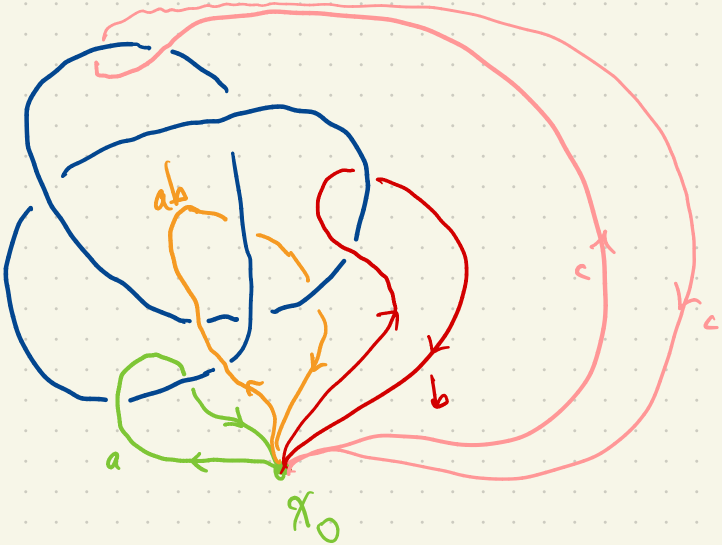
Group Theory

Book 3



$$ab = bc = ca \quad \Rightarrow \quad c = b^{-1}ab = aba^{-1}$$

$$\pi_1(\mathbb{R}^3 - K) = \langle a, b, c : ab = bc = ca \rangle = \langle x, y : x^3 = y^2 \rangle$$

$$\left.\begin{aligned} x &= ab = bc = ca \\ y &= abc \end{aligned}\right\} \Rightarrow x^3 = ab \cdot ca \cdot bc = abc \cdot abc = y^2$$

Observation: $\pi_1(\mathbb{R}^3 - K)$ has no (nontrivial) torsion elements (a "torsion" element in a group is an element of finite order)

If X is a path-connected subset of $\mathbb{R}^2$ then $\pi_1(X)$ has no (nontrivial) torsion elements.
(known for a few decades, this considered "folklore"). Intuitive!

The real proj. plane $P^2\mathbb{R}$ has torsion (its fund. gp. $\pi_1(P^2\mathbb{R})$ has order 2).

← a subspace of $\mathbb{R}^4$, not embeddable in $\mathbb{R}^3$.

For (top. spaces) subsets $X \subset \mathbb{R}^4$, $\pi_1(X)$ <u>can</u> have nontrivial torsion elements.

What about $X \subset \mathbb{R}^3$? Can $\pi_1(X)$ have nontrivial torsion elements? Famous open problem.

We will show $SL_2(\mathbb{Z})$ has a subgroup isomorphic to F_2:

$$\left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right\rangle < SL_2(\mathbb{Z})$$

use the <u>Ping-Pong Lemma</u>:

One version is as follows.

Let G be a group acting on a set X. We can think of G as a subgroup of $\mathrm{Sym}\, X = \{\text{bijections } X \to X\}$

i.e. permutations of X. Here I'm using left action, so we compose "right-to-left"

More generally, to say G acts on X means: For every $g \in G$ we have a permutation of X, $x \mapsto gx$

such that for all $g, h \in G$, $g(hx) = (gh)x$ i.e. the map $G \to \mathrm{Sym}\, X$ is a group homomorphism.

If the map $G \to \mathrm{Sym}\, X$ is one-to-one, we say G acts <u>faithfully</u> on X.

(So G is identified with a subgroup of $\mathrm{Sym}\, X$.)

Eg. $SL_2(F)$ acts on P^1F as fractional linear transformations.

$$\begin{array}{ccc} SL_2(F) & & P^1F = F \cup \{\infty\} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} & & x \end{array} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}(x) = \frac{ax+b}{cx+d}$$

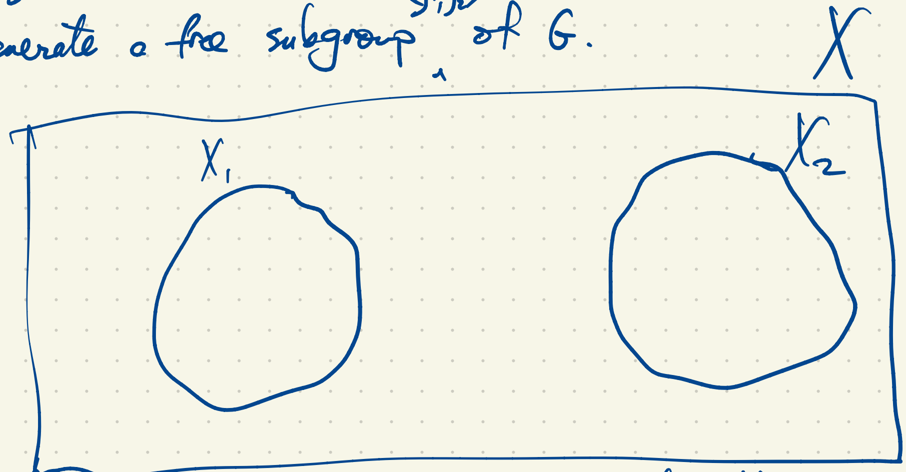
But this action is not faithful: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $-\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ give the same frac. lin. transf.

$PSL_2(F)$ acts faithfully on P^1F.

Given two subsets $X_1, X_2 \subset X$ which are disjoint nonempty and two elements $g_1, g_2 \in G$ such that

$$\left.\begin{aligned} g_1^k(X_2) &\subseteq X_1 \\ \text{and} \quad g_2^k(X_1) &\subseteq X_2 \end{aligned}\right\} \text{ for all integers } k \neq 0$$

then g_1, g_2 generate a free subgroup $\langle g_1, g_2 \rangle$ of G.



Proof A nontrivial word in two generators looks like

$w = g_1^{k_1} g_2^{l_1} g_1^{k_2} g_2^{l_2} \cdots g_1^{k_{r-1}} g_2^{l_{r-1}} g_1^{k_r}$ $\quad k_i, l_j$ nonzero integers up to conjugacy in the free group.

Then $w \neq 1$ since it maps $X \to X_1$.

Distinct words in the generators w_1, w_2 must give distinct permutations of X by considering $w = w_1^{-1} w_2$. $\square$

Application: $SL_2(\mathbb{Z})$ has $F_2 = \langle a,b\rangle$ (free group of rank 2) as a subgroup.

Try $u = \begin{bmatrix}1&1\\0&1\end{bmatrix}$, $v = \begin{bmatrix}1&0\\1&1\end{bmatrix}$. This doesn't work. These generate $\langle u,v\rangle = SL_2(\mathbb{Z})$

but this is not a free group since

$$uv^{-1} = \begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\-1&1\end{bmatrix} = \begin{bmatrix}0&1\\-1&1\end{bmatrix},\quad (uv^{-1})^3 = \begin{bmatrix}-1&0\\0&-1\end{bmatrix},\quad (uv^{-1})^6 = 1.$$

Next try $a = u^2 = \begin{bmatrix}1&2\\0&1\end{bmatrix}$, $b = v^2 = \begin{bmatrix}1&0\\2&1\end{bmatrix}$. These generate $\langle a,b\rangle \cong F_2$.

To prove this, use the Ping-Pong Lemma.

$G = SL_2(\mathbb{Z})$ acts on $P^1\mathbb{Q} = \mathbb{Q} \cup \{\infty\}$ by fractional linear transformations

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}(x) = \frac{ax+b}{cx+d}$$

Let $X_1 = \{x \in P^1\mathbb{Q} : |x| < 1\}$, $X_2 = \{x \in P^1\mathbb{Q} : |x| > 1\}$. Note: $\infty \in X_2$.

$$\begin{tikzcd} X_1 \arrow[r, bend left, "a^*"] & X_2 \arrow[l, bend left, "b^*"] \end{tikzcd}$$

$$b^n = \begin{bmatrix}1&0\\2&1\end{bmatrix}^n = \begin{bmatrix}1&0\\2n&1\end{bmatrix},\quad a^n = \begin{bmatrix}1&2\\0&1\end{bmatrix}^n = \begin{bmatrix}1&2n\\0&1\end{bmatrix} \qquad n \neq 0$$

$$a^n(x) = \frac{1x + 2n}{0x+1} = x + 2n$$

If $x \in X_1$ then $|x_1| < 1$ ie. $-1 < x_1 < 1$ then $a^n(x) \in (2n-1, 2n+1)$ so $|a^n(x)| > 1$ ie. $a^n(x) \in X_2$. for $n \neq 0$.

If $x \in X_2$ then $|x| > 1$

$$b^n(x) = \frac{1x+0}{2nx+1} = \frac{1}{2n + \frac{1}{x}},\quad \left|\frac{1}{x}\right| < 1,\quad \left|2n + \frac{1}{x}\right| > 1,$$

$$|b^n(x)| = \left|\frac{1}{2n+\frac{1}{x}}\right| < 1,\quad b^n(x) \in X_1. \qquad \square$$

In $G = SL_2(\mathbb{Z})$, $\langle a,b \rangle \cong F_2$.

Similarly in $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) / \{\pm I\}$, $\langle a,b \rangle \cong F_2$.

$$\langle a,b \rangle = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} : a_1, a_2, a_3, a_4 \in \mathbb{Z};\ a_1a_4 - a_2a_3 = 1;\ a_1, a_4 \text{ odd};\ a_2, a_3 \text{ even} \right\}.$$

In $SL_2(\mathbb{Z})$ elements have one of the forms

$$\begin{bmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{bmatrix}, \begin{bmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{bmatrix}, \begin{bmatrix} \text{odd} & \text{odd} \\ \text{even} & \text{odd} \end{bmatrix}, \begin{bmatrix} \text{odd} & \text{even} \\ \text{odd} & \text{odd} \end{bmatrix}, \begin{bmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{odd} \end{bmatrix}, \begin{bmatrix} \text{odd} & \text{odd} \\ \text{odd} & \text{even} \end{bmatrix},$$

$$\cancel{\begin{bmatrix} \text{even} & \text{odd} \\ \text{even} & \text{odd} \end{bmatrix}}, \ldots$$ ten choices of parity are excluded in $SL_2(\mathbb{Z})$.

$$|SL_2(\mathbb{F}_2)| = 2(2^2-1) = 6 \qquad SL_2(\mathbb{F}_2) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

Reducing matrix entries mod 2 gives an epimorphism (surjective homomorphism) $SL_2(\mathbb{Z}) \to SL_2(\mathbb{F}_2)$.

The kernel of this map is $\Gamma(2) = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in SL_2(\mathbb{Z}) : a_1, a_4 \text{ odd};\ a_2, a_3 \text{ even} \right\} = \langle a,b \rangle$
(a principal congruence subgroup)

$$[SL_2(\mathbb{Z}) : \Gamma(2)] = 6.$$

Tits Alternative: G linear group over F $\Rightarrow$ either
G virtually solvable
or G has F_2 as a subgroup (exclusive 'or').