Group Theory
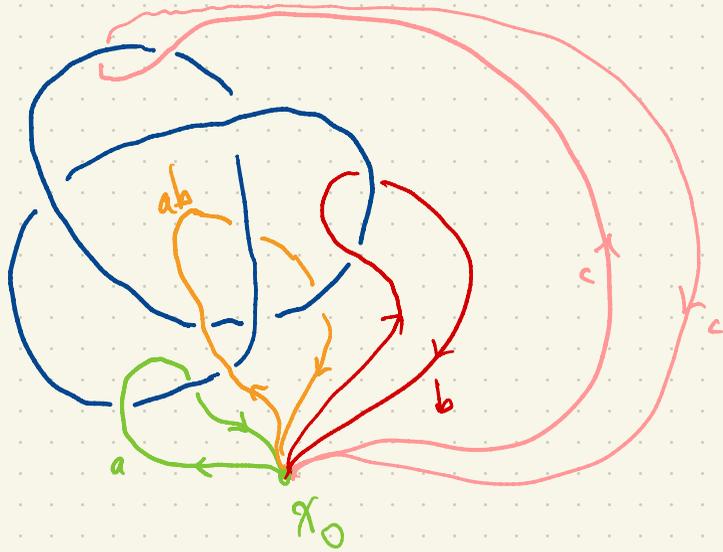
Book 3

$$ab = bc = ca \qquad \Rightarrow c = b^{-1}ab$$
$$= aba^{-1}$$

$$\pi_1(\mathbb{R}^3 - K) = \langle a, b, c : \ ab = bc = ca \rangle$$
$$= \langle x, y : \ x^3 = y^2 \rangle$$

$$\begin{rcases} x = ab = bc = ca \\ y = abc \end{rcases} \Rightarrow \begin{aligned} x^3 &= ab \cdot ca \cdot bc \\ &= abc \cdot abc = y^2 \end{aligned}$$

Observation: $\pi_1(\mathbb{R}^3 - K)$ has no (nontrivial) torsion elements (a "torsion" element in a group is an element of finite order)

If X is a path-connected subset of $\mathbb{R}^2$ then $\pi_1(X)$ has no (nontrivial) torsion elements. (known for a few decades, this considered folklore). Intuitive!

The real proj. plane $P^2\mathbb{R}$ has torsion (its fund. gp. $\pi_1(P^2\mathbb{R})$ has order 2)

(top. spaces) ↖ a subspace of $\mathbb{R}^4$, not embeddable in $\mathbb{R}^3$.

For subsets $X \subset \mathbb{R}^4$, $\pi_1(X)$ can have nontrivial torsion elements.

What about $X \subset \mathbb{R}^3$? Can $\pi_1(X)$ have nontrivial torsion elements? Famous open problem.

We will show $SL_2(\mathbb{Z})$ has a subgroup isomorphic to F_2:
$$\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \rangle < SL_2(\mathbb{Z})$$

use the <u>Ping-Pong Lemma</u>:

One version is as follows.
Let G be a group acting on a set X. We can think of G as a subgroup of
$Sym X = \{$ bijections $X \to X \}$
ie. permutations of X. Here I'm using left action, so we compose "right-to-left"
More generally, to say G acts on X means: For every $g \in G$ we have a
a permutation of X, $\quad x \longmapsto gx$
such that for all $g, h \in G$, $g(hx) = (gh)x$ ie. the map $G \to Sym X$ is a
group homomorphism.

If the map $G \to Sym X$ is one-to-one, we say G acts <u>faithfully</u> on X.
(So G is identified with a subgroup of $Sym X$.)

Eg. $SL_2(F)$ acts on $\mathbb{P}^1 F$ as fractional linear transformations.
$\qquad \Downarrow \qquad\qquad\qquad\quad "\qquad \Downarrow$
$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad\qquad F \cup \{\infty\} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}(x) = \frac{ax+b}{cx+d}$
$\qquad\qquad\qquad\qquad \Downarrow$
$\qquad\qquad\qquad\qquad x$

But this action is not faithful: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $-\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ give the same frac. lin. transf.
$PSL_2(F)$ acts <u>faithfully</u> on $\mathbb{P}^1 F$.

Given two subsets $X_1, X_2 \subset X$ which are disjoint and two elements $g_1, g_2 \in G$
such that nonempty

$$g_1^k(X_2) \subseteq X_1 \quad \Big\} \quad \text{for all integers} \quad k \neq 0$$

and $\quad g_2^k(X_1) \subseteq X_2 \quad \Big\}$

then g_1, g_2 generate a free subgroup $\langle g_1, g_2 \rangle$ of G.
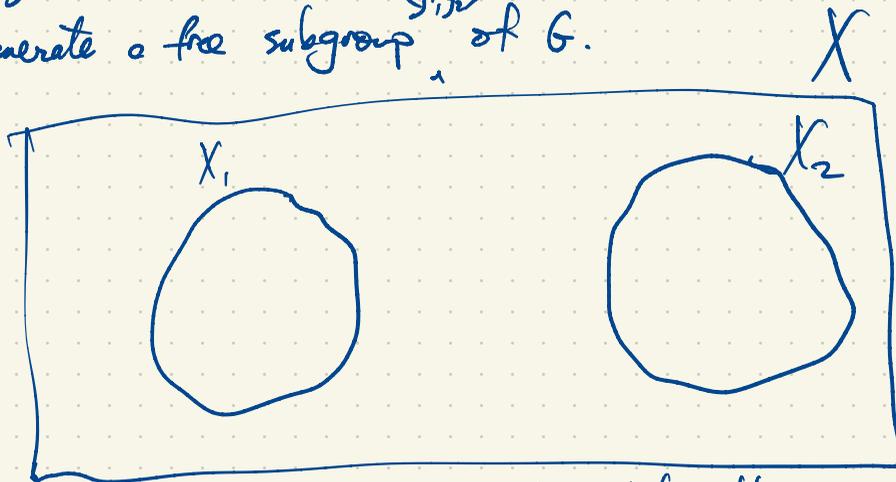
Proof A nontrivial word in two generators looks like

$$w = g_1^{k_1} g_2^{l_1} g_1^{k_2} g_2^{l_2} \cdots g_1^{k_n} g_2^{l_{n-1}} g_1^{k_r} \qquad k_i, l_j \text{ nonzero integers} \qquad \text{up to conjugacy in the free group}.$$

Then $w \neq 1$ since it maps $X \longrightarrow X_1$.
Distinct words in the generators w_1, w_2 must give distinct permutations of X by
considering $w = w_1^{-1} w_2$. $\qquad \square$

Application: $SL_2(\mathbb{Z})$ has $F_2 = \langle a, b \rangle$ (free group of rank 2) as a subgroup.

Try $u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. This doesn't work! These generate $\langle u, v \rangle = SL_2(\mathbb{Z})$ but this is not a free group since

$$u v^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad (u v^{-1})^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (u v^{-1})^6 = 1.$$
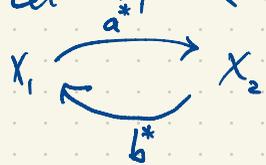
Next try $a = u^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $b = v^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. These generate $\langle a, b \rangle \cong F_2$.

To prove this, use the Ping-Pong lemma.

$G = SL_2(\mathbb{Z})$ acts on $\mathbb{P}^1\mathbb{Q} = \mathbb{Q} \cup \{\infty\}$ by fractional linear transformations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(x) = \frac{ax+b}{cx+d}$$

Let $X_1 = \{x \in \mathbb{P}^1\mathbb{Q} : |x| < 1\}$, $\quad X_2 = \{x \in \mathbb{P}^1\mathbb{Q} : |x| > 1\}$. $\quad$ Note: $\infty \in X_2$.

$b^n = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix}$, $\quad a^n = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$ $\quad n \neq 0$

$a^n(x) = \frac{1x + 2n}{0x + 1} = x + 2n$

If $x \in X_1$ then $|x_1| < 1$ ie. $-1 < x_1 < 1$
then $a^n(x) \in (2n-1, 2n+1)$

So $|a^n(x)| > 1$ ie. $a^n(x) \in X_2$.
$\qquad$ for $n \neq 0$.

If $x \in X_2$ then $|x| > 1$

$b^n(x) = \frac{1x + 0}{2nx + 1} = \frac{1}{2n + \frac{1}{x}}$, $\quad |\frac{1}{x}| < 1$, $|2n + \frac{1}{x}| > 1$,

$|b^n(x)| = |\frac{1}{2n + \frac{1}{x}}| < 1$, $\quad b^n(x) \in X_1$. $\qquad \square$

In $G = SL_2(\mathbb{Z})$, $\langle a, b \rangle \cong F_2$.

Similarly in $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) / \{\pm I\}$, $\langle a, b \rangle \cong F_2$.

$\langle a, b \rangle = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} : a_1, a_2, a_3, a_4 \in \mathbb{Z}; \quad a_1 a_4 - a_2 a_3 = 1; \quad a_1, a_4 \text{ odd}; a_2, a_3 \text{ even} \right\}$

In $SL_2(\mathbb{Z})$ elements have one of the forms

$\begin{bmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{bmatrix}$, $\begin{bmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{bmatrix}$, $\begin{bmatrix} \text{odd} & \text{odd} \\ \text{even} & \text{odd} \end{bmatrix}$, $\begin{bmatrix} \text{odd} & \text{even} \\ \text{odd} & \text{odd} \end{bmatrix}$, $\begin{bmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{odd} \end{bmatrix}$, $\begin{bmatrix} \text{odd} & \text{odd} \\ \text{odd} & \text{even} \end{bmatrix}$

~~$\begin{bmatrix} \text{even} & \text{odd} \\ \text{even} & \text{odd} \end{bmatrix}$~~, $\dots$ ten choices of parity are excluded in $SL_2(\mathbb{Z})$.

$|SL_2(\mathbb{F}_2)| = 2(2^2 - 1) = 6$ $SL_2(\mathbb{F}_2) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

Reducing matrix entries mod 2 gives an epimorphism $SL_2(\mathbb{Z}) \twoheadrightarrow SL_2(\mathbb{F}_2)$.
(surjective homomorphism)

The kernel of this map is $\Gamma(2) = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in SL_2(\mathbb{Z}) : a_1, a_4 \text{ odd}; a_2, a_3 \text{ even} \right\} = \langle a, b \rangle$
(a principal congruence subgroup) $[SL_2(\mathbb{Z}) : \Gamma(2)] = 6.$

Tits Alternative: G linear group over F $\Rightarrow$ either

$\qquad\qquad G$ virtually solvable

$\qquad\qquad$ or $\quad G$ has F_2 as a subgroup $\qquad$ (exclusive 'or').

conjectured originally by Bass & Serre

A linear group is a matrix group over a field F, in particular $GL_n(F)$ and its subgroups such as $SL_n(F)$, $O_n(F)$, $U_n(F)$, $Sp_n(F)$, ...

Tits proved that the alternative holds in GL_n.

$\quad G$ is solvable if it has a composition series with abelian factors ie.

$G_0 = 1 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_n = G \qquad$ with G_k/G_{k-1} abelian.

eg. S_3 is solvable: $\qquad 1 \triangleleft A_3 \triangleleft S_3$

$\quad G$ is virtually solvable if it has a solvable subgroup of finite index.
Every finite group is virtually solvable by considering the trivial subgroup.
Eg. the linear group $O_2(\mathbb{R}) = \{$ orthogonal 2×2 real matrices $\} = \{A \in GL_2(\mathbb{R}) : A^T A = I\}$

If $A \in O_n(\mathbb{R})$ then $AA^T = A^T A = I$, so $(\det A)^2 = 1 \Rightarrow \det A = \pm 1$.

$\quad SO_n(\mathbb{R}) = \{A \in O_n(\mathbb{R}) : \det A = 1\}$ = special orthogonal group = $\{$rotations fixing $0 \in \mathbb{R}^n\}$

$\qquad [O_n(\mathbb{R}) : SO_n(\mathbb{R})] = 2$

$SO_2(\mathbb{R}) \cong S^1$ as Lie groups (abelian)

$\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a^2 + b^2 = 1 \right\}$
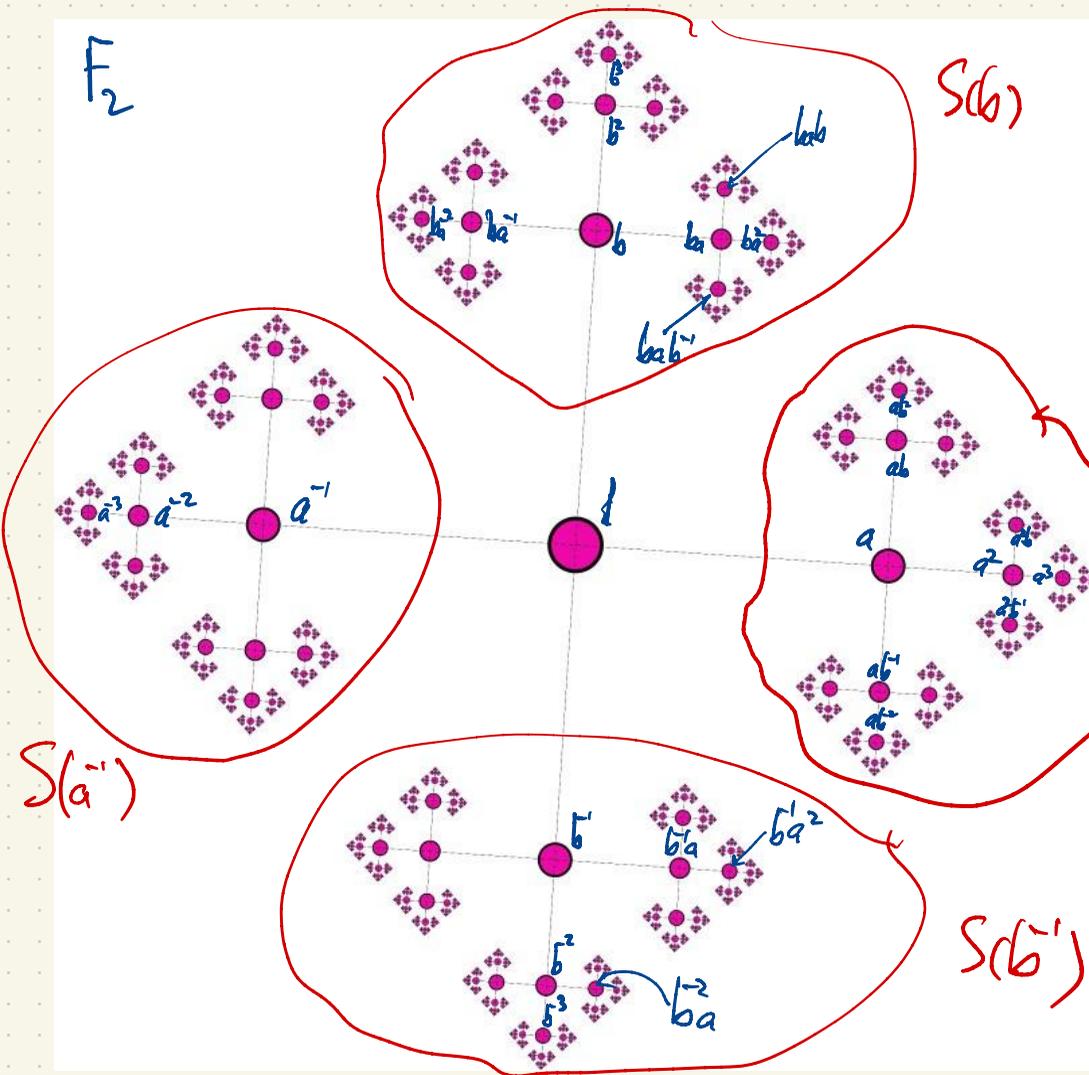
$O_2(\mathbb{R})$ is nonabelian but solvable. $\qquad 1 \lhd SO_2(\mathbb{R}) \lhd O_2(\mathbb{R})$

$O_2(\mathbb{R})$ cannot have a subgroup isomorphic to F_2.

$SO_3(\mathbb{R})$ has a subgroup $\cong F_2$. See handout on free groups.

F_2 has a "paradoxical decomposition" $\quad F_2 = A \sqcup B \sqcup C \sqcup D \quad$ (disjoint union ie. partition)

$F_2 = g_1 A \sqcup g_2 B = g_3 C \sqcup g_4 D \qquad g_i \in F_2$.

F_2

$S(b)$

b^3

b^2

b^2ab

b^2 ba^{-1} b ba ba^2

bab^{-1}

$S(a^{-1})$

a^3 a^{-2} a^{-1}

1

a^2b

ab

a a^2 a^2b a^3

a^2b^{-1}

a^2b^{-2}

$S(a)$

b^{-1} $b^{-1}a$ $b^{-1}a^2$

b^{-2} $b^{-2}a$

b^{-3}

$S(b^{-1})$

If $w \in F_2$, $w \neq 1$ then $S(w) \subset F_2$ is the subset consisting of words starting with w (they have w as a prefix). I'm only considering reduced words.