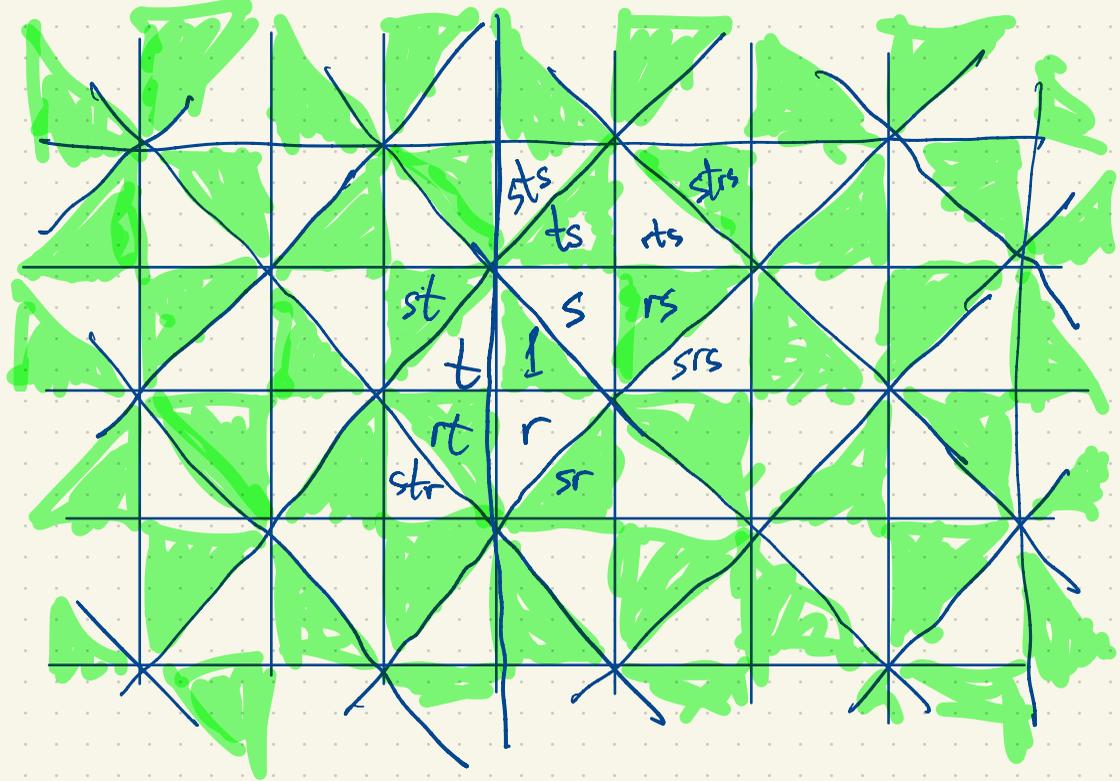
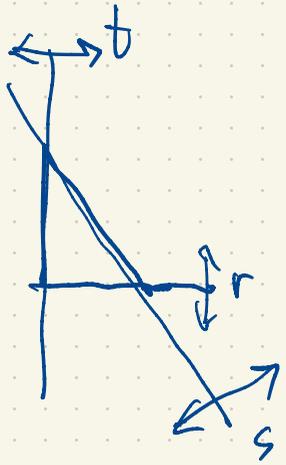


Group Theory

Book 2

reflections interchange
white \leftrightarrow green
triangles

Elements of $G = G(4,4,2)$
white \leftrightarrow map green \leftrightarrow



$G = G(4,4,2)$ labels the green triangles
 W labels all triangles



More generally if $l, m, n \geq 2$ satisfying
 $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1$ then $G = G(l, m, n)$ is a group of
isometries of the Euclidean plane generated by rotations
of order l, m, n .

$$G = \langle a, b, c \mid a^l, b^m, c^n, abc \rangle$$

Moreover $[W:G]$ where $W = \langle r, s, t \mid r^2, s^2, (rs)^l, (st)^m, (tr)^n \rangle$.

If $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1$, $G = G(l, m, n)$ finite then in place of a tiling of the Euclidean plane, we get a tiling of S^2 (Euclidean sphere).

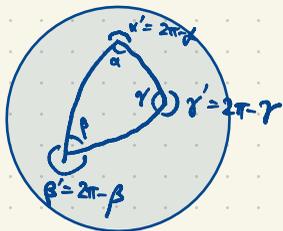
If $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$, then we get a tiling of the hyperbolic plane by congruent triangles. $C = G(l, m, n)$ infinite

A spherical example: $G = G(2, 3, 4)$, $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{6+4+3}{12} = \frac{13}{12} > 1$

Eine kleine spherical geometry

Let $S \subset \mathbb{R}^3$ be a unit sphere; its surface area is $4\pi r^2 = 4\pi$. "Lines" on S are geodesics (great circles).

Triangles in S have area = angular excess = $\alpha + \beta + \gamma - \pi > 0$



$$(\alpha + \beta + \gamma - \pi) + (\alpha' + \beta' + \gamma' - \pi) = 6\pi - 2\pi = 4\pi$$

angular excess of "inside" angular excess outside"

W consists of isometries of S preserving the tiling.

$$[W:G] = 2, \quad G = G(2, 3, 4)$$

$$G = \langle a, b, c \mid a^2, b^3, c^4, abc \rangle$$

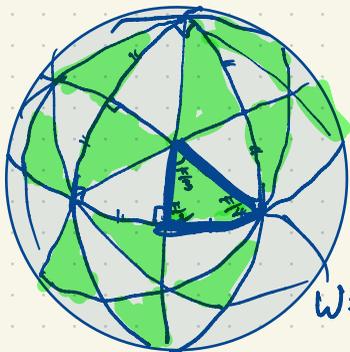
rs st tr



Area = $\alpha + \frac{\pi}{2} + \frac{\pi}{2} - \pi = \alpha$

$|G| = 24$.

$G \cong S_4$



We subdivide S

into 48 congruent spherical triangles, each with angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$ and area $\frac{4\pi}{48} = \frac{\pi}{12} = \frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{4} - \pi$

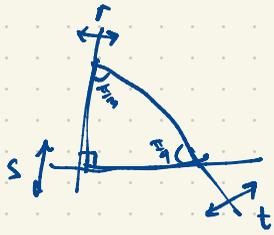
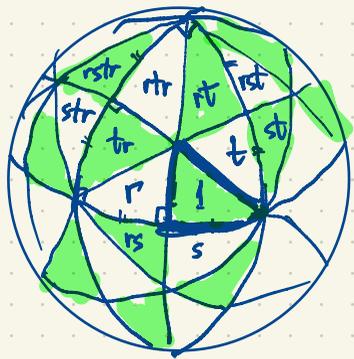
r, s, t : reflections of S in the "lines" bounding one triangle
i.e. reflections in the ~~directions~~ planes through the origin

$$W = W(\overset{3}{\leftarrow} \overset{4}{\rightarrow}) = \langle r, s, t \mid r^2, s^2, t^2, (rs)^2, (rt)^2, (st)^2 \rangle,$$

$W \cong C_2 \times S_4$

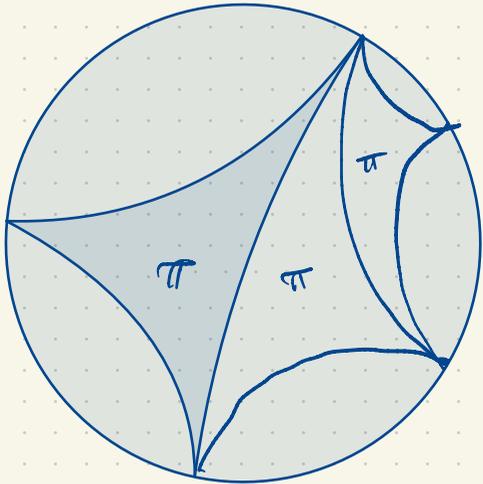
$|W| = 48$

Starting Fri Oct 3 new room is BU24



Case $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$: $G(l, m, n)$ preserves a triangulation of the hyperbolic plane with triangles having angles $\frac{\pi}{l}, \frac{\pi}{m}, \frac{\pi}{n}$

In hyperbolic plane, a triangle (sides are lines = geodesics) having angular defect $\pi - (\alpha + \beta + \gamma) = \text{area}$.



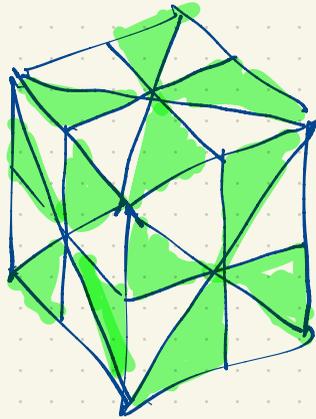
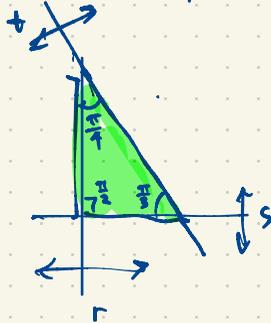
Today: $G = S(2,3,4) = \langle a, b, c : a^2, b^3, c^4, abc \rangle \cong S_4$

$$W = W(\overleftrightarrow{\quad}) \cong C_2 \times S_4 \\ = \langle r, s, t \mid r^2, s^2, t^2, (rs)^2, (rt)^3, (st)^4 \rangle$$

To recognize $G \cong S_4$ without computer:

$$G = \langle rs, st, tr : \dots \rangle$$

rs, st, tr are rotations by angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$ about vertices of triangle shown



$G =$ Group of rotational symmetries of a cube $\cong S_4$

G permutes the four "body diagonals" of the cube in all $4! = 24$ ways. (A "body diagonal" joins two opposite vertices.)

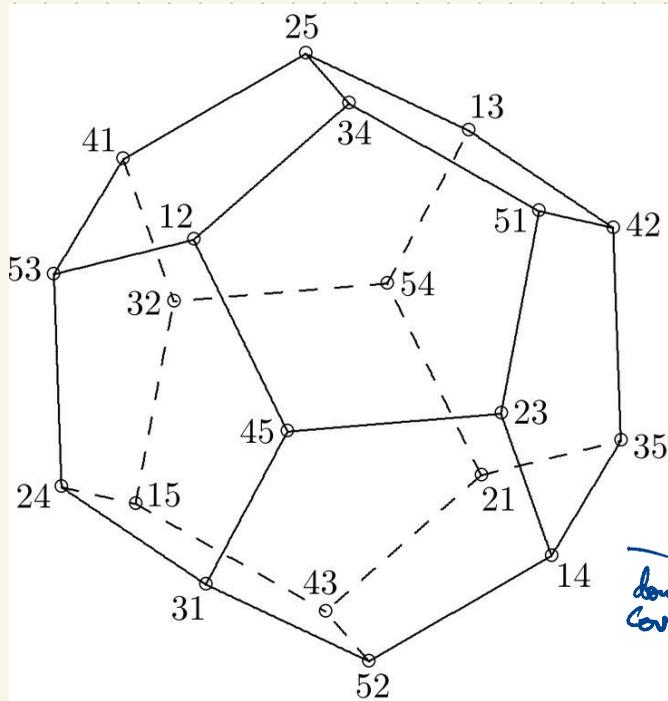
G permutes the 24 green triangles transitively (and regularly).

If $W = HK$ where H and K are normal subgroups with $H \cap K = 1$ (so H and K commute with each other i.e. $hk = kh$ for all $h \in H, k \in K$) then $W \cong H \times K = \{(h, k) : h \in H, k \in K\}$ (direct product)

In our case $|W| = 48, |G| = 24, W = G \cup Gr$, W has a subgroup $H = \langle h \rangle$ of order 2, $H \triangleleft W, h = -I$
 $H = Z(W)$
preserve orientation reverse orientation

Cube has $3 + 6 = 9$ planes of symmetry but altogether 24 orientation-reversing symmetries

Similarly the regular dodecahedron (12 pentagonal faces) has rotational symmetry group G with $|G| = 60$, $G = A_5$ (Alt_5)

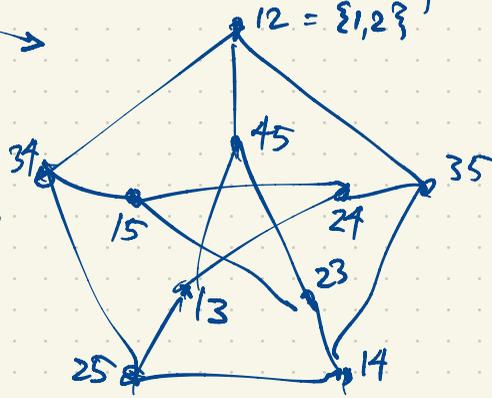


Elements of A_5 give rotational symmetries of the dodecahedron.

The full symmetry group of the regular dodecahedron has order 2×60
 (60 orientation-preserving and 60 orientation-reversing)
 rotations reflections + other

The full group of symmetries has order 120 and it has a subgroup isomorphic to A_5 but the group is not S_5 . S_5 is not a group of isometries of \mathbb{R}^3 .
 Instead the full group of isometries of the regular dodecahedron is isomorphic to $C_2 \times A_5$.

double cover



This graph, the Petersen graph, has isomorphism group S_5

$(i,j) \mapsto \{i,j\}$
 $(j,i) \mapsto \{i,j\}$



Jacques Tits