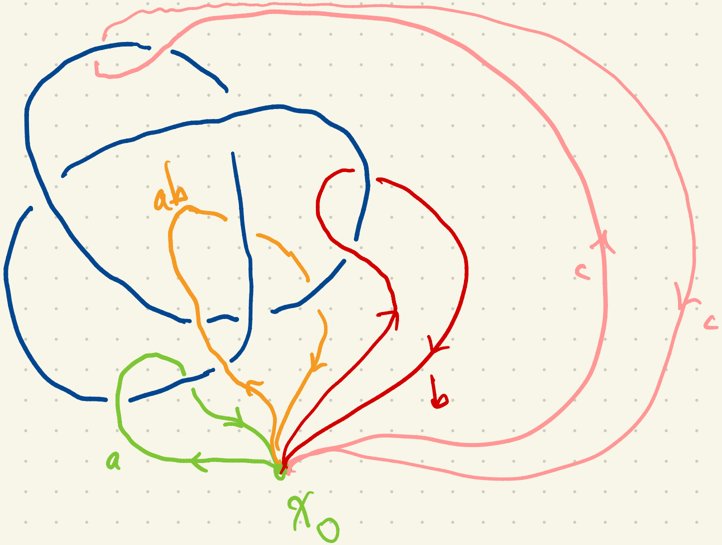
Group Theory

Book 3



$ab = bc = ca \quad \Rightarrow \quad c = b^{-1}ab = aba^{-1}$

$\pi_1(\mathbb{R}^3 - K) = \langle a, b, c : ab = bc = ca \rangle = \langle x, y : x^3 = y^2 \rangle$

$\left.\begin{array}{l} x = ab = bc = ca \\ y = abc \end{array}\right\} \Rightarrow x^3 = ab \cdot ca \cdot bc = abc \cdot abc = y^2$

Observation: $\pi_1(\mathbb{R}^3 - K)$ has no (nontrivial) torsion elements (a "torsion" element in a group is an element of finite order)

If X is a path-connected subset of $\mathbb{R}^2$ then $\pi_1(X)$ has no (nontrivial) torsion elements. (known for a few decades, this considered "folklore"). Intuitive!

The real proj. plane $P^2\mathbb{R}$ has torsion (its fund. gp. $\pi_1(P^2\mathbb{R})$ has order 2) ← a subspace of $\mathbb{R}^4$, not embeddable in $\mathbb{R}^3$.

For (top. spaces) subsets $X \subset \mathbb{R}^4$, $\pi_1(X)$ can have nontrivial torsion elements.

What about $X \subset \mathbb{R}^3$? Can $\pi_1(X)$ have nontrivial torsion elements? Famous open problem.

We will show $SL_2(\mathbb{Z})$ has a subgroup isomorphic to F_2:

$$\left\langle \begin{bmatrix}1&2\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\2&1\end{bmatrix} \right\rangle < SL_2(\mathbb{Z})$$

use the <u>Ping-Pong Lemma</u>:

One version is as follows.

Let G be a group acting on a set X. We can think of G as a subgroup of $\operatorname{Sym} X = \{\text{bijections } X \to X\}$
i.e. permutations of X. Here I'm using left action, so we compose "right-to-left"

More generally, to say G acts on X means: For every $g \in G$ we have a permutation of X, $x \mapsto gx$
such that for all $g, h \in G$, $g(hx) = (gh)x$ i.e. the map $G \to \operatorname{Sym} X$ is a group homomorphism.

If the map $G \to \operatorname{Sym} X$ is one-to-one, we say G acts <u>faithfully</u> on X.
(So G is identified with a subgroup of $\operatorname{Sym} X$.)

Eg. $SL_2(F)$ acts on P^1F as fractional linear transformations.

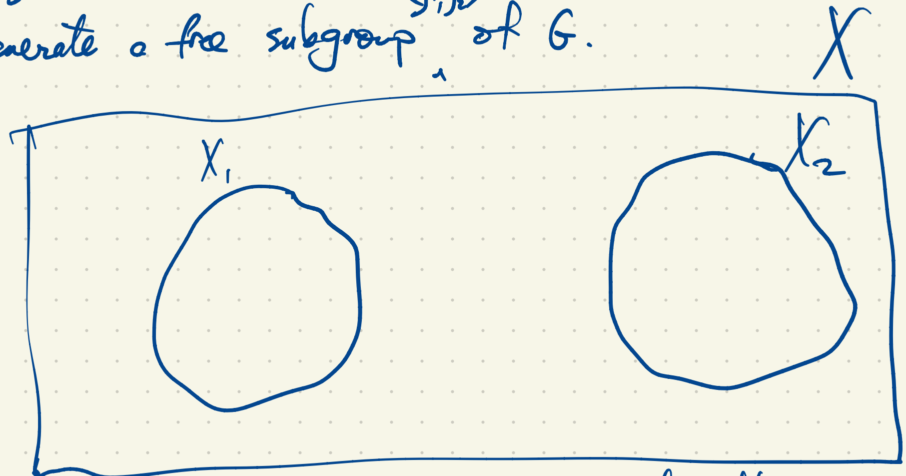
$$\begin{matrix} SL_2(F) & & P^1F = F \cup \{\infty\} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \begin{bmatrix}a&b\\c&d\end{bmatrix} & & x \end{matrix} \qquad \begin{bmatrix}a&b\\c&d\end{bmatrix}(x) = \frac{ax+b}{cx+d}$$

But this action is not faithful: $\begin{bmatrix}a&b\\c&d\end{bmatrix}$ and $-\begin{bmatrix}a&b\\c&d\end{bmatrix}$ give the same frac. lin. transf.
$PSL_2(F)$ acts faithfully on P^1F.

Given two subsets $X_1, X_2 \subset X$ which are disjoint nonempty and two elements $g_1, g_2 \in G$ such that

$$\left.\begin{aligned} & g_1^k(X_2) \subseteq X_1 \\ \text{and} \quad & g_2^k(X_1) \subseteq X_2 \end{aligned}\right\} \text{ for all integers } k \neq 0$$

then g_1, g_2 generate a free subgroup $\langle g_1, g_2 \rangle$ of G.



Proof A nontrivial word in two generators looks like
$g_1^{k_1} g_2^{l_1} g_1^{k_2} g_2^{l_2} \cdots g_1^{k_{r-1}} g_2^{l_{r-1}} g_1^{k_r}$ k_i, l_j nonzero integers up to conjugacy in the free group.