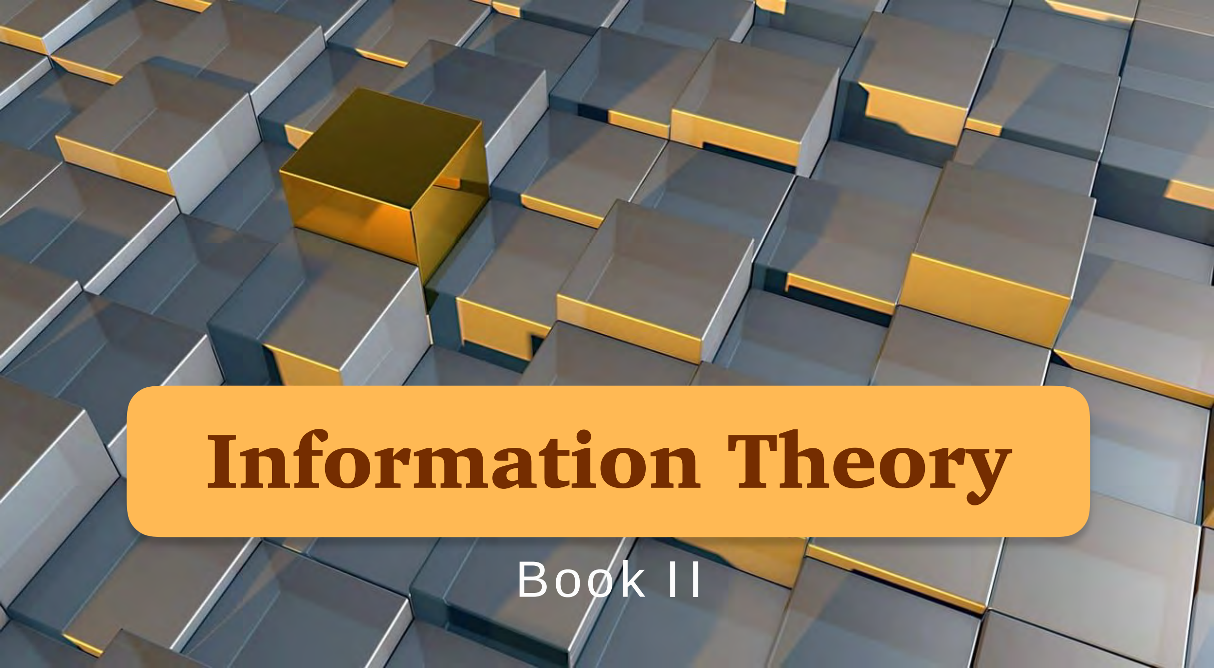
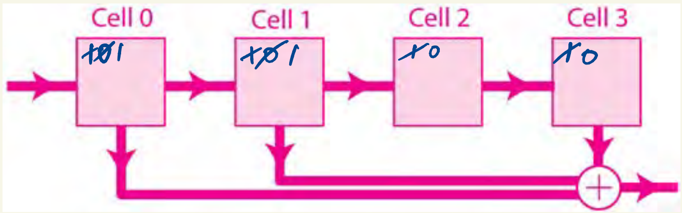


# Information Theory

Book II

Eg. an infinite stream of bits $a_0 a_1 a_2 a_3 a_4 \cdots$ $(a_i \in F)$ can be encoded eg. represent the plaintext bitstream as a $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \in \mathbb{F}_2[[x]]$

$F[[x]]$ = ring of (formal) power series in $x$ with coefficients in $F$.



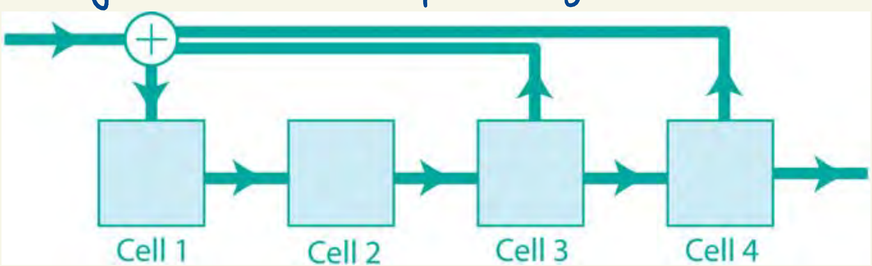

10110...

Eg. consider an input bitstream ~~1~~ ~~1~~ ~~0~~ ~~0~~ ~~1~~ ~~1~~ 0 1 1 1 1 0 0 1 0 ... which is encoded by the shift register above to obtain the output bitstream 1011 00101...

Compare: this is equivalent to multiplication by $1+x+x^3$:

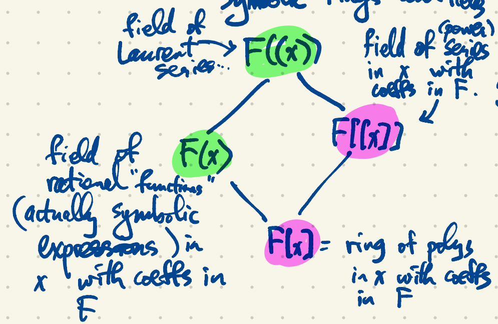
$$(1+x+x^3)(1+x+x^4+x^5+x^7+x^8+x^9+x^{10}+x^{13}+\cdots) = 1 + x^2 + x^3 + x^6 + x^8 + \cdots$$

Decoding of this data is accomplished using backward shift registers eg.




which performs division by $1+x+x^3$ in $\mathbb{F}_2((x))$

Symbolic rings and fields




polynomials vs. polynomial functions

eg. $\mathbb{F}_3 = \{0,1,2\} = \mathbb{Z}/3\mathbb{Z}$

eg. $f(x) = 2+x+x^3 \in \mathbb{F}_3[x]$ is a polynomial of degree 3.

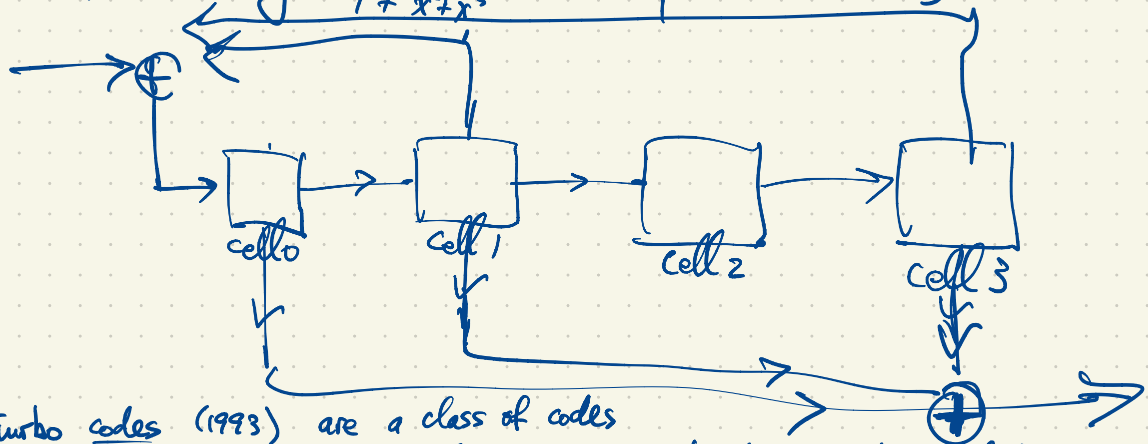
$g(x) = 2+2x \in \mathbb{F}_3[x]$ is a polynomial of degree 1.

| $a$ | $f(a)$ | $g(a)$ |
|---|---|---|
| 0 | 2 | 2 |
| 1 | 1 | 1 |
| 2 | 0 | 0 |

$f(x), g(x)$ are distinct poly's but they represent the same function $\mathbb{F}_3 \to \mathbb{F}_3$.

eg. $f(x) = \dfrac{1+x+x^3}{x+x^2} \in \mathbb{F}_2(x)$

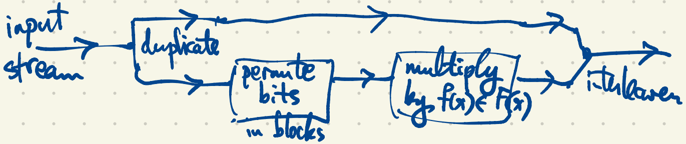
Multiplication by any rational function can be implemented using a single shift register e.g. multiplication by $\frac{1+x+x^3}{1+x^2+x^3}$ is implemented using the shift register




<u>Turbo</u> <u>codes</u> (1993) are a class of codes used for encoding streams of data using combinations of gates including

- multiplication by a rational function in $F(x)$
- splitters & interleavers
- permutations
- puncturing

eg.




$F(x) \subset F((x))$ eg. for $F = \mathbb{F}_2 = \{0,1\}$

First method

$$f(x) = \frac{1+x^2+x^5}{x+x^2+x^4} = \frac{1+x^2+x^5}{x(1+x+x^3)} = \frac{1}{x}\left[\frac{1+x^2+x^5}{1+x+x^3}\right] = \frac{1}{x}\left[1+x+x^3+x^5+\cdots\right] = \frac{1}{x}+1+x^2+x^4+\cdots$$

$$\frac{1+x^2+x^5}{1+x+x^3} = 1+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\cdots$$

$a_1=1$, $a_2=0$, $a_3=1$, $a_4=0$, $a_5=1$

$$1+x^2+x^5 = (1+x+x^3)(1+x+x^3+x^5+\cdots)$$

$(a+b)^2 = a^2+b^2$

$(a+b)^4 = a^4+b^4$

Second method Geometric series $\frac{1}{1-u} = 1+u+u^2+u^3+u^4+\cdots$

$$\frac{1+x^2+x^5}{1+(x+x^3)} = (1+x^2+x^5)\left(1+(x+x^3)+(x+x^3)^2+(x+x^3)^3+(x+x^3)^4+(x+x^3)^5+\cdots\right)$$

$$= (1+x^2+x^5)\left(1+(x+x^3)+(x^2+x^6)+(x^3+x^5+\cdots)+(x^4+\cdots)+(x^5+\cdots)+\cdots\right)$$

$(x^3+3x^5+3x^7+x^9)$

$$= (1+x^2+x^5)(1+x+x^2+x^4+\cdots)$$

$$= 1+x+x^3+x^5+\cdots$$

$$f(x) = \frac{1}{x}(1+x+x^3+x^5+\cdots) = \frac{1}{x}+1+x^2+x^4+\cdots$$

$F = \mathbb{F}_2 = \{0,1\}$ for the time being

The irreducible (monic) polynomials in $F[x]$:

primitive

not primitive

| degree | irred. polys | |
|---|---|---|
| 1 | $x, \quad x+1$ | |
| 2 | $x^2+x+1$ | |
| 3 | $x^3+x+1, \quad x^3+x^2+1$ | |
| 4 | $x^4+x+1, \quad x^4+x^3+1,$ | $x^4+x^3+x^2+x+1$ |
| .... | .... | |

$x^2, \quad x^2+1, \quad x^2+x, \quad x^2+x+1$ all poly's of degree 2.

$x^2 = x\cdot x$, $x^2+1 = (x+1)(x+1)$, $x^2+x = x(x+1)$

$x^4+x^2+1 = (x^2+x+1)^2$

See MacWilliams & Sloane, The Theory of Error-Correcting Codes for more extensive lists of irreducible polynomials.

What are all the cyclic (linear) binary codes of length 7? There are exactly 8 of them. (Why?)

- subspace of $F^7$, $F = \mathbb{F}_2 = \{0,1\}$
- invariant under cyclic shift $(a_0,a_1,a_2,a_3,a_4,a_5,a_6) \longmapsto (a_6,a_0,a_1,\dots,a_5)$ $\quad a_i \in F$

A linear code $\mathcal{C} \subseteq F^n$ is cyclic iff its dual code $\mathcal{C}^\perp \subseteq F^n$ is also cyclic.

$\dim \mathcal{C} + \dim \mathcal{C}^\perp = n$.

eg. $\{(0000000)\}$

$\{0000000, 1111111\}$

$F^7 \leftarrow g(x)=1, \quad h(x)=x^7-1$

$\{\text{words in } F^7 \text{ of even weight}\} = \langle 1100000, 1010000, 1001000, 1000100, 1000010, 1000001\rangle$

$$\begin{array}{r} 1110100 \\ 0101000 \\ \hline 1011100 \end{array}$$

Hamming $[7,4,3]_2$ code $\mathcal{H} = \langle 1101000, 0110100, \dots, 1010001\rangle$ (all cyclic shifts of 1101000 span this code)

$\dim \mathcal{H} = 4$, $|\mathcal{H}| = 2^4 = 16$:

| | | |
|---|---|---|
| 1 | codeword of weight | 0 |
| 7 | ... | 3 |
| 7 | ... | 4 |
| 1 | ... | 7 |

Its dual $\mathcal{H}^\perp$, $\dim \mathcal{H}^\perp = 3$ is a $[7,3,4]_2$ - code.

$\mathcal{H}^\perp$ has 1 codeword of weight 0

7 ... 4

$\mathcal{H}^\perp = \mathcal{H} \cap \langle 1111111\rangle^\perp$

$\mathcal{H}' = \langle 1011000, 0101100, \dots, 0110001\rangle$ also $[7,4,3]_2$

$\mathcal{H}'^\perp$ also $[7,3,4]_2$.

$x^{7}-1 \in F[x]$ ($7 \leftarrow n =$ length)

actually $x^7+1$  $F=\mathbb{F}_2$

$x^7-1 = \underset{\text{i.e. } x+1}{(x-1)}(x^6+x^5+x^4+x^3+x^2+1) = (x-1)\underbrace{(x^3+x+1)}_{(x-\alpha)(x-\alpha^2)(x-\alpha^4)}\underbrace{(x^3+x^2+1)}_{(x-\beta)(x-\beta^2)(x-\beta^4)}$

If $E=\mathbb{F}_q$, $x^q - x = \underset{x-\mathbb{I}_0}{x}(x-1)(x-a_2)(x-a_3)\cdots(x-a_q)$

$a_0=0,\ a_1=1,\ a_2, a_3, \cdots, a_q$ are the field elements.

i.e. $x^{q-1}-1$ has $q-1$ distinct roots which are the nonzero field elements.

If $\alpha \in \mathbb{F}_8$ is a root of $x^3+x+1$

$\mathbb{F}_8 = \mathbb{F}_2[\alpha] = \{a_0+a_1\alpha+a_2\alpha^2 : a_0,a_1,a_2 \in \mathbb{F}_2\}$
$= \{0, 1, \alpha, \alpha+1, \alpha^2, \alpha^2+1, \alpha^2+\alpha, \alpha^2+\alpha+1\}$

Squaring is an automorphism of $\mathbb{F}_8$.

$(u+v)^2 = u^2+v^2$
$(uv)^2 = u^2v^2$

If $f(x) \in \mathbb{F}_p[x]$ is irreducible of degree $d$, then $\mathbb{F}_p[x]/(f(x)) \cong \mathbb{F}_{p^d} = \mathbb{F}_p[\beta]$ where $\beta$ is a root of $f(x)$.

$= \{a_0+a_1\beta+a_2\beta^2+\cdots+a_{d-1}\beta^{d-1} : a_i \in \mathbb{F}_p\}$

($\beta$ generates $\mathbb{F}_{p^d} \supset \mathbb{F}_p$ as an algebra)

If in fact $\mathbb{F}_{p^d} = \{0, 1, \beta, \beta^2, \beta^3, \ldots, \beta^{p^d-2}\}$ then we say $\beta$ is a primitive element and we say $f(x)$ is a primitive polynomial.

If $f(x) = x^4+x^3+x^2+x+1$ and $\beta \in \mathbb{F}_{16} = \mathbb{F}_{2^4}$ is a root of $f(x)$ then $\beta^5=1$ since $\beta$ is a root of $f(x)$

$\beta^5-1 = (\beta-1)\underbrace{(\beta^4+\beta^3+\beta^2+\beta+1)}_{0} = 0$

$0, 1, \beta, \beta^2, \beta^3, \beta^4, 1, \beta, \beta^2, \cdots$ doesn't give all of $\mathbb{F}_{16}$.

There are eight ways to factor $x^7-1 = g(x)h(x)$ in $\mathbb{F}_2[x]$
In each case $g(x)$ is a generator poly. and $h(x)$ is a parity check poly. for a cyclic code of length 7
over $\mathbb{F}_2 = \{0,1\} = F$

Cyclic (linear) codes $\longleftrightarrow$ ideals in $F[x]/(x^n-1)$

$g(x)=1$, $h(x)=x^7-1$ gives $F^7$
$g(x)=x^7-1$, $h(x)=1$ gives $\{0000000\}$
$g(x)=x+1$, $h(x)=x^6+x^5+x^4+x^3+x^2+x+1$ gives all words of even weight i.e. $\langle 1100000, 1010000, \cdots, 1000001 \rangle$
$g(x)=x^6+x^5+\cdots+1$, $h(x)=x+1$ gives $\langle 1111111 \rangle = \{0000000, 1111111\}$
$g(x)=1+x+x^3$, $h(x)=1+x^2+x^3+x^4$ gives $\mathcal{H}$ $[7,4,3]_2$ code

**BCH bound** : a lower bound for performance of a cyclic code.

Consider a cyclic code of length $n$ over $\mathbb{F}_q$, i.e. an ideal in $\mathbb{F}_q[x]/(x^n-1)$ with gen. poly. $g(x)$, parity check poly. $h(x)$, $x^n-1=g(x)h(x)$, $g(x)$ primitive, $\beta$ root of $g(x)$ in $\mathbb{F}_{q^r}$, $r=\deg g(x)$, and $\beta, \beta^2, \cdots, \beta^{s-1}$ are roots of $g(x)$, then the code has min. distance $\geq s$.

For Hamming $[7,4,3]_2$ code, $\beta$ root of $g(x)=1+x+x^3 \in F[x]$, $\beta \in \mathbb{F}_8 = \mathbb{F}_2[\beta]$
Also $\beta^2$ - - - - - - - - by Freshman's Dream

$$1+\beta+\beta^3=0$$
$$(1+\beta+\beta^3)^2 = 1+\beta^2+\beta^6 = 0 = 1+\beta^2+(\beta^2)^3 \quad \Rightarrow \quad \mathcal{H} \text{ has min. dist.} \geq 3.$$

BCH : R.C. Bose
Dijen Ray-Chaudhuri
Hocquenghem

The Gilbert-Varshamov Bound (GV-bound): a lower bound for existence of good codes

$A_q(n,d) = \max. |\mathcal{C}|$ s.t. $\mathcal{C} \subseteq A^n$, $|A|=q$ with min. distance $\geq d$ i.e. $d(w,w') \geq d$ for all $w \neq w'$ in $\mathcal{C}$.

Ball of radius $r$ in $A^n$ centered at "0" $\in A^n$ $\qquad e = \lfloor \frac{d-1}{2} \rfloor$ = error-correcting capability.

has cardinality $|B_r(0)| = \sum_{k=0}^{r} \binom{n}{k} (q-1)^k$ $\qquad 1 = |B_0| < |B_1| < |B_2| < \cdots < |B_n| = |A^n| = q^n$

Hamming bound: $A_q(n,d) \leq \dfrac{q^n}{|B_e|}$ : balls of radius $e$ centered at codewords $w \in \mathcal{C}$ are required to be disjoint

$$\bigsqcup_{w \in \mathcal{C}} B_e(w) \subseteq A^n. \Rightarrow |\mathcal{C}| \cdot |B_e(w)| \leq q^n$$

$$\Rightarrow |\mathcal{C}| \leq \frac{q^n}{|B_e(w)|}$$

In the other direction the GV-bound

$$A_q(n,d) \geq \frac{q^n}{|B_{d-1}(0)|} \qquad \text{so} \qquad \frac{q^n}{|B_{d-1}(0)|} \leq A_q(n,d) \leq \frac{q^n}{|B_e(0)|}$$

We regard the GV bound as an existence proof only.

Proof: Let $\mathcal{C} \subseteq A^n$ be any $q$-ary code with $|\mathcal{C}| = A_q(n,d)$. We claim

$$\bigcup_{w \in \mathcal{C}} B_{d-1}(w) \supseteq A^n.$$

Codes satisfying this condition by greedy construction. But such codes are usually not practical because membership & decoding are not efficient.

If not, there exists $w' \in A^n$, $w' \notin \bigcup_{w \in \mathcal{C}} B_{d-1}(w)$ so $d(w',w) > d-1$ for all $w \in \mathcal{C}$.

But then $\mathcal{C} \cup \{w'\}$ has min. distance $\geq d$. This contradicts the maximality of $\mathcal{C}$ among all $q$-ary codes of length $n$ having min. distance $d$.

So $|\mathcal{C}|\,|B_{d-1}(0)| \geq |A^n| = q^n$. $\square$

Recommended viewing:
YouTube videos on coding & info. theory (including alg. geom. codes) by Mary Wootters

Asymptotic version of GV-bound due to Shannon:

Fix $0 < \delta < 1$. $\quad |B_{\delta n}(0)| \approx |A^n|^{h_q(\delta)} = q^{n h_q(\delta)}$, $\quad 0 \le h_q(\delta) \le 1$.

$$\log_q |B_{\delta n}(0)| \approx n h_q(\delta)$$

This is a true asymptotic formula: for fixed $q$ and $\delta \in (0,1)$,

$$\frac{\log_q |B_{\delta n}(0)|}{n h_q(\delta)} \longrightarrow 1 \quad \text{as } n \to \infty.$$

$$\log_q |B_{\delta n}(0)| \sim n h_q(\delta).$$

More precisely,

$$n h_q(\delta) - o(1) \le \log_q |B_{\delta n}(0)| \le n h_q(\delta)$$

The $q$ary entropy function

binary entropy function $\quad h_2(q) = -\delta \log_2 \delta - (1-\delta)\log_q (1-\delta) = \delta \log_2 \frac{1}{\delta} + (1-\delta)\log_2 \frac{1}{1-\delta}$

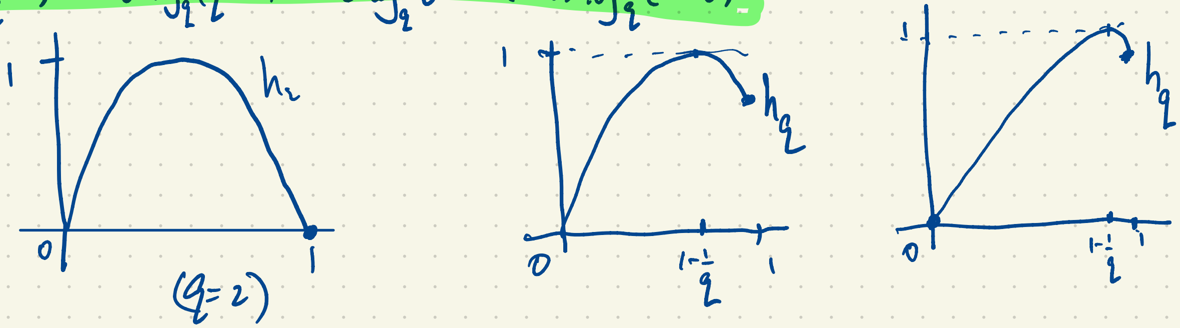
Eg. consider a random stream of information coming from letters in $A$, $|A| = q$, $A = \{x_1, \dots, x_q\}$

with letter $x_1$ having frequency $1-p$

$x_i$ .. .. $\frac{p}{q-1}$ $\quad (2 \le i \le q)$ $\quad$ & $(1-p) + \frac{p}{q-1} + \frac{p}{q-1} + \dots + \frac{p}{q-1} = 1$.

$H(\underset{\text{single char. from}}{\text{this stream}}) = \sum p \log \frac{1}{p} = -\sum p \log p = -(1-p)\log(1-p) - \cancel{(q-1)}\frac{p}{\cancel{q-1}} \cdot \log \frac{p}{q-1} = p\log(q-1) - p\log p - (1-p)\log(1-p)$

$$h_q(\delta) = \delta \log_q(q-1) - \delta \log_q \delta - (1-\delta)\log_q(1-\delta)$$




increasing $q$

Singleton bound:

$d \le n-k+1$

$\frac{d}{n} \le 1 - \frac{k}{n} + \cancel{\frac{1}{n}}$

$\frac{d}{n} + \frac{k}{n} \le 1$

$R + \delta \le 1$

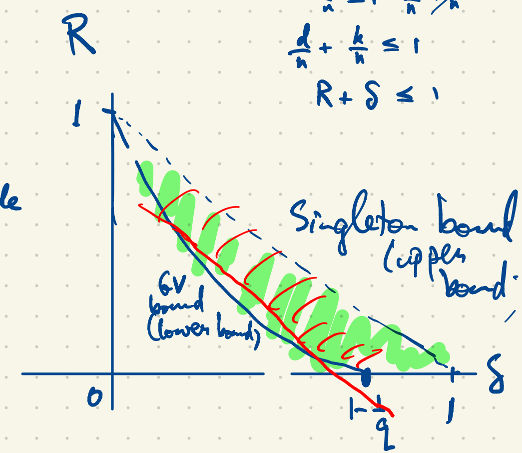
$h_q(x) = x\log_q(q-1) + \frac{\log 2}{\log q} h_2(x)$ Let $x \to 1^-$.

$h_q(x) \to \log_q(q-1)$ as $x \to 1^-$.

For long codes $(n \gg 0)$ over a fixed alphabet $|A| = q$, we consider the information rate $R = \frac{\log_q |C|}{n} = \frac{k}{n}$ in the case of an $[n,k]_q$-code

relative distance $\delta = \frac{d}{n}$

relative error-correcting capability $\frac{e}{n} \approx \frac{d}{2n} = \frac{\delta}{2}$




$R \ge 1 - h_q(\delta)$

For $q \ge 49$ we have a new (1982) lower bound for asymptotically good explicit codes using algebraic geometry (Tsfasman, Vladut, Zink)

The 1982 theorem literally says: There exists a family $X_i$ of algebraic curves over $\mathbb{F}_q$ $(i=1,2,3,\dots)$ such that $X_i$ has $n_i+1$ (rational) points over $\mathbb{F}_q$, genus $g_i$ with

$$\frac{g_i}{n_i} \to \frac{1}{\sqrt{q}-1} \quad \text{as } i\to\infty.$$

The Reed-Solomon codes come from the simplest curve of all, the projective line $\mathbb{P}^1 F = F \cup \{\infty\}$ ($F$: field) of genus 0.

$S^2$ $\quad$ $T^2 = S^1\times S^1$

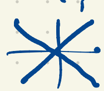
$g=0$ $\quad$ $g=1$ $\quad$ $g=2$ $\quad$ $g=3$

On a curve $X$, $\Omega_X = \{\text{smooth global differential 1-forms}\}$ is a vector space of dimension $\dim \Omega_X = g$.
The number of $\mathbb{F}_q$-points on the curve (if it's defined over $\mathbb{F}_q$), $N_q$, satisfies $|N_q - (q+1)| \le 2g\sqrt{q}$
Hasse-Weil bound.

Eg. $\mathbb{P}^1 F$ has $N = q+1$ points, $g=0$

For a ~~plane~~ irreducible plane curve of degree $d$ (defined by a poly. equation of degree $d$) has genus $g \le \binom{d-1}{2} = \frac{(d-1)(d-2)}{2}$
(equality for smooth curve; $g = \binom{d-1}{2} - \sum\limits_{\text{singular points}} (\ )$.

$y^2 = x^2 \iff y = \pm x$

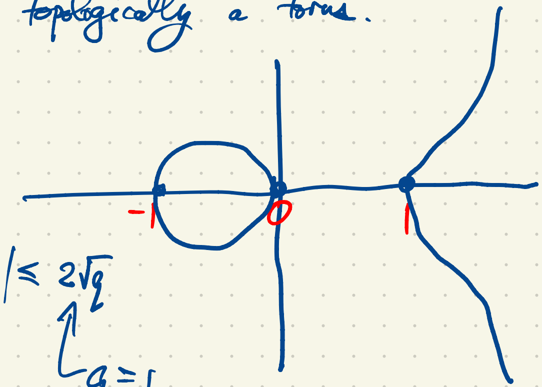


has $2q+1$ points

$y^2 - x^2 = (y+x)(y-x) = 0$

Irreducible conic:
$y = x^2$ $\quad$ $(t, t^2)$ $\quad$ $t\in F$ $\quad$ genus $g=0$
plus one point at infinity
$q+1$ points

Smooth curve of degree $d=3$ (elliptic curve) has genus $g = \binom{3-1}{2} = 1$ is topologically a torus.




eg. $y^2 =$ cubic in $x$ with no repeated roots is an elliptic curve.

$y^2 = x^3 - x = x(x+1)(x-1)$

$g=1$ (torus)

H·W bound: over $\mathbb{F}_q$ the number of points satisfies $|\underline{N} - (\underline{q}+1)| \le 2\sqrt{q}$

$g=1$

~~$q>3$~~ $q =$ prime $p \ge 5$

$N = q+1$ if $q =$ prime $p \equiv 3 \mod 4$

$q+1 \pm \varepsilon$ if $q =$ prime $p \equiv 1 \mod 4$

$|\varepsilon| \le 2\sqrt{q}$

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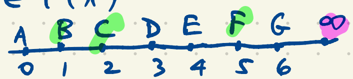
Projective line $\mathbb{P}^1 F = F \cup \{\infty\} = X$

We consider rational functions $f(x) \in F(X)$ defined on a curve $X$ (e.g. $X = \mathbb{P}^1 F$)

Eg. $F = \mathbb{F}_7 = \{0,1,2,3,4,5,6\}$




$= X$

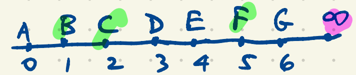
Formal integer-linear combinations of points $A, B, C, D, E, F, G, \infty$ on $X$ are called <u>divisors</u> as a book-keeping device for keeping track of zeroes and poles of functions on $X$.

eg. $f(x) = (x-1)(x-2)(x-5) = x^3 + 3x^2 - x - 3 = x^3 + 3x^2 + 6x + 4$ has simple zeroes at $B, C, F$ and a triple pole at $\infty$

Near $\infty$, $z = \frac{1}{x}$ ; $f(x) = f(\frac{1}{z}) = \frac{1}{z^3} + \frac{3}{z^2} + \frac{6}{z} + 4 = \frac{1 + 3z + 6z^2 + 4z^3}{z^3}$ so $f$ has a triple pole at $z=0$ (i.e. at $x = \infty$).

The divisor of $f(x)$ is $B + C + F - 3\infty =: \mathrm{Div}(f)$ (Sometimes abbreviated $(f)$).

More complicated: $f(x) = \dfrac{(x-1)^2(x+3)^4(x+5)}{(x+2)(x+1)^3}$ $\qquad$ $\mathrm{Div}(f) = 2B + 4E + C - F - 3G - 3\infty$



$= X$

$z = \frac{1}{x}$

$$f(x) = f(\tfrac{1}{z}) = \frac{(\frac{1}{z}-1)^2(\frac{1}{z}+3)^4(\frac{1}{z}+5)}{(\frac{1}{z}+2)(\frac{1}{z}+1)^3}\cdot\frac{z^7}{z^7} = \frac{(1-z)^2(1+3z)^4(1+5z)}{(1+2z)(1+z)^3\cdot z^3}$$

triple pole at $z=0$ (i.e. at $x=\infty$)

The degree of $D = \sum_i m_i P_i$ is $\deg D = \sum_i m_i$, $\quad (m_i \in \mathbb{Z})$

For any $f(x) \in F(X)$, $\deg(\mathrm{Div}\, f) = 0$. (equally many poles as zeroes)

Given a divisor $D = \sum_i m_i P_i - \sum_j n_j Q_j$ $\quad (m_i, n_j \geq 1)$,

we consider the vector space $\mathcal{L}(D) = \{ f(x) \in F(X) :$ $f$ has a zero of multiplicity at least $m_i$ at $P_i$,
$f$ has a pole of order at most $n_j$ at $Q_j$,
and possibly other zeroes but no other poles$\}$.

In the case of $\mathbb{P}^1 F = F \cup \{\infty\}$ consider $D = k\infty$, $\deg D = k$.

$\mathcal{L}(k\infty) = \{ f(x) \in F(x) : f$ has a pole of order at most $k$ at $\infty$; no other poles$\}$
(there can be as many zeroes as you like)

$\dim \mathcal{L}(k\infty) = k+1$. $\qquad \mathcal{L}(D) = \{ f : \mathrm{Div} f + D \geq 0 \}$

$\mathcal{L}(-k\infty) = \{$polynomials in $x$ of degree at most $k\}$ $:= \{ a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k : a_i \in F \}$.

has basis $\{1, x, x^2, \cdots, x^k\}$.

The Riemann-Roch theorem gives a relation for determining $\ell(D) = \dim \mathcal{L}(D)$.

$$\underbrace{\ell(D)}_{\text{non-negative}} - \underbrace{\ell(K-D)}_{\text{integers}} = \deg D - g + 1$$

where $K$ is a "canonical divisor"

$\ell(D) \geq \deg D - g + 1$ is Riemann's bound

The genus of a smooth curve $X$ is the dimension $g = \dim \Omega_X$ where $\Omega_X$ is the vector space of (globally) smooth differential 1-forms on $X$.

eg. $X$ = projective line $\mathbb{P}^1 F$ over $F$, $\mathbb{P}^1 F = F \cup \{\infty\}$ $\qquad y = \frac{1}{x}$, $x = \frac{1}{y}$ $\qquad \frac{d(\frac{1}{y})}{dy} = -\frac{1}{y^2}$

A 1-form has the (shape) form $\omega = f(x)\,dx = f(\frac{1}{y})\,d(\frac{1}{y}) = -\frac{f(\frac{1}{y})\,dy}{y^2}$. $\qquad d(\frac{1}{y}) = -\frac{dy}{y^2}$

On $\mathbb{P}^1 F$ there is no (nonzero) global 1-form

If $f(x)$ is a poly of degree $k$ in $x$ then it has a pole of order $k$ at $\infty$ (and $k$ zeroes in $F$).

So $\omega = f(x)\,dx$ has a pole of order $k+2$ at $\infty$.

$\frac{1}{x^2}dx$ has no pole at $\infty$ but it has a double pole at the origin.

$= -y^2 \frac{dy}{y^2} = -dy$

$\operatorname{Div} \omega = \sum(\text{zeroes of } \omega) - \sum(\text{poles of } \omega)$ the divisor of $\omega$

For $\omega$ a 1-form on $\mathbb{P}^1 F$, $\deg(\omega) = -2$ (2 more poles than divisors, counting multiplicity).

$\Omega_{\mathbb{P}^1 F} = \{0\}$, $g = \dim \Omega_{\mathbb{P}^1 F} = 0$.

---

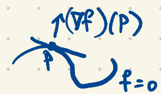
For an elliptic curve e.g. the curve $X: y^2 = x^3 - x$ has $\Omega_X = \operatorname{span}\{\omega\}$ ($\omega$ is a global smooth 1-form)

$g = \dim \Omega_X = 1$.

First, why is $X$ a smooth cubic curve?

$X: f = y^2 - x^3 + x = 0$

$\nabla f = (1 - 3x^2, 2y)$




$\begin{cases} 1 - 3x^2 = 0 \\ 2y = 0 \\ y^2 - x^3 + x = 0 \end{cases}$ has no solutions.


$y^2 = x^3$ is singular at $(0,0)$

$f(x,y) = y^2 - x^3$

$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (-3x^2, 2y)$

$y^2 = x^3 - x$ points $(x, y, 1)$ of a cubic curve with $z \neq 0$.

$y^2 z = x^3 - xz^2$ points $(x, y, z)$ in homogeneous coords

If $z = 0$: $x = 0$, $y = 1$.

Near this point, $y \neq 0$, divide by $y$ to get $z = x^3$.

$$(x, y, z) \mapsto \left(\tfrac{x}{y}, 1, \tfrac{z}{y}\right) \quad \tfrac{x}{y} = x', \ \tfrac{z}{y} = z'$$

$$\frac{y^2 z}{y^3} = \frac{x^3 - xz^2}{y^3}$$

$$z' = (x')^3 - x'(z')^2$$

$$y = x^3 - xy^2$$

$(0,0)$ is a smooth point.

$$f = y^2 - x^3 + x = 0$$

$$y^2 = x^3 - x$$

$$dy^2 = d(x^3 - x)$$

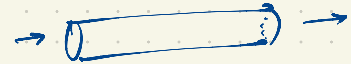
$$2y\,dy = (3x^2 - 1)\,dx$$

$$\omega = \frac{dy}{3x^2 - 1} = \frac{dx}{2y}$$

$$\frac{z\,dy}{3x^2 - z^2} = \frac{dx}{2y}$$

This equation is preserved under scalar multiples $(x, y, z) \mapsto (\lambda x, \lambda y, \lambda z)$ $(\lambda \neq 0)$

# Shannon's Theorem for noisy channels

Imagine a pipe in which we can send 1 liter of water per second. 

Now for the same pipe imagine that a certain amount of sludge/silt/gravel is carried along at a rate of $\varepsilon$ liters per second. This means that only $1-\varepsilon$ liters of water per second can be transmitted by this same channel/pipe.

Amazingly, the same simplistic reasoning applies to send information reliably.

Suppose we transmit information using strings of symbols from an alphabet $A$, $|A| = q$.
If there were no noise, we could reliably send 1 character (letter in $A$) per unit time.

If instead error is introduced to the channels having entropy rate $h_q(\varepsilon)$ (characters per unit time) $0 \le h_q(\varepsilon) \le 1$
then the rate at which useful information can be reliably transmitted in this channel is asymptotically $1 - h_q(\varepsilon)$ characters per unit time.

$$|B_d| = \sum_{k=0}^{d} \binom{n}{k} (q-1)^k, \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \quad \text{as } n \to \infty$$

Stirling's formula

Back to Shannon's first theorem: optimal compression of information for noiseless channel

Source of information is a random variable $X = \begin{cases} x_1 & \text{with prob. } p_1 \\ x_2 & \cdots\cdots\ p_2 \\ \vdots & \\ x_k & \cdots\cdots\ p_k \end{cases}$ $0 \le p_i \le 1$, $\sum p_i = 1$

We ask for the optimal compression of info. from this source using strings over alphabet $A$, $|A| = q$

A code for this source is a map $X \to A^*$

$$\mathcal{C}: \begin{array}{l} x_1 \longmapsto w_1 \in A^{l_1} \\ x_2 \longmapsto w_2 \in A^{l_2} \\ \vdots \end{array}$$

i.e. $w_i$ is a word in $A^*$ of length $l_i$.

The expected length of $\mathcal{C}(X)$ is $\sum_{i=1}^{k} p_i l_i$

<u>Theorem</u> $\sum p_i l_i \geq H_q(X) := \sum_i p_i \log_q \frac{1}{p_i} = -\sum_i p_i \log_q p_i$. Moreover, we can asymptotically achieve compression having $\sum_i p_i l_i$ as close as desired to $H_q(X)$.

$\mathcal{C}$ must be an injective map (the code is uniquely decodable). We will discuss the proof under the stronger assumption that $\mathcal{C}$ is prefix-free: none of the codewords $w_1, \dots, w_k$ is a prefix (initial substring) of any of the other codewords.

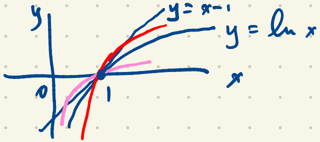
<u>Lemma</u> (Kraft's inequality) $\sum_{i=1}^{k} \frac{1}{q^{l_i}} \le 1$.

<u>Proof</u> Elements in $[0,1]$ (real interval) can be written in base $q$ as infinite strings over $A$ as

$0.a_1 a_2 a_3 a_4 \cdots$, $a_j \in A$

Each $w_i \in \mathcal{C}(X)$ determines a subinterval of $[0,1]$ given by all real numbers whose first $l_i$ "digits" agree with $w_i$ i.e. $r \in [0,1]$ s.t. $r \restriction l_i = w_i$. These real numbers form a subinterval of width $\frac{1}{q^{l_i}}$.

These subintervals are disjoint. $\square$

Proof 


$\ln x \le x-1$ for all $x > 0$.

$\log_q x = \dfrac{\ln x}{\ln q}$ because $x = q^y$, $y = \log_q x$

$\ln x = y \ln q \Rightarrow y = \log_q x = \dfrac{\ln x}{\ln q}$

$$\ln \frac{1}{p_i q^{l_i}} \le \frac{1}{p_i q^{l_i}} - 1$$

$$p_i \ln \frac{1}{p_i q^{l_i}} \le \frac{1}{q^{l_i}} - p_i \qquad (1 \le i \le k)$$

$$\sum_{i=1}^{k} p_i \ln \frac{1}{p_i q^{l_i}} \le \underbrace{\sum_{i=1}^{k} \frac{1}{q^{l_i}}}_{\le 1} - \underbrace{\sum_{i=1}^{k} p_i}_{=1} \le 0$$

, divide both sides by $\ln q > 0$

$$\sum_{i=1}^{k} p_i \log_q \frac{1}{p_i q^{l_i}} \le 0$$

$$\sum_{i=1}^{k} p_i \left( \log_q \frac{1}{p_i} + \log_q \frac{1}{q^{l_i}} \right) \le 0$$

$\log_q \frac{1}{q^{l_i}} = -l_i$

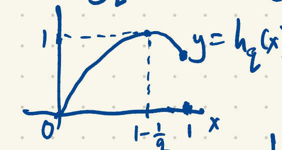
$$\underbrace{\sum_{i=1}^{k} p_i \log_q \frac{1}{p_i}}_{H(X)} \le \underbrace{\sum_{i=1}^{k} p_i l_i}_{E(\text{length of a codeword})} \qquad \square$$

Outline of ~~of~~ Shannon's second $\overset{\text{coding}}{\cancel{\text{theorem}}}$ ( $\overset{\text{source}}{\text{coding}}$ for noisy channels )

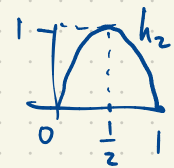
If $0 \le \varepsilon \le 1$ then $|B_{\varepsilon n}(v_0)| \approx q^{n h_q(\varepsilon)}$

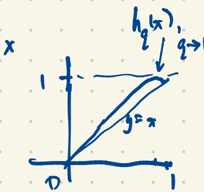
$$\{ v \in A^n : d(v, v_0) \le \varepsilon n \}$$

$$h_q(x) = x \log_q(q-1) - x \log_q x - (1-x)\log_q(1-x)$$




$$\frac{1}{n} \log_q |B_{\varepsilon n}(v_0)| \sim h_q(\varepsilon) \quad \text{as } n \to \infty \quad (q \text{ fixed})$$




Hamming Bound: ~~for~~ $\mathcal{C} \subseteq A^n$ e-error correcting

$$|\mathcal{C}||B_e| \le |A^n| = q^n.$$

$$\log_q |\mathcal{C}| + \log_q |B_e| \le n$$

$$R = \underbrace{\frac{1}{n} \log_q |\mathcal{C}|}_{\text{info. rate}} + \frac{1}{n} \log_q |B_e| \le 1 \implies R \le 1 - \frac{1}{n} \log_q |B_e|$$

As $n \to \infty$, $q$ fixed

$$R \le 1 - h_q(\varepsilon)$$

relative error

$e = \varepsilon n$ ($0 < \varepsilon < 1$ fixed)

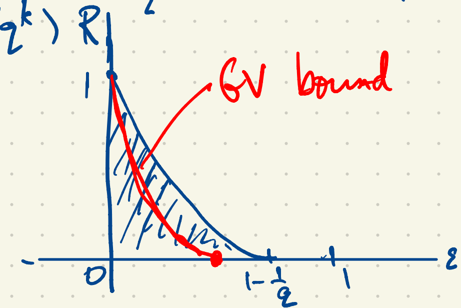
$d \approx 2e$

$d = 2e$ or $2e+1$

$e = \lfloor \frac{d-1}{2} \rfloor \approx \frac{d}{2}$

$d = \delta n$ relative distance

If $\mathcal{C}$ is linear, $\dim \mathcal{C} = k \le n$, info. rate $\frac{k}{n} = \frac{1}{n} \cdot \log_q(q^k)$



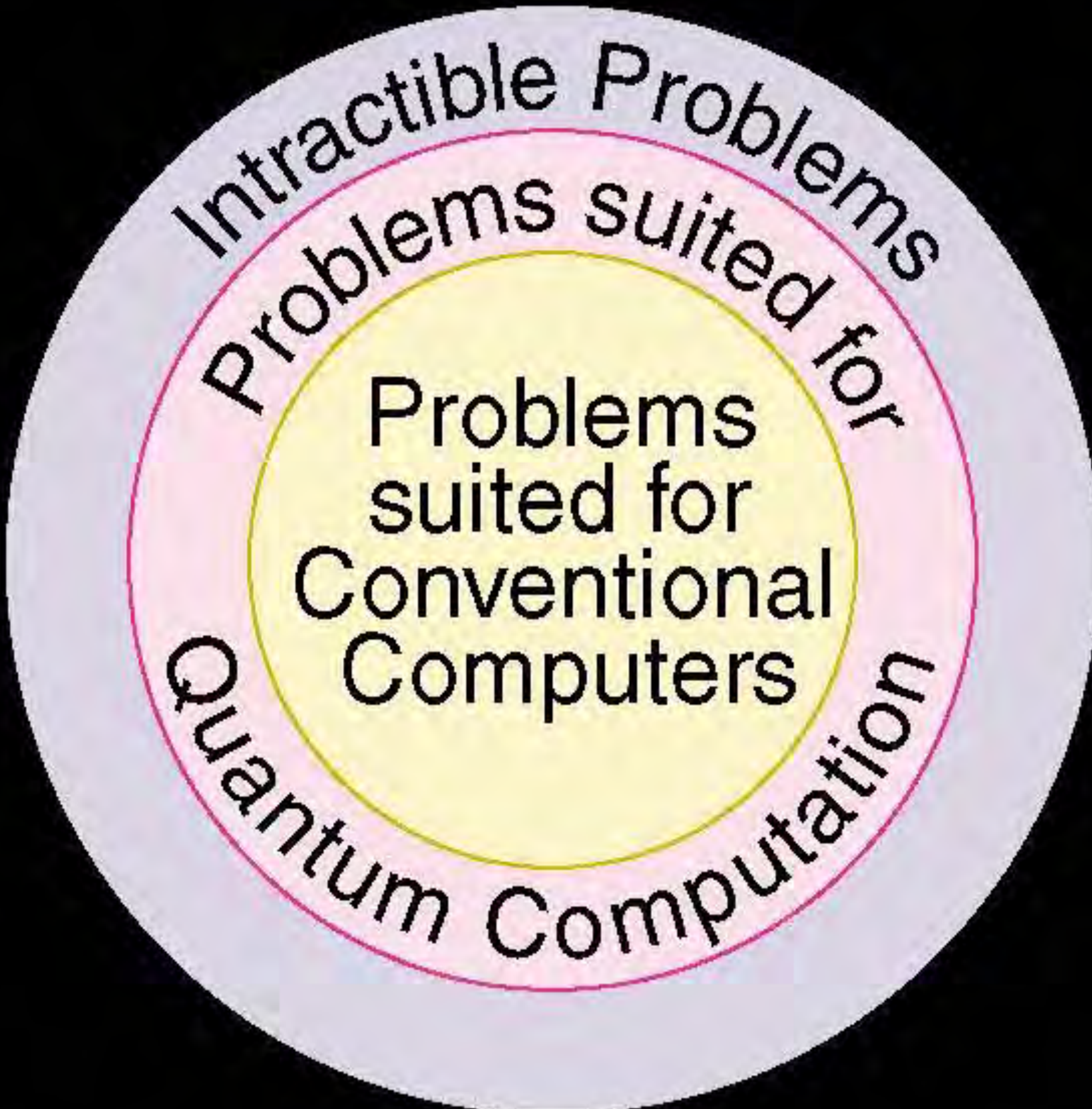
Quantum Information

- quantum mechanics (the physical foundations)
- quantum entanglement, tensors
- quantum teleportation
- quantum cryptography
- quantum computation — quantum circuits / quantum Turing machines
- quantum error correction

# Quantum Information and Computation

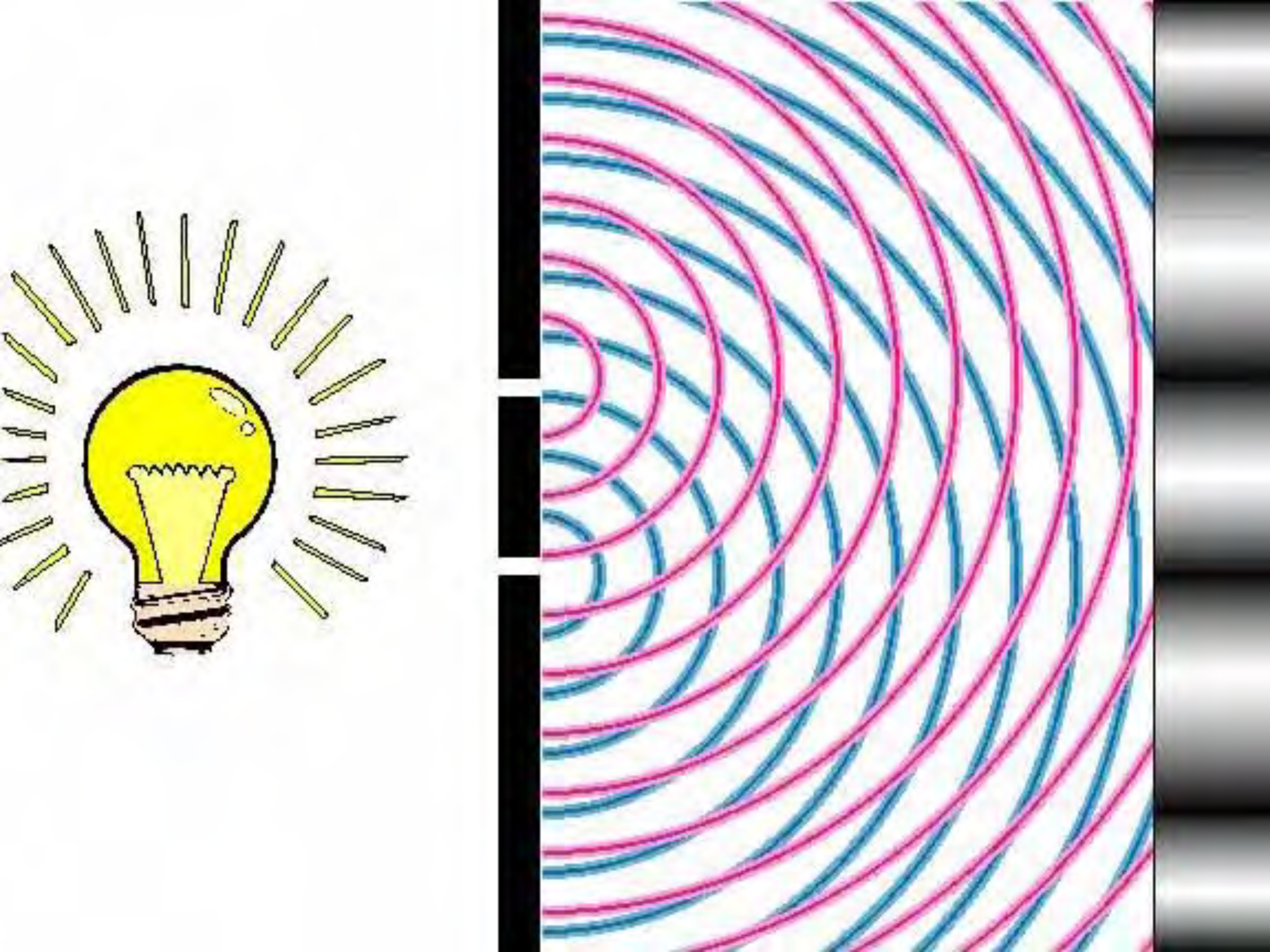
Eric Moorhouse

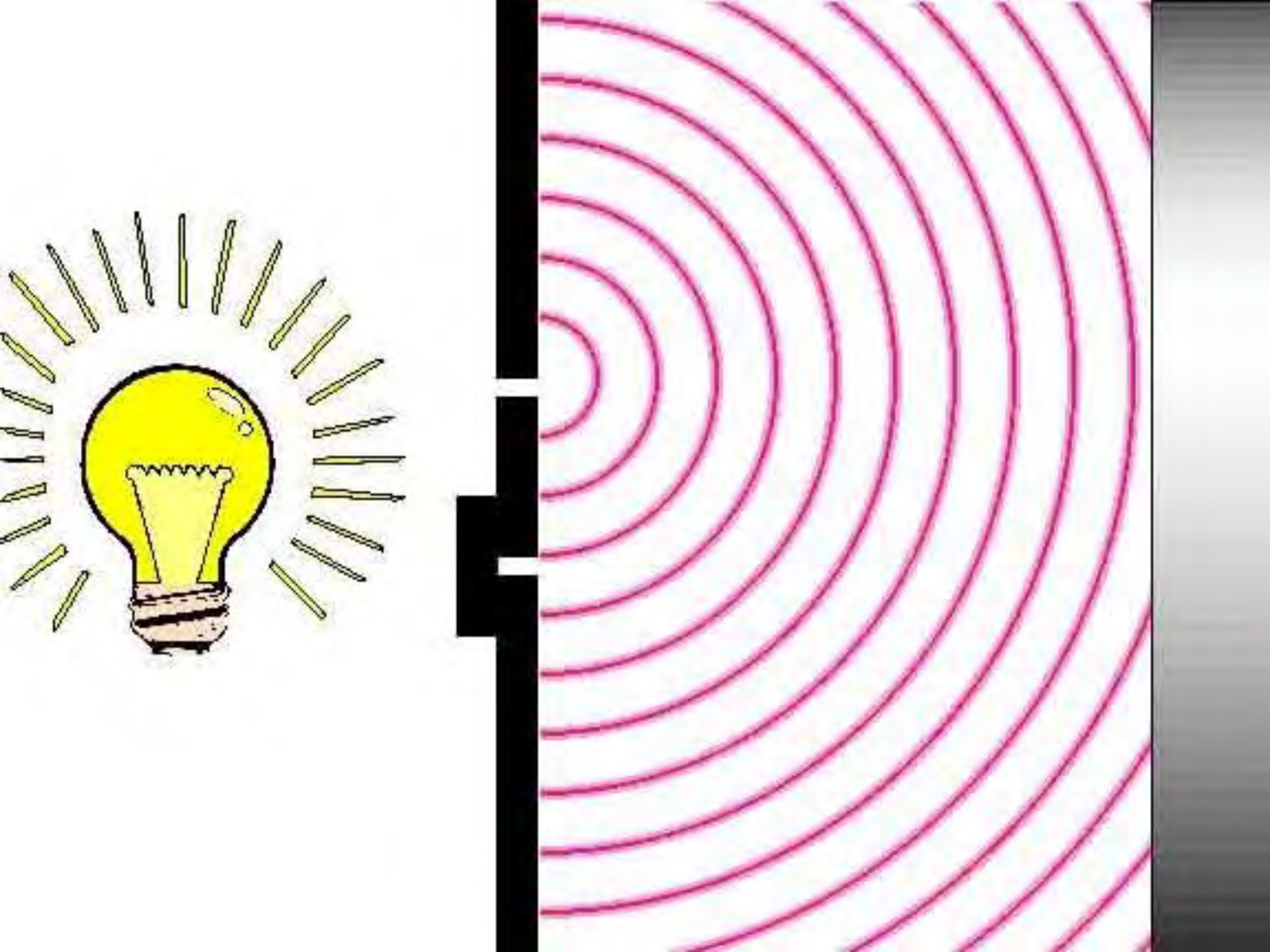
Department of Mathematics
University of Wyoming

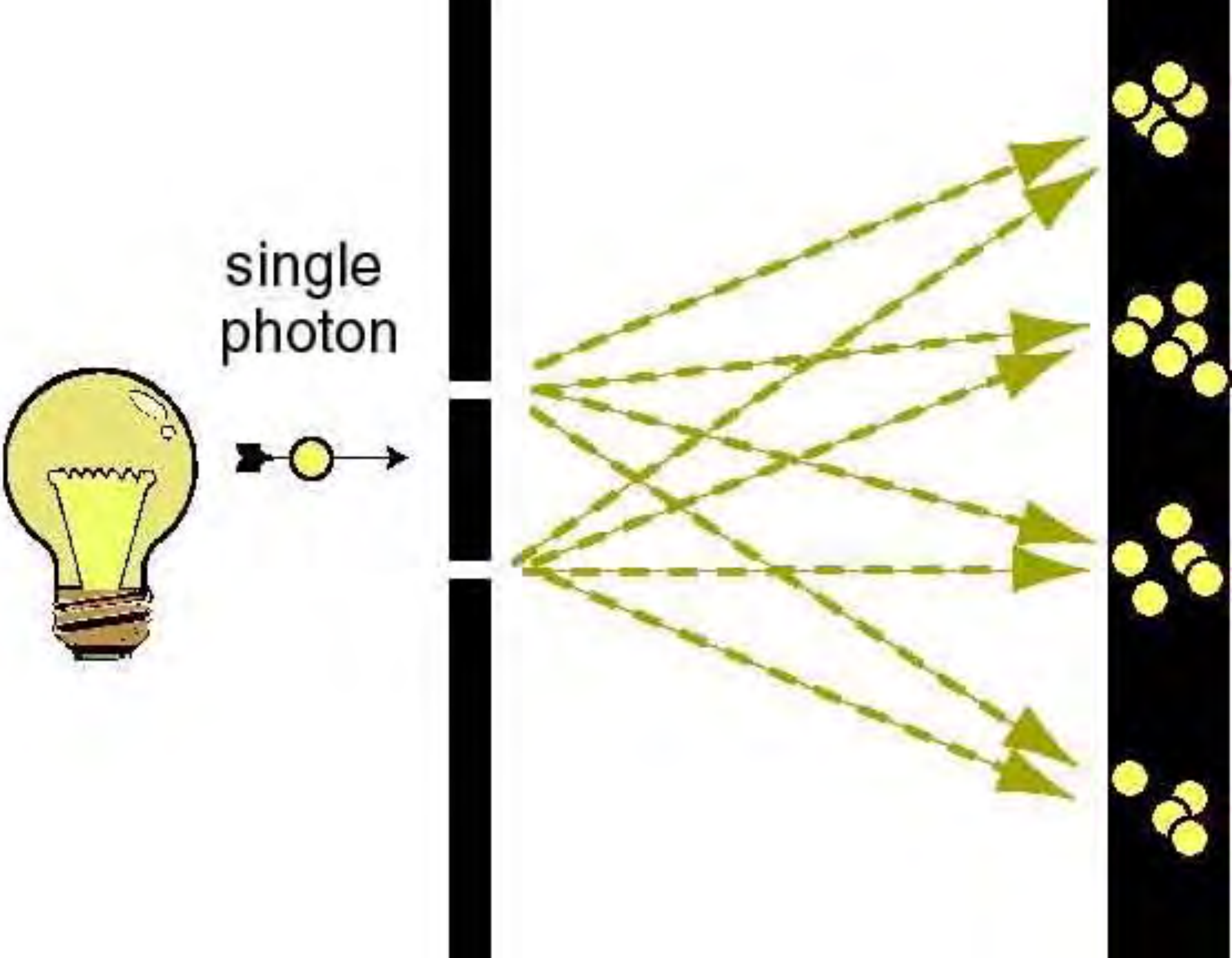

Intractible Problems
Problems suited for Quantum Computation
Problems suited for Conventional Computers

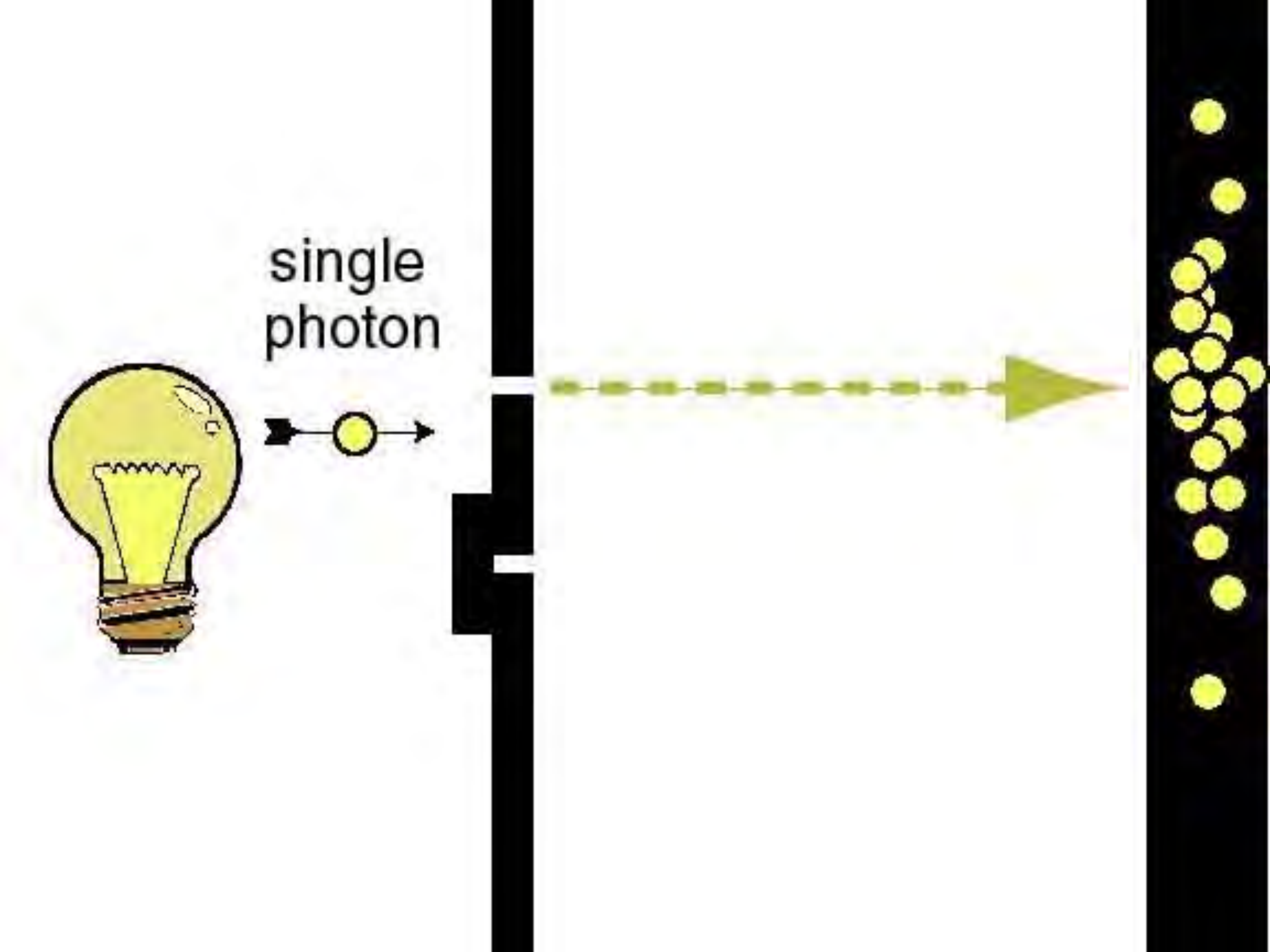
# Peter Shor

# A Crash Course in Quantum Mechanics






single
photon


single
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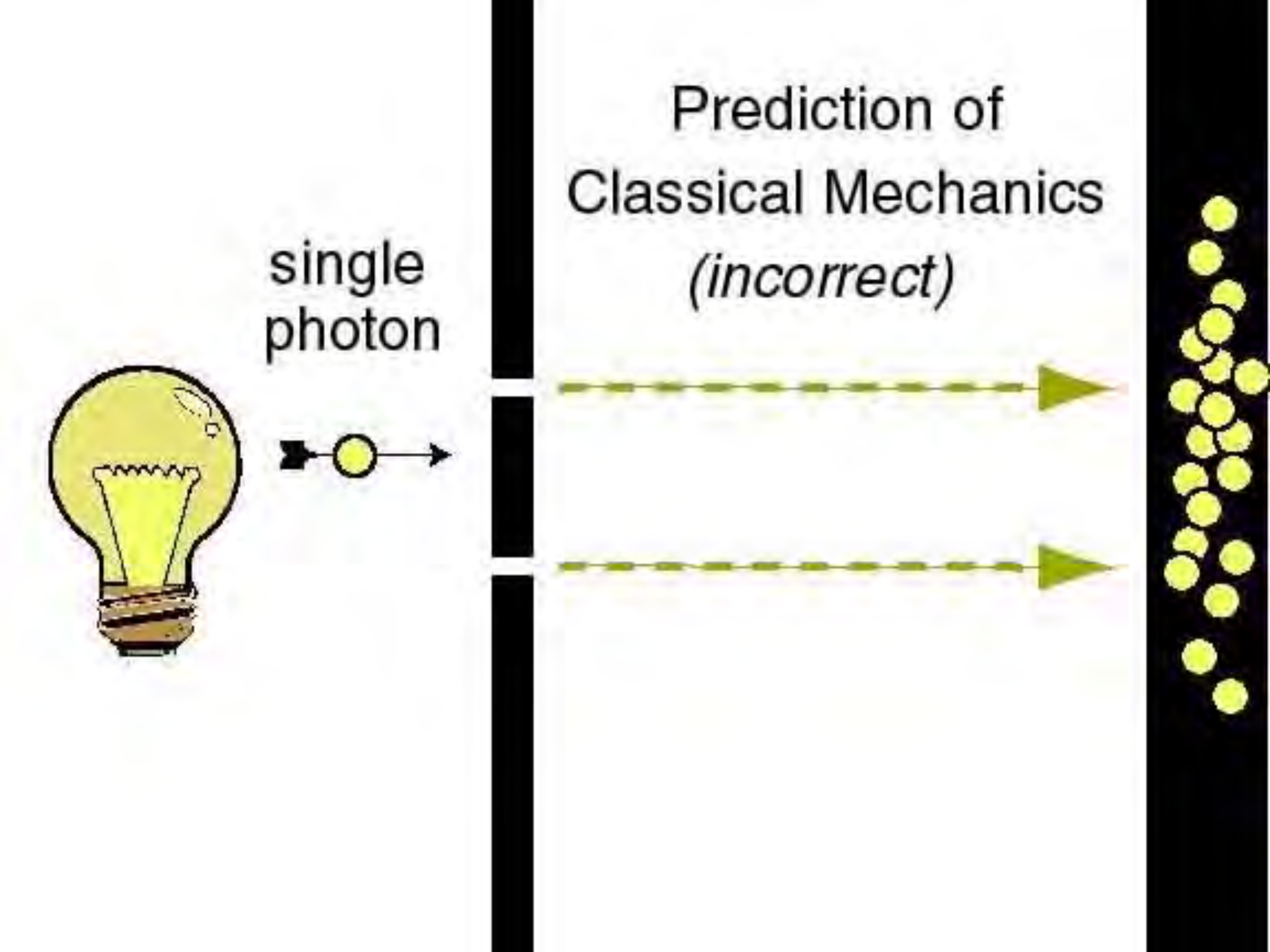
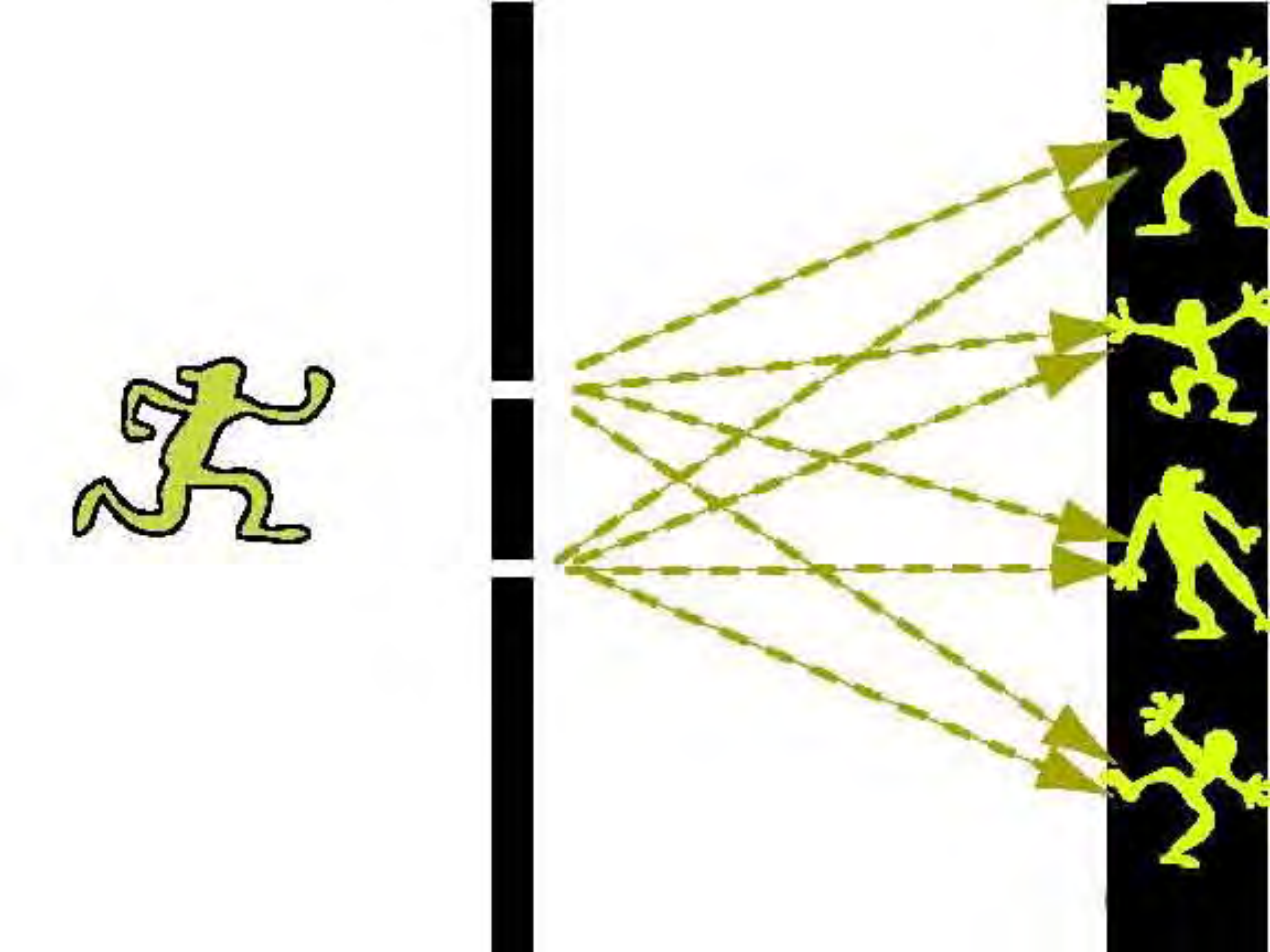
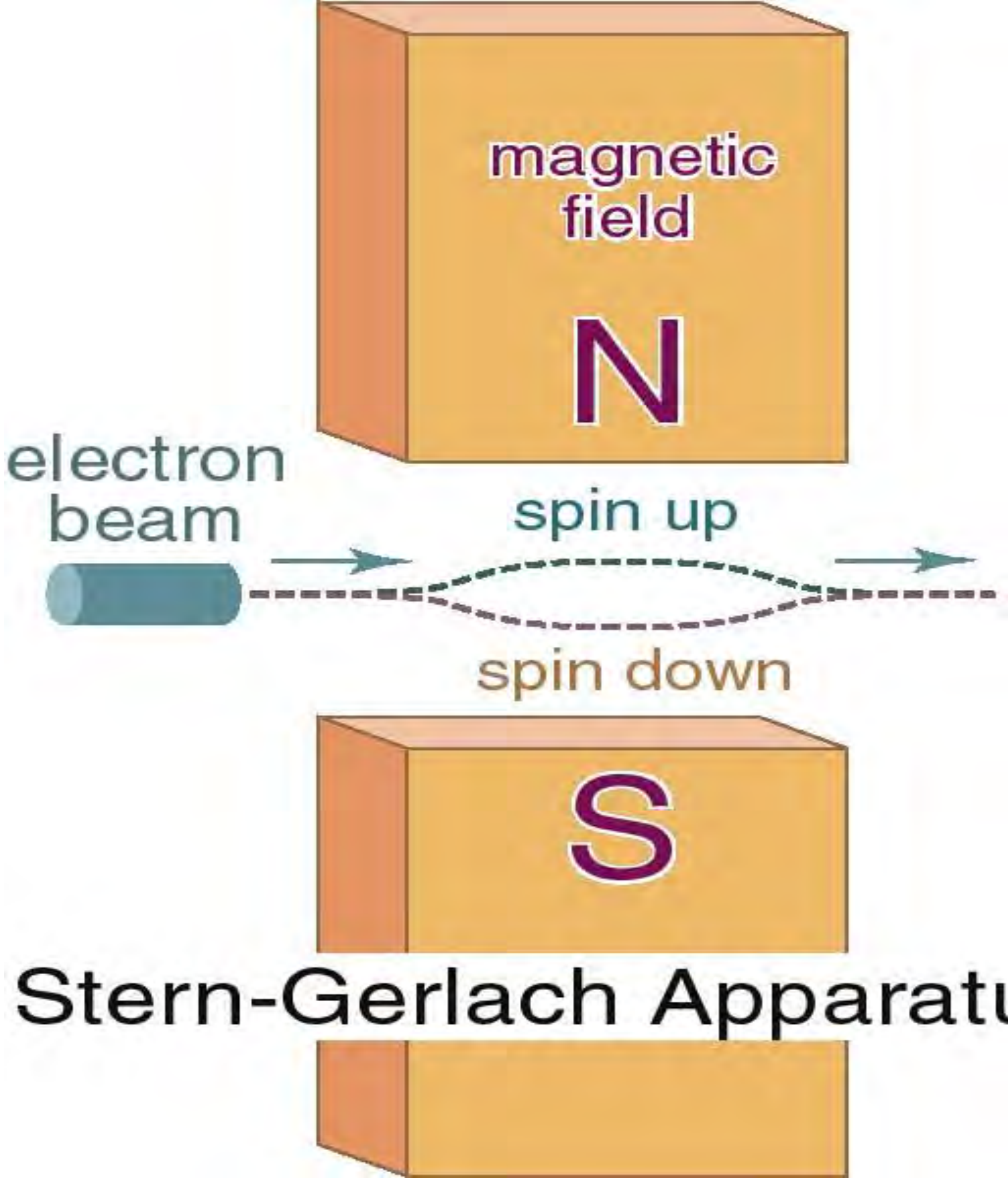

single
photon
Prediction of
Classical Mechanics
(incorrect)






Stern-Gerlach Apparatus