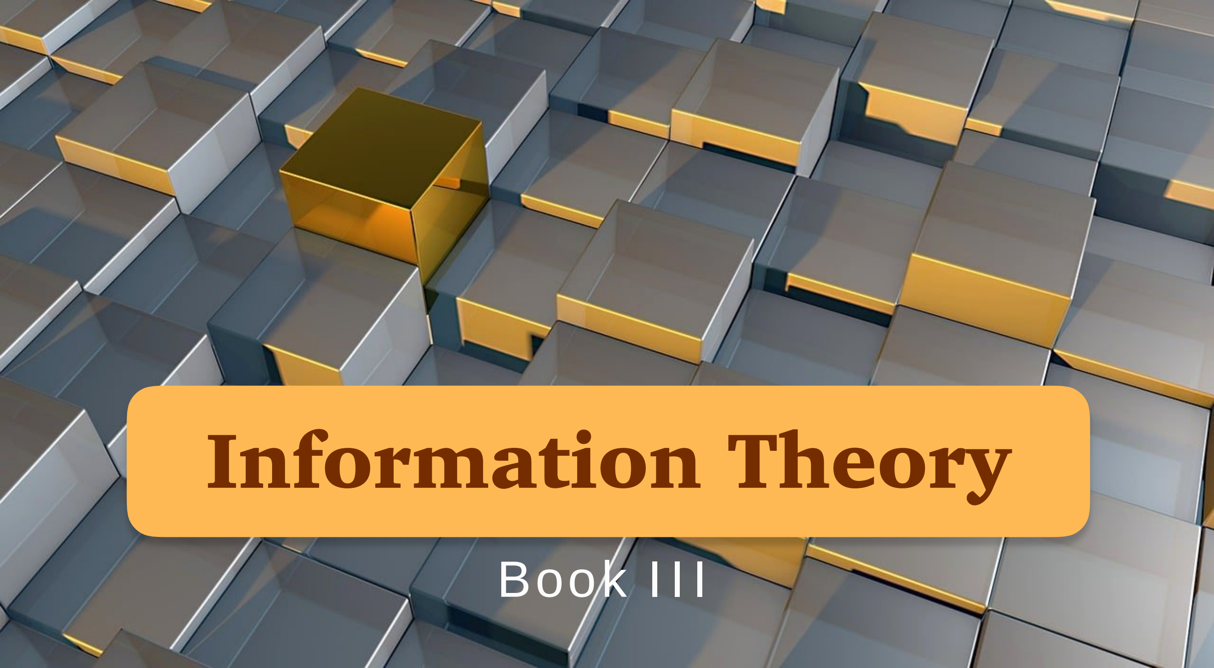


Information Theory

Book III

Spin state of an electron (disregard position and momentum) is an example of a qubit, which is a vector $|\psi\rangle \in \mathbb{C}^2 = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$.

ket notation

Standard basis of $\mathbb{C}^2$: $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
"spin up" "spin down"

$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha|+\rangle + \beta|-\rangle$, $|\alpha|^2 + |\beta|^2 = 1$.

An electron in this spin state is in a superposition of spin up and spin down states

A measurement of an electron in this spin state yields a single bit of classical information:

- spin up, with probability $|\alpha|^2$;
- spin down, with probability $|\beta|^2$.

This says what happens when we measure with respect to the z-axis. (For measurement in a different direction/axis, we'll say later.)

As soon as the measurement is taken, the spin state collapses; all knowledge of α, β is then lost.

A linear functional on $\mathbb{C}^2$ is a linear transformation

$\langle\phi| : \mathbb{C}^2 \to \mathbb{C}$

bra notation

$\langle\phi| = (\gamma\ \delta) : \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longmapsto (\gamma\ \delta)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \underset{\langle\phi|\psi\rangle}{\underset{\|}{\gamma\alpha + \delta\beta}} \in \mathbb{C}$.

Dual basis:

$\langle +| = |+\rangle^* = (1\ 0)$ (conjugate transpose)

$\langle -| = |-\rangle^* = (0\ 1)$

$|\psi\rangle^* = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^* = (\bar\alpha\ \bar\beta) = \bar\alpha\langle +| + \bar\beta\langle -|$

$\langle +|\psi\rangle = \langle +|\left(\alpha|+\rangle + \beta|-\rangle\right) = \alpha$

$\langle -|\psi\rangle = \beta$.

Spin states are unit vectors in $\mathbb{C}^2$ i.e. $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$.

i.e. in $\mathbb{R}^4$

$\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$

so $|\psi\rangle \in S^3$ = unit sphere in $\mathbb{R}^4$.

$\left.\begin{array}{l} \alpha = \alpha_1 + \alpha_2 i \\ \beta = \beta_1 + \beta_2 i \end{array}\right\} \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$

Any time we measure a spin state $|\psi\rangle \in S^3$, it collapses.
But it is possible to perform certain reversible operations $|\psi\rangle \longmapsto A|\psi\rangle$ where A is a 2×2 unitary matrix $\left(AA^* = A^*A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$ over $\mathbb{C}$.

Special examples of unitary matrices are scalar matrices $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$

These perform an operation on $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ whose only effect is to alter the phase of α, β

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longmapsto A|\psi\rangle = \begin{pmatrix} \lambda\alpha \\ \lambda\beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$\lambda = e^{i\theta}$ $(\theta \in [0, 2\pi))$

which has no physical significance. For this reason the so-called density matrix

$$\underbrace{\underbrace{|\psi\rangle}_{2\times 1} \underbrace{\langle\psi|}_{1\times 2}}_{2\times 2} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (\bar{\alpha} \ \ \bar{\beta}) = \begin{pmatrix} \alpha\bar{\alpha} & \alpha\bar{\beta} \\ \beta\bar{\alpha} & \beta\bar{\beta} \end{pmatrix}$$

which holds all the physically significant information of the single qubit.

Hermitian 2×2 matrix

$H \in \mathbb{C}^{2\times 2}$ (2×2 complex matrix)

satisfying $H^* = H$

The map $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longmapsto \begin{pmatrix} \lambda\alpha \\ \lambda\beta \end{pmatrix}$ does not change this density matrix.

Entanglement typically occurs when we include multiple electrons in our system.

Start by reviewing statistical dependence works:

Let's say we take a random individual A from a population.

Imagine the population is 40% male, 60% female; 30% short, 70% tall.

Sampling by selecting one person gives two bits: MS, MT, FS, or FT.

Combinations of attributes: 12%, 28%, 18%, 42% if gender is independent of height.

In this example, gender and height are independent.

Gender / Height

	S	T	
M	0.12	0.28	0.4
F	0.18	0.42	0.6
	0.3	0.7	1

$$\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.12 & 0.28 \\ 0.18 & 0.42 \end{bmatrix}$$

Outer product of two vectors. is a rank 1.

More typical distribution

	S	T	
M	0.1	0.3	0.4
F	0.2	0.4	0.6
	0.3	0.7	1

The matrix $\begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{bmatrix}$ has rank 2.

In this second example gender and height are (statistically) dependent.

If one electron has (spin) state $|\psi_1\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathbb{C}^2$ and a second electron has spin state $|\psi_2\rangle = \begin{pmatrix}\gamma\\ \delta\end{pmatrix} \in \mathbb{C}^2$

$|\alpha|^2+|\beta|^2=1$ $\qquad$ $|\gamma|^2+|\delta|^2=1$

the pair of electrons has state $|\psi_{12}\rangle = \alpha_{11}|{++}\rangle + \alpha_{12}|{+-}\rangle + \alpha_{21}|{-+}\rangle + \alpha_{22}|{--}\rangle \in \mathbb{C}^4$

$\alpha_{ij} \in \mathbb{C}$, $\quad |\alpha_{11}|^2 + |\alpha_{12}|^2 + |\alpha_{21}|^2 + |\alpha_{22}|^2 = 1.$

prob. of both electrons having spin up

If the two electrons are <u>not</u> entangled then

$$\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\begin{pmatrix}\gamma & \delta\end{pmatrix} \qquad \text{rank } 1.$$

If the matrix has rank 2 then the two electrons are entangled.

Eg. $|\psi\rangle = \frac{1}{\sqrt2}(|{++}\rangle + |{--}\rangle)$ i.e. $\begin{pmatrix}\frac{1}{\sqrt2} & 0\\ 0 & \frac{1}{\sqrt2}\end{pmatrix}$

$|\psi'\rangle = \frac{1}{\sqrt2}(|{+-}\rangle + |{-+}\rangle)$ i.e. $\begin{pmatrix}0 & \frac{1}{\sqrt2}\\ \frac{1}{\sqrt2} & 0\end{pmatrix}$

} Examples of EPR pairs

One way to talk about the spin state of a set of n electrons is

$$|\psi\rangle = \sum_{\substack{i_1\in\{0,1\}\\ i_2\in\{0,1\}\\ \vdots\\ i_n\in\{0,1\}}} \alpha_{i_1 i_2 i_3\cdots i_n} \underbrace{|\pm\pm\pm\cdots\pm\rangle}_{\text{all } 2^n \text{ combinations of } \pm} \in \mathbb{C}^{2^n} \qquad \sum |\alpha_{i_1 i_2\cdots i_n}|^2 = 1$$

$(\alpha_{i_1 i_2 \cdots i_n} : i_1, i_2, \dots, i_n \in \{0,1\})$ is a $\underbrace{2\times2\times2\times\cdots\times2}_{n}$ array or tensor

$\mathbb{C}^{2^n} = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}}$ tensor product. Take basis $|+\rangle = \binom{1}{0}$, $|-\rangle = \binom{0}{1}$

has basis $|++\cdots ++\rangle = |+\rangle \otimes |+\rangle \otimes \cdots \otimes |+\rangle$

$|++\cdots +-\rangle = |+\rangle \otimes |+\rangle \otimes \cdots \otimes |-\rangle$

$\vdots$

$|--\cdots -\rangle = |-\rangle \otimes |-\rangle \otimes \cdots \otimes |-\rangle$.

More generally if $v_i \in \mathbb{C}^2$ $(i = 1, 2, \cdots, n)$

then $v_1 \otimes v_2 \otimes \cdots \otimes v_n \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. (pure tensors)
simple

$\mathbb{C}^2 \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^2$ $\qquad$ $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 = \mathbb{C}^{2^n}$

$(v_1, \cdots, v_n) \longmapsto v_1 \otimes v_2 \otimes \cdots \otimes v_n$ this map is multilinear
i.e. linear in each argument separately.

Bell's Theorem
Gleason's Theorem
Kochen-Specker Theorem

In $\mathbb{C}^m \otimes \mathbb{C}^n \cong \mathbb{C}^{mn}$ every vector is a sum of at most $\min\{m,n\}$ pure tensors.

The corresponding result for $\mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2} \otimes \cdots \otimes \mathbb{C}^{m_k}$ is not known and extremely hard.

(In Algebraic Geometry look up Higher Secant varieties of Segre Varieties)

Recall: the spin state of a single electron is a qubit $|\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathbb{C}^2$, $|\alpha|^2 + |\beta|^2 = 1$.

Standard basis $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

spin up/down with respect to the z-axis

How do we measure the spin in an arbitrary direction?

In the vertical direction we make use of basis $|+_z\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|-_z\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ basis of eigenvectors for the Pauli spin operator $\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.

$$\sigma_z |+_z\rangle = |+_z\rangle$$

$$\sigma_z |-_z\rangle = -|-_z\rangle$$

Any electron with spin state $|\psi\rangle = \alpha|+_z\rangle + \beta|-_z\rangle$ can be measured in the vertical direction

$$\sigma_z|\psi\rangle = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha\\ -\beta\end{pmatrix}$$

Follow this by a linear functional eg. $|+_z\rangle^* = \langle +_z|$

$\langle +_z|\sigma_z|\psi\rangle = (1\ 0)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = (1\ 0)\begin{pmatrix}\alpha\\ -\beta\end{pmatrix} = \alpha$, the amplitude for the electron to be spin up.

Once the measurement is performed, the state collapses into that spin state $|\psi\rangle \longmapsto |+_z\rangle$.

$$\langle +_z|\underbrace{\sigma_z|+_z\rangle}_{|+_z\rangle} = 1.$$

If we measure $|\psi\rangle \longmapsto \langle +_z|\sigma_z|\psi\rangle$ and find spin down, the state collapses to spin down $|-_z\rangle$

$\langle +_z|\sigma_z|\psi\rangle = -\beta$, $|-\beta|^2 = |\beta|^2$

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Eigenvectors: $|+_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $|-_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $|+_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$, $|-_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $|+_z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|-_z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ eigenvalues $+1, -1$

Hermitian: $\sigma^* = \sigma$

eg. $\sigma_y \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} = |+_y\rangle$

$\sigma_y \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ i \end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = -|-_y\rangle$

If we measure an electron having spin $|+_y\rangle$ (in the pos. y-direction) with respect to the x-axis

$\langle +_x | +_y \rangle = \frac{1}{\sqrt{2}}(1 \;\; 1)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+i \\ 1+i \end{pmatrix}$ $\left|\frac{1+i}{2}\right|^2 = \frac{2}{4} = \frac{1}{2}$ $|a+bi|^2 = a^2+b^2$

Density matrix of $|\psi\rangle$ is $|\psi\rangle\langle\psi| = |\psi\rangle|\psi\rangle^* = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}(\bar{\alpha} \;\; \bar{\beta}) = \begin{pmatrix} \alpha\bar{\alpha} & \alpha\bar{\beta} \\ \bar{\alpha}\beta & \beta\bar{\beta} \end{pmatrix}$ $|\alpha|^2 + |\beta|^2 = 1$

is Hermitian having eigenvalues $1, 0$; corresponding eigenvectors $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $|\psi'\rangle = \begin{pmatrix} \bar{\beta} \\ -\bar{\alpha} \end{pmatrix}$

$|\psi\rangle\langle\psi|\psi\rangle = |\psi\rangle$ since $\langle\psi|\psi\rangle = 1$ $\langle\psi'|\psi\rangle = (\beta \; -\alpha)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha\beta - \alpha\beta = 0.$

$|\psi\rangle\langle\psi|\psi'\rangle = 0. = 0|\psi'\rangle.$

$\langle\psi|\psi\rangle = 1 = \mathrm{tr}\, |\psi\rangle\langle\psi|$

$\mathrm{tr}\, AB = \mathrm{tr}\, BA.$

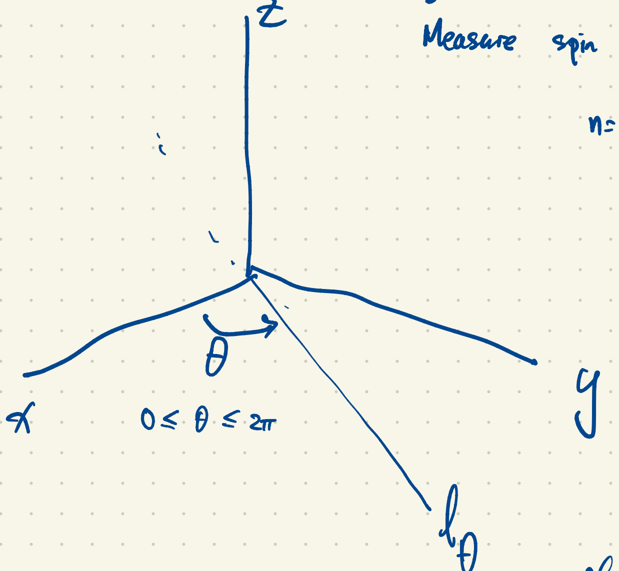
What is the corresponding Pauli spin operator in an arbitrary direction $n = (n_x, n_y, n_z) \in \mathbb{R}^3$

$n_x^2 + n_y^2 + n_z^2 = 1.$

$\sigma_n = n \cdot \sigma = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z$

$\sigma = (\sigma_x, \sigma_y, \sigma_z)$

Measure spin wrt line l_θ in x-y plane at angle θ as shown.



$n = (\cos\theta, \sin\theta, 0)$

σ_θ : Pauli spin operator for the direction n

$$\sigma_\theta = n \cdot (\sigma_x, \sigma_y, \sigma_z) = \cos\theta \underbrace{\sigma_x}_{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} + \sin\theta \underbrace{\sigma_y}_{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}} = \begin{bmatrix} 0 & \cos\theta - i\sin\theta \\ \cos\theta + i\sin\theta & 0 \end{bmatrix} = \begin{bmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{bmatrix}$$

using de Moivre's formula $e^{i\theta} = \cos\theta + i\sin\theta$

Eigenvectors $|+_\theta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix}$

$|-_\theta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\theta/2} \\ -e^{i\theta/2} \end{bmatrix}$

Check: $\sigma_\theta |+_\theta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{bmatrix} \begin{bmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix} = |+_\theta\rangle$ eigenvector with eigenvalue +1

$\sigma_\theta |-_\theta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{bmatrix} \begin{bmatrix} e^{-i\theta/2} \\ -e^{i\theta/2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix} = -|-_\theta\rangle$ " " " " " " −1.

The map $l_\theta \longmapsto \begin{cases} |+_\theta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix} \\ |-_\theta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\theta/2} \\ -e^{i\theta/2} \end{bmatrix} \end{cases}$

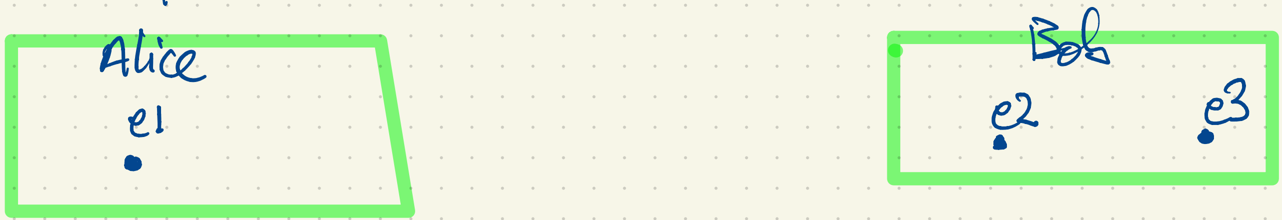
is 2-to-1.

Spin vectors go around "full circle" in $\mathbb{C}^2$ as θ goes from 0 to 4π; the "+" direction of l_θ goes twice around a circle in this same θ-interval.

If we measure an electron in spin state $|\psi\rangle = \alpha|+_\theta\rangle + \beta|-_\theta\rangle$ with respect to the direction l_θ, we get $|+_\theta\rangle$ with prob. $|\alpha|^2$, spin $|-_\theta\rangle$ with prob. $|\beta|^2$.

Spin states actually lie in S^3 = unit vector in $\mathbb{C}^2$ which is a double cover of $SO_3(\mathbb{R})$ = { rotations of $\mathbb{R}^3$ about the origin } = { 3x3 real matrices A : $AA^T = I$, $\det A = 1$ }.

Bob has an electron in spin state $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle = \binom{\alpha}{\beta} \in \mathbb{C}^2$, $|\alpha|^2 + |\beta|^2 = 1$.
He wants to send this to Alice. Bob doesn't know α, β and he cannot directly measure them.
Analogy: transporting Captain Kirk from enterprise to planet's surface.
In advance of this teleportation process, Alice and Bob have stockpiled some EPR pairs



Electrons e1, e2 are entangled: their joint spin state $|\psi_{12}\rangle = \frac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle)$

$|{+}{+}\rangle = |+\rangle \otimes |+\rangle$
$|{-}{-}\rangle = |-\rangle \otimes |-\rangle$
e1 e2

Electron e3 is in state $|\psi_3\rangle = \binom{\alpha}{\beta} = \alpha|+\rangle + \beta|-\rangle$, $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$.
e3 is not (currently) entangled with e1, e2.
The combined state of e1, e2, e3 is

$$|\psi_{123}\rangle = |\psi_{12}\rangle \otimes |\psi_3\rangle = \tfrac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle) \otimes (\alpha|+\rangle + \beta|-\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^8$$
$$= \tfrac{1}{\sqrt{2}}(\alpha|{+}{+}{+}\rangle + \beta|{+}{+}{-}\rangle + \alpha|{-}{-}{+}\rangle + \beta|{-}{-}{-}\rangle)$$

$$\left|\tfrac{\alpha}{\sqrt{2}}\right|^2 + \left|\tfrac{\beta}{\sqrt{2}}\right|^2 + \left|\tfrac{\alpha}{\sqrt{2}}\right|^2 + \left|\tfrac{\beta}{\sqrt{2}}\right|^2 = 1.$$

$|+\rangle = \binom{1}{0} \in \mathbb{C}^2$
$|-\rangle = \binom{0}{1} \in \mathbb{C}^2$

Spin of the pair e1, e2 lives in $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$ which has orthonormal basis $|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle$

$$|\psi_{123}\rangle = \frac{1}{\sqrt{2}}\left(\alpha|{+}{+}{+}\rangle + \beta|{+}{+}{-}\rangle + \alpha|{-}{-}{+}\rangle + \beta|{-}{-}{-}\rangle\right)$$

Bob performs a reversible (unitary) transformation with respect to e2, e3 defined by

$$|{+}{+}\rangle \mapsto \tfrac{1}{\sqrt{2}}\left(|{+}{+}\rangle + |{-}{-}\rangle\right)$$
$$|{-}{-}\rangle \mapsto \tfrac{1}{\sqrt{2}}\left(|{+}{+}\rangle - |{-}{-}\rangle\right)$$
$$|{+}{-}\rangle \mapsto \tfrac{1}{\sqrt{2}}\left(|{+}{-}\rangle + |{-}{+}\rangle\right)$$
$$|{-}{+}\rangle \mapsto \tfrac{1}{\sqrt{2}}\left(|{+}{-}\rangle - |{-}{+}\rangle\right)$$

This transforms $|\psi_{123}\rangle$ to

$$|\psi_{123}\rangle \mapsto \tfrac{1}{2}\left[\left(\alpha|{+}{+}{+}\rangle + \alpha|{+}{-}{-}\rangle\right) + \left(\beta|{+}{+}{-}\rangle + \beta|{+}{-}{+}\rangle\right) + \left(\alpha|{-}{+}{-}\rangle - \alpha|{-}{-}{+}\rangle\right) + \left(\beta|{-}{+}{+}\rangle - \beta|{-}{-}{-}\rangle\right)\right]$$
$$= \left(\alpha|+\rangle + \beta|-\rangle\right) \otimes \tfrac{1}{2}|{+}{+}\rangle + \left(\alpha|+\rangle - \beta|-\rangle\right) \otimes \tfrac{1}{2}|{-}{-}\rangle + \left(\beta|+\rangle + \alpha|-\rangle\right) \otimes \tfrac{1}{2}|{+}{-}\rangle + \left(\beta|+\rangle - \alpha|-\rangle\right) \otimes \tfrac{1}{2}|{-}{+}\rangle$$

Now Bob measures e2, e3 with respect to the basis $|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle$ of $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$. e2, e3 collapse into one of these four states. At this moment we know e1 is in one of the four states.

Bob sends this classical information (2 classical bits) to Alice.

Alice applies the appropriate unitary 2×2 matrix to e1 which transforms e1 into the correct state.

$\begin{bmatrix}1&0\\0&1\end{bmatrix} : \begin{pmatrix}\alpha\\\beta\end{pmatrix} \mapsto \begin{pmatrix}\alpha\\\beta\end{pmatrix} = \alpha|+\rangle + \beta|-\rangle$

$\begin{bmatrix}1&0\\0&-1\end{bmatrix} : \begin{pmatrix}\alpha\\\beta\end{pmatrix} \mapsto \begin{pmatrix}\alpha\\-\beta\end{pmatrix} : \alpha|+\rangle - \beta|-\rangle$

$\begin{bmatrix}0&1\\1&0\end{bmatrix} : \begin{pmatrix}\alpha\\\beta\end{pmatrix} \mapsto \begin{pmatrix}\beta\\\alpha\end{pmatrix} = \beta|+\rangle + \alpha|-\rangle$

$\begin{bmatrix}0&-1\\1&0\end{bmatrix} : \begin{pmatrix}\alpha\\\beta\end{pmatrix} \mapsto \begin{pmatrix}\beta\\-\alpha\end{pmatrix} = \beta|+\rangle - \alpha|-\rangle$

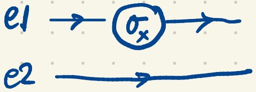
Note: Alice's operations on e1 can be described using Pauli spin matrices.

$\begin{bmatrix}1&0\\0&1\end{bmatrix}$ identity $|0\rangle \mapsto |0\rangle$, $|1\rangle \mapsto |1\rangle$ $\qquad |0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$\sigma_x = \begin{bmatrix}0&1\\1&0\end{bmatrix} : \begin{pmatrix}\alpha\\\beta\end{pmatrix} \mapsto \begin{pmatrix}\beta\\\alpha\end{pmatrix}$ i.e. $|0\rangle \longleftrightarrow |1\rangle$ 'bit flip' or 'NOT'

$\sigma_z = \begin{bmatrix}1&0\\0&-1\end{bmatrix} : \begin{pmatrix}\alpha\\\beta\end{pmatrix} \mapsto \begin{pmatrix}\alpha\\-\beta\end{pmatrix}$ i.e. $|0\rangle \mapsto |0\rangle$, $|1\rangle \mapsto -|1\rangle$ 'phase shift'

$\sigma_y = \begin{bmatrix}0&-i\\i&0\end{bmatrix} : \begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}-i\beta\\i\alpha\end{pmatrix} = -i\begin{pmatrix}\beta\\-\alpha\end{pmatrix}$



$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e1$

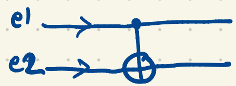
e2

$|00\rangle \mapsto |10\rangle$
$|01\rangle \mapsto |11\rangle$
$|10\rangle \mapsto |00\rangle$
$|11\rangle \mapsto |01\rangle$

$$\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] = \sigma_x \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

wrt basis $|00\rangle, |10\rangle, |01\rangle, |11\rangle$

CNOT gate: $|00\rangle \mapsto |00\rangle$
$|01\rangle \mapsto |01\rangle$
$|10\rangle \mapsto |11\rangle$
$|11\rangle \mapsto |10\rangle$



$$\left[\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right]$$

wrt basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

In quantum computation, quantum information is often modeled as qubits.

An ensemble of n electrons has spin state $|\psi\rangle \in \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n} \cong \mathbb{C}^{2^n}$ (unit vector).

Can initialize $|\psi\rangle$ in a particular state, usually $|000\cdots0\rangle = |0\rangle \otimes \cdots \otimes |0\rangle$ (all electrons spin up)

Cannot clone a qubit. Measurement of a qubit yields at most n classical bits of information.

Can perform reversible processes $|\psi\rangle \mapsto U|\psi\rangle$, U unitary $2^n \times 2^n$ matrix.

Can measure $|\psi\rangle$, typically by measuring spin of each electron separately.

Kochen-Specker Theorem 1967, as simplified by Peres shortly after.

Electrons vs. Photons

Spin $\in \mathbb{C}^2$ (unit vector)
Measurement yields one classical bit $+$ or $-$
up : down

Spin $\in \mathbb{C}^3$
Measurement wrt orthonormal frame (x, y, z)
Spin (when measured) gives 0,1,1 or 1,0,1 or 1,1,0
classical trit (ternary digit)

We designate 33 choices of axis in $\mathbb{R}^3$ prescribed by the Peres configuration containing 24 different orthogonal frames (an orthogonal frame is a choice of (x,y,z) coordinate system with perpendicular axes).

Take an EPR pair of photons (photons 1 and 2) separated by a vast distance.

Spin is actually measured in units of $\frac{\hbar}{2}$, $\hbar = h/2\pi$

Electron spin: $-\frac{\hbar}{2}$ or $\frac{\hbar}{2}$ spin $\frac{1}{2}$ particle (fermion)

Photon spin: $-\hbar$, 0, $\hbar$ spin 1 particle (boson)

CHSH game: Alice and Bob.

$$\boxed{\text{Alice}} \;\underset{\pm}{\overset{x\in\{0,1\}}{\rightleftarrows}}\; \text{Referee} \;\underset{\pm 1}{\overset{y\in\{0,1\}}{\rightleftarrows}}\; \boxed{\text{Bob}}$$

when $(x,y) = \{(0,0), (1,0), (0,1)\}$, Alice and Bob win the round when their guesses agree

when $(x,y) = (1,1)$, Alice and Bob win the round when their guesses disagree.

Strategy for winning 75% of the time: Both Alice and Bob guess '+' every round.

Can't beat this (apparently) without EPR pairs.

Consider an electron pair e1, e2 in spin state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}|0\rangle\otimes|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\otimes|1\rangle$

Alice has e1, Bob has e2. Alice chooses a pair of states $|+_A\rangle, |-_A\rangle$ which is an orthonormal basis for $\mathbb{C}^2$. Bob has $|+_B\rangle, |-_B\rangle$.

$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ $\begin{pmatrix}\bar\beta\\ -\bar\alpha\end{pmatrix}$

$|\alpha|^2+|\beta|^2 = 1 = \langle +_A|+_A\rangle = 1$, $|\gamma|^2+|\delta|^2 = 1 = \langle -_A|-_A\rangle$

$\bar\alpha\gamma + \bar\beta\delta = 0 = \langle +_A|-_A\rangle = 0$

Probability that Alice and Bob agree in their spin measurement is $|\langle +_A|+_B\rangle|^2$.

Born's Rule: If Alice measures an electron in state $|\psi\rangle \in \mathbb{C}^2$ with respect to her choice of basis, the prob. of spin up is $|\langle +_A|\psi\rangle|^2$
" " " " " " down is $|\langle -_A|\psi\rangle|^2$ So $|\langle +_A|\psi\rangle|^2 + |\langle -_A|\psi\rangle|^2 = 1. \leftarrow \||\psi\rangle\|^2 = \langle\psi|\psi\rangle$

In particular $|\langle +_A|+_B\rangle|^2 = 1 - |\langle -_A|+_B\rangle|^2 = |\langle -_A|-_B\rangle|^2$

If Alice and Bob measure their electrons, what is the probability they agree on spin direction?

Assume Alice measures first:

- Suppose Alice measures e1 to be spin up. This occurs with probability $|\langle +_A|\psi\rangle|^2$.
  In this case e1 collapses into state $|+_A\rangle$. Instantly we know e2 is also in this state $|+_A\rangle$. The prob. that Bob also measures e2 to be 'up' is $|\langle +_B|+_A\rangle|^2$.
  The prob. that all this occurs is $|\langle +_A|\psi\rangle|^2 |\langle +_B|+_A\rangle|^2$.
- Suppose Alice measures e1 to be spin down. Prob. (this) $= |\langle -_A|\psi\rangle|^2$.
  In this case e1 and e2 are in state $|-_A\rangle$. The prob. that Bob also finds e2 to be in 'down' state is $|\langle -_B|-_A\rangle|^2$. All this occurs with prob. $|\langle -_A|\psi\rangle|^2 |\langle -_B|-_A\rangle|^2$.

Total prob. that Alice and Bob 'agree' is

$$|\langle +_A|\psi\rangle|^2 |\langle +_A|+_B\rangle|^2 + |\langle -_A|\psi\rangle|^2 |\langle +_A|+_B\rangle|^2 = \underbrace{\left(|\langle +_A|\psi\rangle|^2 + |\langle -_A|\psi\rangle|^2\right)}_{1} |\langle +_A|+_B\rangle|^2 = |\langle +_A|+_B\rangle|^2!$$

Alice and Bob can win 85.4% of the time at the CHSH game using EPR pairs.
Say each EPR pair is in spin state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ e1, e2

Alice uses $|+A_0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$, $|-A_0\rangle = \begin{bmatrix}0\\1\end{bmatrix}$ if the referee sends her $x=0$

" " $|+A_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$, $|-A_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$ " — — — — — — $x=1$

Bob use $|+B_0\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}-1\\1+\sqrt{2}\end{bmatrix}$, $|-B_0\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}1+\sqrt{2}\\1\end{bmatrix}$ if the ref sends him $y=0$

" " $|+B_1\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}1\\1+\sqrt{2}\end{bmatrix}$, $|-B_1\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}1+\sqrt{2}\\-1\end{bmatrix}$ — — — — — — $y=1$.

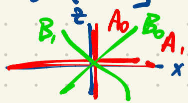
If Alice and Bob measure their respective electrons e1, e2 with their choice of bases A_x, B_y then the probability that they "agree"

$$|\langle +_{A_x} | +_{B_y}\rangle|^2 = \begin{cases} \frac{2+\sqrt{2}}{4} \approx 0.854 & \text{if } (x,y) \in \{(0,0),(1,0),(0,1)\} \\ 1 - \frac{2+\sqrt{2}}{4} & \text{if } (x,y) = (1,1). \end{cases}$$

In all cases $(x,y) \in \{0,1\}^2$ Alice and Bob have a probability $\frac{2+\sqrt{2}}{4} \approx 0.854$ of winning their round at CHSH game.

Bell's Theorem: Under certain reasonable assumption, hidden variable theories give at most 75% chance of Alice and Bob winning at the CHSH game.

A_0: $\sigma_z = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$, A_1: $\sigma_x = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$, $n = (n_x, n_y, n_z) \in \mathbb{R}^3$ $n \cdot (\sigma_x, \sigma_y, \sigma_z)$ B_0, B_1



"Superdense coding" or "dense coding"

A way to store and retrieve 2 classical bits in one qubit. (Using EPR pairs)

Alice and Bob are far apart. They have a shared EPR pair $e1, e2$ in state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

At some later time, Alice wants to send a pair of classical bits 00, 01, 10 or 11 to Bob.

Alice performs a reversible operation U (2×2 unitary matrix) on her electron $e1$.

She then sends $e1$ to Bob. (no faster than the speed of light)

Bob has $e1$ now as well as $e2$. He performs a measurement and retrieves Alice's bit pair