogether 24 orientation-reversing symmetries

Similarly the regular dodecahedron (12 pentagonal faces) has rotational symmetry group G with $|G| = 60$, $G = A_5$ (Alt_5)



Elements of A_5 give rotational symmetries of the dodecahedron.

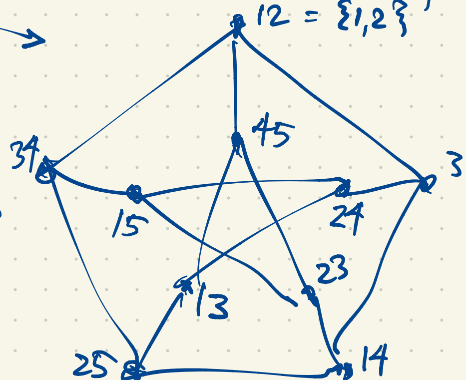
The full symmetry group of the regular dodecahedron has order 2×60 (60 orientation-preserving and 60 ~~orientation-reversing~~)

rotations   reflections + other

The full group of symmetries has order 120 and it has a subgroup isomorphic to A_5 but the group is not S_5. S_5 is not a group of isometries of $\mathbb{R}^3$.

Instead the full group of isometries of the regular dodecahedron is isomorphic to $C_2 \times A_5$.

double cover



This graph, the Petersen graph, has isomorphism group S_5

$$(i\ j) \mapsto \{i, j\}$$
$$(j\ i) \mapsto \{i, j\}$$

A presentation of a group G is an expression $G = \langle X : R \rangle$ where X is a set of symbols (letters) and $R \subset F(X) =$ free group on $X = \{x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} : x_j \in X,\ i_j \in \mathbb{Z}\}$

$x^j x^k = x^{j+k}$ $(j,k \in \mathbb{Z};\ x \in X)$

$x^0 = 1 =$ identity.

X = set of "generators"

R = set of "relators" (words in the generators)

If X is finite then G is <u>finitely generated</u>.

If X and R are both finite then G is <u>finitely presented</u>.

Burnside groups are finitely generated (usually) but not finitely presented.

$G = F/$ subgroup of F generated by R and their conjugates

= "<u>largest</u>" homomorphic image of $F = F(X)$ having R in kernel

<u>universal</u> as we'll discuss later – see handout.

Every group has a presentation. Given G, for every $g \in G$, introduce a generator x_g. So $X = \{x_g : g \in G\}$. For every pair $g, h \in G$ we want to force $x_g x_h = x_{gh}$ but this doesn't happen in $F = F(X)$ so introduce relators $x_g x_h x_{gh}^{-1} \in R$. $R = \{x_g x_h x_{gh}^{-1} : g, h \in G\}$. Then $F/\{\ldots R \ldots\} \cong G$.

If G is finitely generated then G is countable i.e. finite or countably infinite.

If $X = \{x_1, \ldots, x_n\}$ then $F = F(X) = \{$<u>products</u> of $x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\} = \bigcup_{\ell=0}^{\infty} S_\ell$ where

finite products

F is a countable union of finite sets, hence countable.

$S_\ell = \{x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_\ell}^{e_\ell} :$ $i_1, \ldots, i_\ell \in \{1, \ldots, n\}$, $e_j \in \{\pm 1\}\}$

$|S_\ell| \le 2^\ell$

If G is countably generated (X countable) then $F = F(X)$ is countable.

$S_\ell = \{$words of length $\ell\}$ is countable.

If $|X| = m$ and $|R| = n$, m, n pos. integers, what can we say about $|G|$ where $G = \langle X \mid R \rangle$?
If $n < m$ then G is infinite. That is (contrapositive form) in order for $|G| < \infty$, we need at least as many relators as generators.

eg. $m = 1$. $X = \{x\}$. If $R = \emptyset$ then $G = \langle x \mid \emptyset \rangle = \{\ldots, x^{-2}, x^{-1}, 1, x, x^2, \ldots\}$

If $R = \{1\}$ then $G = \langle x \mid 1 \rangle = F(x) = \{\ldots, x^{-2}, x^{-1}, 1, x, x^2, \ldots\}$

If $R = \{x^{15}\}$ then $G = \langle x \mid x^{15} \rangle = \{1, x, x^2, \ldots, x^{14}\} \cong C_{15}$

If $R = \{x^{15}, x^{40}\}$ then $G = \langle x \mid x^{15}, x^{40} \rangle = \langle x \mid x^5 \rangle = \{1, x, x^2, x^3, x^4\}$

$$x^5 = (x^{15})^3 (x^{40})^{-1}$$

If G is a group then a <u>homomorphic image</u> of G is the image of G under a homomorphism

$G \overset{\phi}{\twoheadrightarrow} H$ (surjective, ~~otherwise~~ replace H by $\phi(G) \leq H$.)

Note: $H \cong G/K$ where $K = \ker \phi \trianglelefteq G$. A homomorphic image of G is the same thing as a quotient group G/K, $K \trianglelefteq G$.

In particular the <u>abelianization</u> of G is the largest homo. image of G which is abelian.
A normal subgroup $K \trianglelefteq G$ yields an abelian quotient group G/K iff $K \geq G'$ = derived subgroup of G
i.e. $G' = \langle [g,h] : g, h \in G \rangle$, $[g,h] = g^{-1}h^{-1}gh$.
If $K \geq G'$ then in G/K, take any two elements $gK, hK \in G/K$, we have

$hg[g,h] = \cancel{hg}\,\cancel{g^{-1}h^{-1}}gh = gh$

$(hK)(gK) = hgK = ghK = (gK)(hK)$ and conversely.

eg. the abelianization of an abelian gp. is itself.

eg. the abelianization of S_n, $n \geq 2$ is C_2 (group of order 2).

The abelianization of $GL_n(F)$ $(n \geq 1)$ is $F^\times = \{$ nonzero elements of $F\}$

$$GL_n(F) \xrightarrow{\det} F^\times \qquad \ker(\det) = SL_n(F)$$

If $F = F(X)$ free group on m generators $X = \{x_1, \cdots, x_m\}$ e.g. $m=3$

The abelianization of F $\quad w = x_1 x_2^3 x_1^{-4} x_3^2 x_1 x_2^{-1} \longmapsto (-2, 2, 2) \in \mathbb{Z}^3$

$F/F' \cong \mathbb{Z}^m$ where $F' = \{w \in F :$ every x_i has exponents in w adding to $0\}$ $\quad$ (Multiplicatively, $x_1^{-2} x_2^2 x_3^2$.

$i = 1, 2, \cdots, m$.

(free abelian group on m generators) $\qquad \mathbb{Z}^3 / \langle (1,2,3), (4,9,-1) \rangle$

Now consider $G = \langle x_1, \cdots, x_m \mid r_1, \cdots, r_n \rangle$, $\quad r_1, \cdots, r_n \in F = F(x_1, \cdots, x_m)$

$= F/K$, $\quad K$ is the normal subgp of F generated by $r_1, \cdots, r_n$ and their conjugates in F.

We want to show that if $n < m$ then $|G| = \infty$.

To prove this, first consider $F/F'K$ $\qquad$ Here $F', K \trianglelefteq F$.

$F'K = \{ab : a \in F', b \in K\} \trianglelefteq F$.

$$F/F'K \cong \frac{F/F'}{F'K/F'}$$ (Third Isomorphism Theorem also sometimes called the Second Isomorphism theorem).

$F/F'K$ is a homomorphic image of $F/F' \cong \mathbb{Z}^m$.

$F/F'K \cong \mathbb{Z}^m /$ subgp. generated by n elements.

$F'K/F' \cong K/K \cap F'$ (Second Iso. Thm.)

$$\mathbb{Z}^3 / \langle (1,0,0), (0,2,0) \rangle \cong \left(\mathbb{Z}/2\mathbb{Z}\right) \oplus \mathbb{Z}$$

$$\mathbb{Z}^3 / \langle (1,0,0), (0,2,0), (0,0,3) \rangle \cong \left(\mathbb{Z}/2\mathbb{Z}\right) \oplus \left(\mathbb{Z}/3\mathbb{Z}\right)$$