HW1

(Due Wednesday, October 2, 2019)

Instructions: See the course syllabus for general expectations regarding homework. Note especially that you may discuss the problems with other students, but you should write up solutions yourself. Giving complete solutions to all problems is less important than showing me what *you* can do.

1. Let X be an infinite set. Denote by $\mathsf{Sym}\,X$ the group of all permutations of X (i.e. bijections $X \to X$ under composition). Also, $\mathsf{FinSym}\,X$ the subgroup consisting of all permutations with finite support (where the *support* of $g \in \mathsf{Sym}\,X$ is $\{x \in X : g(x) \neq x\}$).
   (a) Show that $\mathsf{FinSym}\,X$ is a nontrivial normal subgroup of $\mathsf{Sym}\,X$. Conclude that $\mathsf{Sym}\,X$ is not simple.
   (b) Show that $\mathsf{FinSym}\,X$ has a normal subgroup $\mathsf{Alt}\,X$ of index two. Is $\mathsf{Alt}\,X$ simple?
   (c) Is the quotient group $\mathsf{Sym}\,X/\,\mathsf{FinSym}\,X$ simple?

2. Let $G = GL_2(F)$ where F is an arbitrary field.
   (a) Show that the center of G is the subgroup $Z = \left\{\left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right] : 0 \neq a \in F\right\}$.

   Denote by $\mathbb{P}^1F$ the set of all one-dimensional subspaces of F^2, i.e.

$$\mathbb{P}^1F = \{[x] : x \in F\} \cup \{[\infty]\}$$

   where $[x]$ is the span of $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] \in F^2$ and $[\infty]$ is the span of $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \in F^2$. This is known as the *projective line over* F. The group $G = GL_2(F)$ permutes the elements of the projective line $\mathbb{P}^1F$; for example the element $\left[\begin{smallmatrix} 2 & 4 \\ 1 & 7 \end{smallmatrix}\right]$ maps $[1] \mapsto \left[\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\right]$.

   (b) Show that an element $g \in G$ fixes more than two points of the projective line $\mathbb{P}^1F$, iff $g \in Z$.

   The action of G on the projective line gives a homomorphism $G \to \mathsf{Sym}\,\mathbb{P}^1F$.

   (c) Show that the kernel of this action is precisely the subgroup Z. (Recall: the *kernel* of a homomorphism is the set of elements mapping to the identity—in this case, the identity permutation.)

The image of this homomorphism, i.e. the group of all permutations of the projective line induced by G, is the quotient group $PSL_2(F) = G/Z$. This is the *projective special linear group*, whose elements are represented as invertible 2×2 matrices; but where two matrices are identified whenever one is a scalar multiple of the other. Informally, $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ maps $[x] \mapsto \left[\frac{ax+b}{cx+d}\right]$ whenever the denominator is nonzero; and exceptional cases are handled by the heuristic method of 'taking limits'. This action is made precise by our formalism above.

3. Every complex $n \times n$ matrix B defines a quadratic form $Q : \mathbb{C}^n \to \mathbb{C}$ given by

$$Q(\mathbf{x}) = \mathbf{x}^T B \mathbf{x}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$. (Note: no complex conjugation is used here!) There is no harm in assuming the matrix B is symmetric; otherwise we would replace it by the symmetric matrix $\frac{1}{2}(B + B^T)$, which defines the same quadratic form. In fact, we will assume more: *We will assume B is an invertible $n \times n$ real symmetric matrix.*

The *isometry group* of the quadratic form Q is the group

$$\begin{aligned} O(Q) = O(B, \mathbb{C}) &= \{A \in GL_n(\mathbb{C}) \,:\, Q(A\mathbf{x}) = Q(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbb{C}^n\} \\ &= \{A \in GL_n(\mathbb{C}) \,:\, A^T B A = B\}. \end{aligned}$$

Note that $O(I, \mathbb{C}) = O_n(\mathbb{C})$ is the usual orthogonal group over $\mathbb{C}$.

Show that $O(B, \mathbb{C}) \cong O(I, \mathbb{C}) \cong O_n(\mathbb{C})$, independent of the choice of B. (*Hint:* The matrix B is diagonalizable by a real orthogonal change of basis.)

Remarks: The choice to work over $\mathbb{C}$ here is somewhat arbitrary; similar results hold over other fields. The ground field $\mathbb{C}$ is a little easier because it is algebraically closed and it does not have characteristic two. In particular on $\mathbb{C}^n$, any two nondegenerate quadratic forms are isometric. Over the reals this is no longer true: there are $n+1$ nondegenerate quadratic forms on $\mathbb{R}^n$ up to isometry, determined by the number of positive and negative eigenvalues of B. On $\mathbb{Q}^n$ there are infinitely many isometry types of quadratic forms for each $n \geqslant 1$.

4. Let $G = SO_3(\mathbb{C})$, the group of all 3×3 complex matrices A satisfying $AA^T = I$. Here we are guided through a proof that $G \cong PSL_2(\mathbb{C})$.

Let V be the complex vector space consisting of all homogeneous polynomials of degree 2 in two variables x, y, i.e. V is the set of all expressions $ax^2 + bxy + cy^2$ where $a, b, c \in \mathbb{C}$. Note that V is 3-dimensional with basis $\{x^2, xy, y^2\}$. The *discriminant* of $f(x, y) = ax^2 + bxy + cy^2 \in V$ is

$$Q(f) = b^2 - 4ac = \begin{bmatrix} a & b & c \end{bmatrix} B \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{where } B = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}.$$

The group $GL_2(\mathbb{C})$ acts on V by linear changes in x, y. For example, the element $g = \left[\begin{smallmatrix} 2 & 4 \\ 1 & 7 \end{smallmatrix}\right] \in GL_2(\mathbb{C})$ maps $f(x,y) = ax^2 + bxy + cy^2$ to

$$\begin{aligned}(gf)(x,y) &= f(2x{+}4y,\, x{+}7y) \\ &= a(2x{+}4y)^2 + b(2x{+}4y)(x{+}7y) + c(x{+}7y)^2 \\ &= (4a{+}2b{+}c)x^2 + (16a{+}18b{+}14c)xy + (16a{+}28b{+}49c)y^2\end{aligned}$$

which has discriminant given by $Q(gf) = 100Q(f)$.

(a) Show that for an arbitrary $h \in GL_2(\mathbb{C})$, we have $Q(hf) = (\det h)^2 Q(f)$. (This can be shown by a routine technical computation; or it can be done using higher level reasoning. Use whichever method you are most comfortable with.)

(b) Show that the matrices $h \in SL_2(\mathbb{C})$ act on V, preserving the discriminant.

If we denote by $R_h \in GL(V)$ the map $V \to V$, $f \mapsto hf$, then we have a homomorphism $SL_2(\mathbb{C}) \to GL(V) \cong GL_3(\mathbb{C})$. Step (b) shows that in fact $R_h \in O(Q) \cong O_3(\mathbb{C})$. The latter isomorphism follows from #3, which says that the isometry group of the discriminant form Q is isomorphic to the isometry group of any other nondegenerate form on $\mathbb{C}^3$, including the standard form.

(c) Show that in fact $R_h \in SO_3(\mathbb{C})$. So we have a homomorphism $SL_2(\mathbb{C}) \to SO_3(\mathbb{C})$, $h \mapsto R_h$.

(d) Show that the homomorphism in (c) is onto.

(e) Find the kernel of the homomorphism in (c), and use this to prove that $SO_3(\mathbb{C}) \cong PSL_2(\mathbb{C})$.

Remarks: Over $\mathbb{R}$, we have $O(Q) \cong PSL_2(\mathbb{R})$ where the discriminant form Q is a nondegenerate but *indefinite* form (it is *not* positive definite). In fact, diagonalizing B gives two positive and one negative eigenvalue. A similar isomorphism holds over other fields, if we are careful about the choice of quadratic form we are using—different quadratic forms have different isometry groups, so there are different types of orthogonal group in general. You might notice that unlike the group $SO_3(\mathbb{R}) \cong PSU_2(\mathbb{C})$ considered in class (a *compact* 3-dimensional Lie group), the group $SO_3(\mathbb{C}) \cong PSL_2(\mathbb{C})$ is *not compact*.