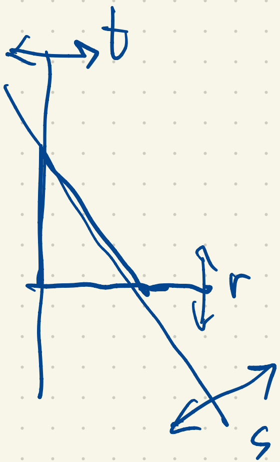
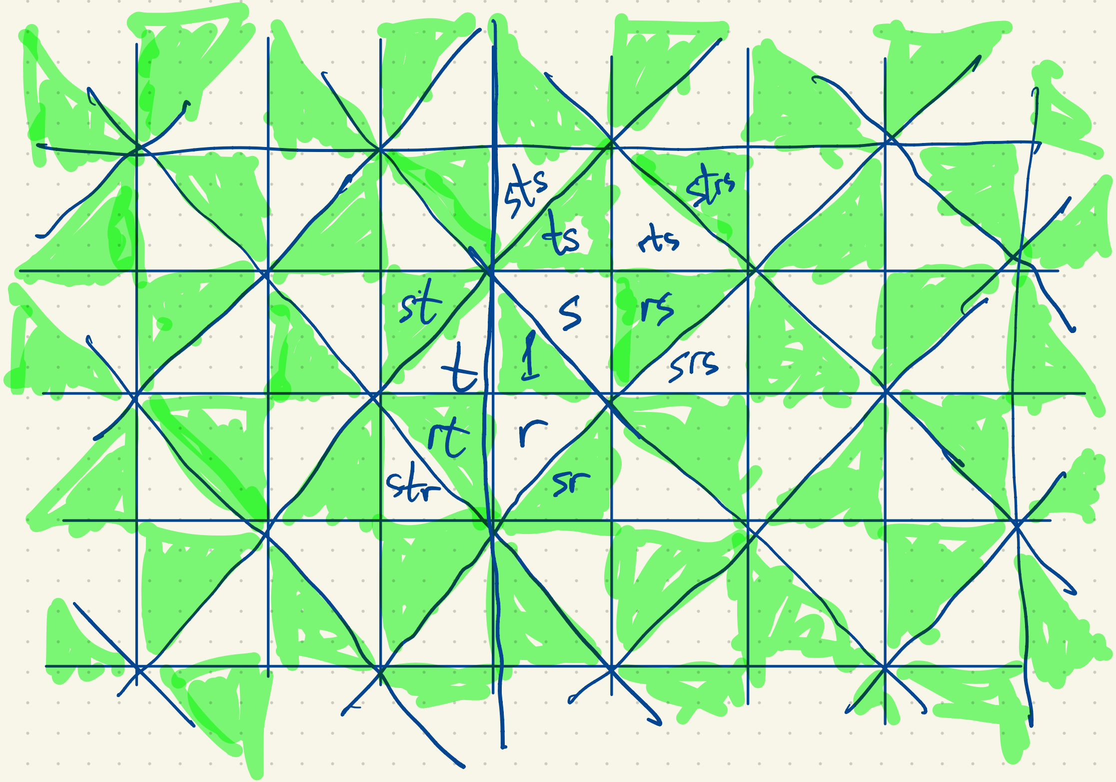
Group Theory

Book 2

reflections interchange white $\longleftrightarrow$ green
triangles triangles

Elements of $G = G(4,4,2)$ map
white? green?





$G = G(4,4,2)$ labels the green triangles
W labels all triangles

More generally if $l, m, n \geq 2$ satisfying $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1$ then $G = G(l,m,n)$ is a group of isometries of the Euclidean plane generated by rotations of order l, m, n:

$$G = \langle a, b, c \mid a^l, b^m, c^n, abc \rangle$$

Moreover $[W:G]$ where $W = \langle r, s, t \mid r^2, s^2, t^2, (rs)^l, (st)^m, (tr)^n \rangle$.

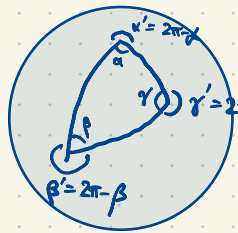
If $\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}>1$, $G=G(\ell,m,n)$ finite, then in place of a tiling of the Euclidean plane, we get a tiling of S^2 (Euclidean sphere).

If $\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}<1$, then we get a tiling of the hyperbolic plane by congruent triangles.
$G=G(\ell,m,n)$ infinite

A spherical example: $G=G(2,3,4)$, $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{6+4+3}{12}=\frac{13}{12}>1$

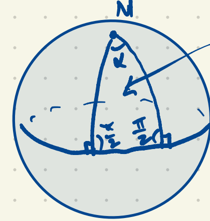
Eine Kleine spherical geometry

Let $S\subset\mathbb{R}^3$ be a unit sphere; its surface area is $4\pi r^2=4\pi$. "Lines" on S are geodesics (great circles).
Triangles in S have area = angular excess = $\alpha+\beta+\gamma-\pi>0$

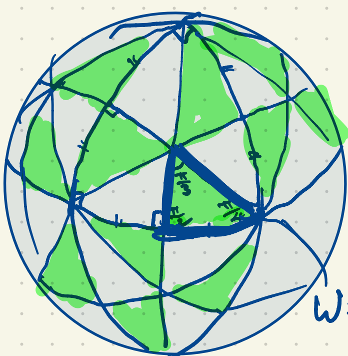


$$\underbrace{(\alpha+\beta+\gamma-\pi)}_{\text{angular excess of "inside"}}+\underbrace{(\alpha'+\beta'+\gamma'-\pi)}_{\text{angular excess "outside"}}=6\pi-2\pi=4\pi$$

W consists of isometries of S preserving the tiling:
$[W:G]=2$, $G=G(2,3,4)$
$G=\langle a,b,c \mid a^2,b^3,c^4,abc\rangle$
$a=rs$, $b=st$, $c=tr$



Area $=\alpha+\frac{\pi}{2}+\frac{\pi}{2}-\pi=\alpha$
$|G|=24$.
$G\cong S_4$



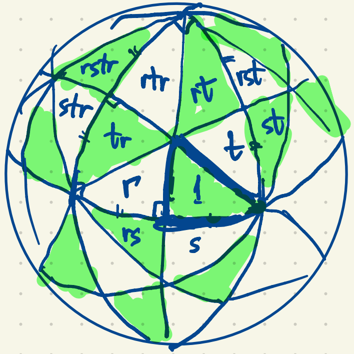
We subdivide S into 48 congruent spherical triangles each with angles $\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{4}$ and area $\frac{4\pi}{48}=\frac{\pi}{12}=\frac{\pi}{2}+\frac{\pi}{3}+\frac{\pi}{4}-\pi$

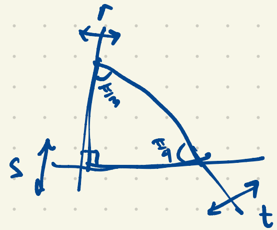
r,s,t: reflections of S in the "lines" bounding one triangle
i.e. reflections in the corresponding planes through the origin

$W=W(\bullet\overset{3}{-\!-}\bullet\overset{4}{=\!=}\bullet)=\langle r,s,t\mid r^2,s^2,t^2,(rs)^2,(rt)^3,(st)^4\rangle$,

$W\cong C_2\times S_4$
$|W|=48$

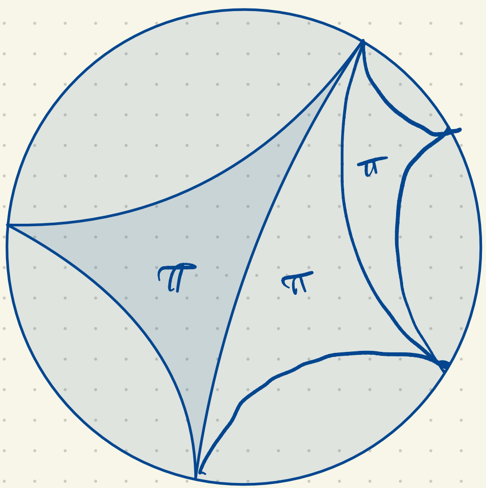
Starting Fri Oct 3 new room is BU24





Case $\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}<1$: $G(\ell,m,n)$ preserves a triangulation of the hyperbolic plane with triangles having angles $\frac{\pi}{\ell}, \frac{\pi}{m}, \frac{\pi}{n}$

In hyperbolic plane, a triangle (sides are lines = geodesics) having angular defect $\pi-(\alpha+\beta+\gamma)$ = area.

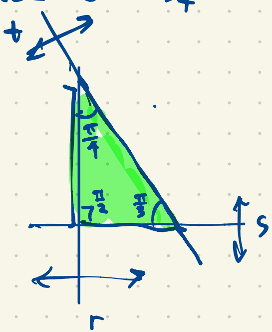


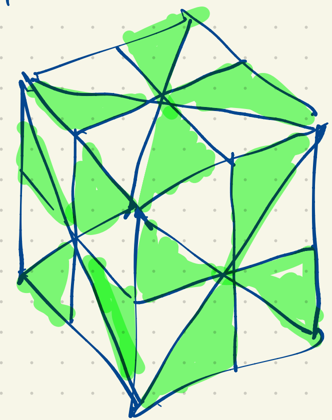
Today: $G = G(2,3,4) = \langle a,b,c : a^2, b^3, c^4, abc \rangle \cong S_4$

$W = W(\underset{B_2}{\bullet\!\!-\!\!\bullet\!\!=\!\!\bullet}) \cong C_2 \times S_4$

$= \langle r,s,t \mid r^2, s^2, t^2, (rs)^2, (rt)^3, (st)^4 \rangle$

To recognize $G \cong S_4$ without computer: $G = \langle rs, st, tr$: rs, st, tr are rotations by angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$ about vertices of triangle shown$\rangle$





G = Group of rotational symmetries of a cube $\cong S_4$.

G permutes the four "body diagonals" of the cube in all $4! = 24$ ways.
(A "body diagonal" joins two opposite vertices.)

G permutes the 24 green triangles transitively (and regularly).

If $W = HK$ where H and K are normal subgroups with $H \cap K = 1$ (so H and K commute with each other i.e. $hk = kh$ for all $h \in H$, $k \in K$) then $W \cong H \times K = \{(h,k) : h \in H, k \in K\}$ (direct product)

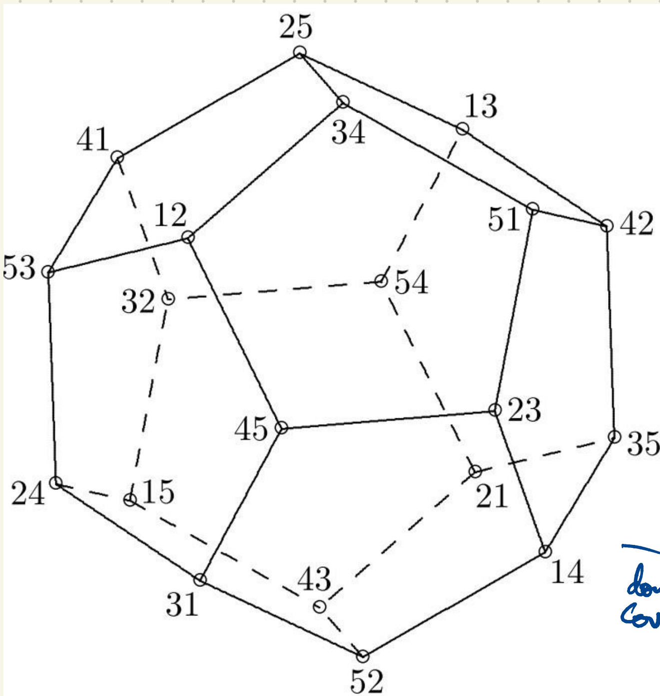
In our case $|W| = 48$, $|G| = 24$, $W = \underset{\text{preserve orientation}}{G} \cup \underset{\text{reverse orientation}}{Gr}$, W has a subgroup $H = \langle h \rangle$ of order 2, $H \triangleleft W$, $h = -I$
$H = Z(W)$

Cube has $3 + 6 = 9$ planes of symmetry
but altogether 24 orientation-reversing symmetries

Similarly the regular dodecahedron (12 pentagonal faces) has rotational symmetry group G with $|G| = 60$, $G = A_5$ (Alt_5)

Elements of A_5 give rotational symmetries of the dodecahedron.

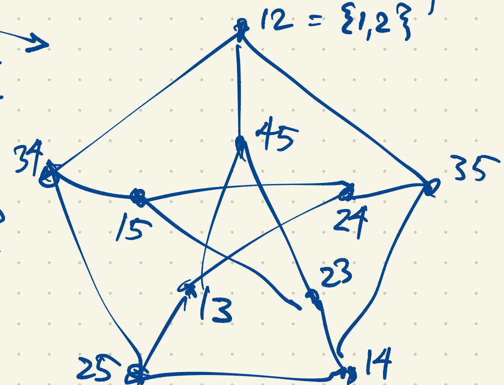
The full symmetry group of the regular dodecahedron has order 2×60 (60 $\underbrace{\text{orientation-preserving}}_{\text{rotations}}$ and 60 $\underbrace{\text{orientation-reversing}}_{\text{reflections + other}}$)



The full group of symmetries has order 120 and it has a subgroup isomorphic to A_5 but the group is not S_5. S_5 is not a group of isometries of $\mathbb{R}^3$.

Instead the full group of isometries of the regular dodecahedron is isomorphic to $C_2 \times A_5$.

double cover



This graph, the Petersen graph, has isomorphism group S_5

$$(i\ j) \mapsto \{i, j\}$$
$$(j\ i) \mapsto \{i, j\}$$

A presentation of a group G is an expression $G = \langle X : R \rangle$ where X is a set of symbols (letters) and $R \subset F(X) =$ free group on $X = \{x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} : x_j \in X,\ i_j \in \mathbb{Z}\}$

$x^j x^k = x^{j+k}$ $(j,k \in \mathbb{Z};\ x \in X)$
$x^0 = 1 =$ identity.

X = set of "generators"
R = set of "relators" (words in the generators)

If X is finite then G is <u>finitely generated</u>.
If X and R are both finite then G is <u>finitely presented</u>.
Burnside groups are finitely generated (usually) but not finitely presented.

$G = F/$ subgroup of F generated by R and their conjugates
$=$ "largest" homomorphic image of $F = F(X)$ having R in kernel
universal as we'll discuss later – see handout.

Every group has a presentation. Given G, for every $g \in G$, introduce a generator x_g. So $X = \{x_g : g \in G\}$.
For every pair $g, h \in G$ we want to force $x_g x_h = x_{gh}$ but this doesn't happen in $F = F(X)$ so introduce relators $x_g x_h x_{gh}^{-1} \in R$. $R = \{x_g x_h x_{gh}^{-1} : g, h \in G\}$. Then $F/\{\ldots R \ldots\} \cong G$.

If G is finitely generated then G is countable i.e. finite or countably infinite.
If $X = \{x_1, \ldots, x_n\}$ then $F = F(X) = \{\text{products of } x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\} = \bigcup_{l=0}^{\infty} S_l$ where
finite products

F is a countable union of finite sets, hence countable.

$S_l = \{x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_l}^{e_l} : i_1, \ldots, i_l \in \{1, \ldots, n\},\ e_j \in \{\pm 1\}\}$
$|S_l| \leq 2^l$

If G is countably generated (X countable) then $F = F(X)$ is countable.
$S_l = \{\text{words of length } l\}$ is countable.

If $|X| = m$ and $|R| = n$, m, n pos. integers, what can we say about $|G|$ where $G = \langle X \mid R \rangle$?

If $n < m$ then G is infinite. That is (contrapositive form) in order for $|G| < \infty$, we need at least as many relators as generators.

eg. $m = 1$. $X = \{x\}$. If $R = \emptyset$ then $G = \langle x \mid \emptyset \rangle = \{\dots, x^{-2}, x^{-1}, 1, x, x^2, \dots\}$.

If $R = \{1\}$ then $G = \langle x \mid 1 \rangle = F(x) = \{\dots, x^{-2}, x^{-1}, 1, x, x^2, \dots\}$

If $R = \{x^{15}\}$ then $G = \langle x \mid x^{15} \rangle = \{1, x, x^2, \dots, x^{14}\} \cong C_{15}$

If $R = \{x^{15}, x^{40}\}$ then $G = \langle x \mid x^{15}, x^{40} \rangle = \langle x \mid x^5 \rangle = \{1, x, x^2, x^3, x^4\}$

$$x^5 = (x^{15})^3 (x^{40})^{-1}$$