

A 3D perspective view of a grid of cubes. Most cubes are grey, but one cube in the upper-left quadrant is gold. A yellow banner with rounded corners is positioned horizontally across the middle of the image.

# Information Theory

Book III

Spin state of an electron (disregard position and momentum) is an example of a qubit, which is a vector  $|\psi\rangle \in \mathbb{C}^2 = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$ .

Standard basis of  $\mathbb{C}^2$ :  $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$   
 "spin up" "spin down"

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha|+\rangle + \beta|-\rangle, \quad |\alpha|^2 + |\beta|^2 = 1.$$

An electron in this spin state is in a superposition of spin up and spin down states

A linear functional on  $\mathbb{C}^2$  is a linear transformation

$$\langle\phi| : \mathbb{C}^2 \rightarrow \mathbb{C}$$

bra notation

$$\langle\phi| = (r \ s) : \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto (r \ s) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = r\alpha + s\beta \in \mathbb{C}$$

Dual basis:

$$\langle+| = |+\rangle^* = (1 \ 0) \quad \langle\phi|\psi\rangle$$

$$\langle-| = |-\rangle^* = (0 \ 1)$$

$$|\psi\rangle^* = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^* = (\bar{\alpha} \ \bar{\beta}) = \bar{\alpha}\langle+| + \bar{\beta}\langle-|$$

$$\langle+|\psi\rangle = \langle+|(\alpha|+\rangle + \beta|-\rangle) = \alpha$$

$$\langle-|\psi\rangle = \beta$$

Spin states are unit vectors in  $\mathbb{C}^2$  i.e.  $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ ,  $\alpha, \beta \in \mathbb{C}$ ,  $|\alpha|^2 + |\beta|^2 = 1$ .

i.e. in  $\mathbb{R}^4$

so  $|\psi\rangle \in S^3 =$  unit sphere in  $\mathbb{R}^4$ .

$$\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$$

$$\begin{cases} \alpha = \alpha_1 + \alpha_2 i \\ \beta = \beta_1 + \beta_2 i \end{cases} \} \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$$

A measurement of an electron in this spin state yields a single bit of classical information:

- spin up, with probability  $|\alpha|^2$ ;
- spin down, with probability  $|\beta|^2$ .

This says what happens when we measure with respect to the z-axis. (For measurement in a different direction/axis, we'll say later.)

As soon as the measurement is taken, the spin state collapses; all knowledge of  $\alpha, \beta$  is then lost.

Any time we measure a spin state  $|\psi\rangle \in S^3$ , it collapses.

But it is possible to perform certain reversible operations  $|\psi\rangle \mapsto A|\psi\rangle$  where  $A$  is a  $2 \times 2$  unitary matrix ( $AA^* = A^*A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ) over  $\mathbb{C}$ .

Special examples of unitary matrices are scalar matrices  $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ ,  $\lambda \in \mathbb{C}$ ,  $|\lambda| = 1$

These perform an operation on  $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$  whose only effect is to alter the phase of  $\alpha, \beta$

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto A|\psi\rangle = \begin{pmatrix} \lambda\alpha \\ \lambda\beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\lambda = e^{i\theta} \quad (\theta \in [0, 2\pi))$$

which has no physical significance. For this reason the so-called density matrix

$$\underbrace{|\psi\rangle}_{2 \times 1} \underbrace{\langle\psi|}_{1 \times 2} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \end{pmatrix} = \begin{pmatrix} \alpha\bar{\alpha} & \alpha\bar{\beta} \\ \beta\bar{\alpha} & \beta\bar{\beta} \end{pmatrix}$$

$2 \times 2$

Hermitian  $2 \times 2$  matrix

$$H \in \mathbb{C}^{2 \times 2} \quad (2 \times 2 \text{ complex matrix})$$

$$\text{satisfying } H^\dagger = H$$

which holds all the physically significant information of the single qubit.

The map  $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \lambda\alpha \\ \lambda\beta \end{pmatrix}$  does not change this density matrix.

Entanglement typically occurs when we include multiple electrons in our system.

Start by reviewing statistical dependence works:

Let's say we take a random individual A from a population.

Imagine the population is 40% male, 60% female; 30% short, 70% tall.

Sampling by selecting one person gives two bits: MS, MT, FS, or FT.

Combinations of attributes:

12%, 28%, 18%, 42% if gender is independent of height.

In this example, gender and height are independent.

		Height		
		S	T	
Gender	M	0.12	0.28	0.4
	F	0.18	0.42	0.6
		0.3	0.7	1

$$\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.12 & 0.28 \\ 0.18 & 0.42 \end{bmatrix}$$

Outer product of two vectors is a rank 1.

More typical distribution

		Height		
		S	T	
Gender	M	0.1	0.3	0.4
	F	0.2	0.4	0.6
		0.3	0.7	1

The matrix  $\begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{bmatrix}$  has rank 2.

In this second example gender and height are (statistically) dependent.



If one electron has <sup>(spin)</sup> state  $|\psi_1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2$  and a second electron has spin state  $|\psi_2\rangle = \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \in \mathbb{C}^2$   
 $|\alpha|^2 + |\beta|^2 = 1$   $|\gamma|^2 + |\delta|^2 = 1$

the pair of electrons has state  $|\psi_n\rangle = \alpha_n |++\rangle + \alpha_{n1} |+-\rangle + \alpha_{n2} |-+\rangle + \alpha_{n3} |--\rangle \in \mathbb{C}^4$

If the two electrons are not entangled then

$$\begin{pmatrix} \alpha_{n1} & \alpha_{n2} \\ \alpha_{n3} & \alpha_{n4} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \gamma & \delta \end{pmatrix} \quad \text{rank 1.}$$

$$\alpha_{ij} \in \mathbb{C}, \quad |\alpha_{11}|^2 + |\alpha_{12}|^2 + |\alpha_{21}|^2 + |\alpha_{22}|^2 = 1.$$

↑  
prob. of  
both electrons  
having spin up

If the matrix has rank  $\geq 2$  then the two electrons are entangled.

Ex.  $|\psi\rangle = \frac{1}{\sqrt{2}} (|++\rangle + |--\rangle)$  i.e.  $\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$  } Examples of EPR pairs  
 $|\psi'\rangle = \frac{1}{\sqrt{2}} (|+-\rangle + |-+\rangle)$  i.e.  $\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$

One way to talk about the spin state of a set of  $n$  electrons is

$$|\psi\rangle = \sum_{\substack{i_1 \in \{0,1\} \\ i_2 \in \{0,1\} \\ \vdots \\ i_n \in \{0,1\}}} \alpha_{i_1 i_2 \dots i_n} |\pm \pm \dots \pm\rangle \in \mathbb{C}^{2^n} \quad \sum |\alpha_{i_1 i_2 \dots i_n}|^2 = 1$$

all  $2^n$  combinations of  $\pm$

$(\alpha_{i_1 i_2 \dots i_n} : i_1, i_2, \dots, i_n \in \{0,1\})$  is a  
 $\underbrace{2 \times 2 \times 2 \times \dots \times 2}_n$  array or tensor

$\mathbb{C}^2 = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{n \text{ times}}$  tensor product. Take basis  $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

has basis  $|+++ \dots ++\rangle = |+\rangle \otimes |+\rangle \otimes \dots \otimes |+\rangle$   
 $|+++ \dots +-\rangle = |+\rangle \otimes |+\rangle \otimes \dots \otimes |-\rangle$   
 $\vdots$   
 $|-----\rangle = |-\rangle \otimes |-\rangle \otimes \dots \otimes |-\rangle$

In  $\mathbb{C}^m \otimes \mathbb{C}^n \cong \mathbb{C}^{mn}$   
 every vector is a sum of at most  $\min\{m, n\}$  pure tensors.

More generally if  $v_i \in \mathbb{C}^2$  ( $i=1, 2, \dots, n$ )

then  $v_1 \otimes v_2 \otimes \dots \otimes v_n \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2$ . (pure tensors)  
 $\mathbb{C}^2 \times \mathbb{C}^2 \times \dots \times \mathbb{C}^2$   $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2 = \mathbb{C}^{2^n}$  simple

$(v_1, \dots, v_n) \mapsto v_1 \otimes v_2 \otimes \dots \otimes v_n$  this map is multilinear  
 i.e. linear in each argument separately.

The corresponding result for  $\mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2} \otimes \dots \otimes \mathbb{C}^{m_k}$  is not known and extremely hard.

(In Algebraic Geometry look up Higher Secant varieties of Segre Varieties)

- Bell's Theorem
- Gleason's Theorem
- Kochen-Specker Theorem

Recall: the spin state of a single electron is a qubit  $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2$ ,  $|\alpha|^2 + |\beta|^2 = 1$ .

Standard basis  $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

How do we measure the spin in an arbitrary direction?   
 spin up/down with respect to the z-axis

In the vertical direction we make use of basis  $|+\frac{z}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $|-\frac{z}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  basis of eigenvectors for the Pauli spin operator  $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

$$\sigma_z |+\frac{z}{2}\rangle = |+\frac{z}{2}\rangle$$

$$\sigma_z |-\frac{z}{2}\rangle = -|-\frac{z}{2}\rangle$$

Any electron with spin state  $|\psi\rangle = \alpha |+\frac{z}{2}\rangle + \beta |-\frac{z}{2}\rangle$  can be measured in the vertical direction

$$\sigma_z |\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$$

Follow this by a linear functional eg.  $|+\frac{z}{2}\rangle^* = \langle +\frac{z}{2} |$

$$\langle +\frac{z}{2} | \sigma_z | \psi \rangle = (1 \ 0) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (1 \ 0) \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \alpha$$
, the amplitude for the electron to be spin up.

Once the measurement is performed, the state collapses into that spin state  $|\psi\rangle \mapsto |+\frac{z}{2}\rangle$ .

$$\langle +\frac{z}{2} | \underbrace{\sigma_z | +\frac{z}{2} \rangle}_{|+\frac{z}{2}\rangle} = 1.$$

If we measure  $|\psi\rangle \mapsto \langle +\frac{z}{2} | \sigma_z | \psi \rangle$  and find spin down, the state collapses to spin down  $|-\frac{z}{2}\rangle$

$$\langle +\frac{z}{2} | \sigma_z | \psi \rangle = -\beta, \quad |-\beta|^2 = |\beta|^2$$

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Hermitian:  $\sigma^* = \sigma$

Eigenvectors:  $|+\rangle_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $|-\rangle_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$|+\rangle_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$ ,  $|-\rangle_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$|+\rangle_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $|-\rangle_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

eigenvalues  $+1, -1$

eg.  $\sigma_y \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ i \end{pmatrix} = |+\rangle_y$

$\sigma_y \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ i \end{pmatrix} = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = -|-\rangle_y$

If we measure an electron having spin  $|+\rangle_y$  (in the pos. y-direction) with respect to the x-axis

$\langle +\rangle_x |+\rangle_y = \frac{1}{\sqrt{2}} (1 \ 1) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+i \\ 1+i \end{pmatrix}$   $|\frac{1+i}{2}|^2 = \frac{2}{4} = \frac{1}{2}$   $|a+bi|^2 = a^2+b^2$

Density matrix of  $|\psi\rangle$  is  $|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi|^* = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (\bar{\alpha} \ \bar{\beta}) = \begin{pmatrix} \alpha\bar{\alpha} & \alpha\bar{\beta} \\ \bar{\alpha}\beta & \beta\bar{\beta} \end{pmatrix}$   $|\alpha|^2 + |\beta|^2 = 1$

is Hermitian having eigenvalues  $1, 0$ ; corresponding eigenvectors  $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ ,  $|\psi^\perp\rangle = \begin{pmatrix} \bar{\beta} \\ -\bar{\alpha} \end{pmatrix}$

$|\psi\rangle\langle\psi| |\psi\rangle = |\psi\rangle$  since  $\langle\psi|\psi\rangle = 1$

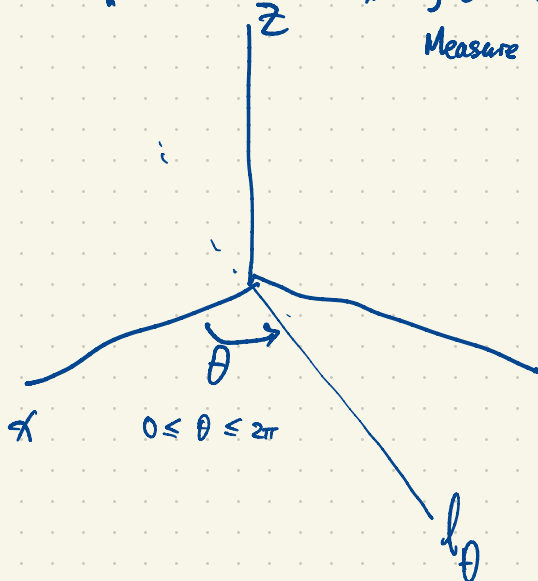
$\langle\psi^\perp|\psi\rangle = (\beta - \alpha) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha\beta - \alpha\beta = 0$

$|\psi\rangle\langle\psi| |\psi^\perp\rangle = 0 = 0|\psi^\perp\rangle$

$\langle\psi^\perp|\psi^\perp\rangle = 1 = \langle\psi|\psi\rangle$   
 for  $AB = BA$ .

What is the corresponding Pauli spin operator in an arbitrary direction  $n = (n_x, n_y, n_z) \in \mathbb{R}^3$   
 $n_x^2 + n_y^2 + n_z^2 = 1$ .

$$\sigma_n = n \cdot \sigma = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z$$



Measure spin wrt line  $l_0$  in x-y plane at angle  $\theta$  as shown.

$$\sigma = (\sigma_x, \sigma_y, \sigma_z)$$

$$n = (\cos\theta, \sin\theta, 0)$$

$\sigma_\theta$ : Pauli spin operator for the direction  $n$

$$\sigma_\theta = n \cdot (\sigma_x, \sigma_y, \sigma_z) = \cos\theta \sigma_x + \sin\theta \sigma_y = \begin{bmatrix} 0 & \cos\theta - i\sin\theta \\ \cos\theta + i\sin\theta & 0 \end{bmatrix} = \begin{bmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{bmatrix}$$

using de Moivre's formula  $e^{i\theta} = \cos\theta + i\sin\theta$

Eigen vectors  $|+\theta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix}$

$$|-\theta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\theta/2} \\ -e^{i\theta/2} \end{bmatrix}$$

Check:  $\sigma_\theta |+\theta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{bmatrix} \begin{bmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix} = |+\theta\rangle$  eigenvector with eigenvalue  $+1$

$$\sigma_\theta |-\theta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{bmatrix} \begin{bmatrix} e^{-i\theta/2} \\ -e^{i\theta/2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix} = -|-\theta\rangle$$

The map  $l_\theta \mapsto \begin{cases} |+\theta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix} \\ |-\theta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\theta/2} \\ -e^{i\theta/2} \end{bmatrix} \end{cases}$

is 2-to-1.

Spin vectors go around "full circle" in  $\mathbb{C}^2$  as  $\theta$  goes from 0 to  $4\pi$ ; the "+" direction of  $l_\theta$  goes twice around a circle in this same  $\theta$ -interval.

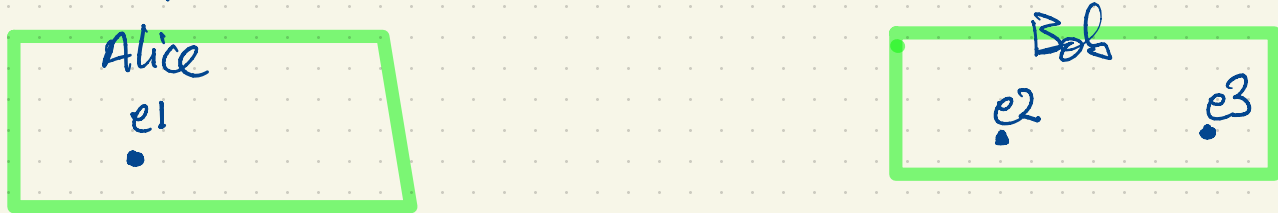
If we measure an electron in spin state

$|\psi\rangle = \alpha |+\theta\rangle + \beta |-\theta\rangle$  with respect to the direction  $l_\theta$ , we get  $|+\theta\rangle$  with prob.  $|\alpha|^2$ , spin  $|-\theta\rangle$  with prob.  $|\beta|^2$ .

Spin states actually lie in  $S^3 =$  unit vector in  $\mathbb{C}^2$  which is a double cover of  
of  $SO_3(\mathbb{R}) = \{ \text{rotations of } \mathbb{R}^3 \text{ about the origin} \} = \{ 3 \times 3 \text{ real matrices } A : AA^T = I, \det A = 1 \}$ .

Bob has an electron in spin state  $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2$ ,  $|\alpha|^2 + |\beta|^2 = 1$ .  
He wants to send this to Alice. Bob doesn't know  $\alpha, \beta$  and he cannot directly measure them.

Analogy: transporting Captain Kirk from enterprise to planet's surface.  
In advance of this teleportation process, Alice and Bob have stockpiled some EPR pairs



Electrons  $e1, e2$  are entangled: their joint spin state  $|\psi_{12}\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle)$

Electron  $e3$  is in state  $|\psi_3\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha|+\rangle + \beta|-\rangle$ ,  $\alpha, \beta \in \mathbb{C}$ ,  $|\alpha|^2 + |\beta|^2 = 1$ .  
 $e3$  is not (currently) entangled with  $e1, e2$ .

The combined state of  $e1, e2, e3$  is

$$|\psi_{123}\rangle = |\psi_{12}\rangle \otimes |\psi_3\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle) \otimes (\alpha|+\rangle + \beta|-\rangle) \\ = \frac{1}{\sqrt{2}}(\alpha|+++ \rangle + \beta|++-\rangle + \alpha|+-- \rangle + \beta|+-- \rangle)$$

$$\in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^8$$

$$|\frac{\alpha}{\sqrt{2}}|^2 + |\frac{\beta}{\sqrt{2}}|^2 + |\frac{\alpha}{\sqrt{2}}|^2 + |\frac{\beta}{\sqrt{2}}|^2 = 1.$$

$$\begin{aligned} |++\rangle &= |+\rangle \otimes |+\rangle \\ |--\rangle &= |-\rangle \otimes |-\rangle \end{aligned}$$

$$\begin{aligned} e1 \quad e2 \\ |+\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2 \\ |-\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2 \end{aligned}$$

Spin of the pair  $e1, e2$  lives in  $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$  which has orthonormal basis  $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$



$$|\psi_{123}\rangle = \frac{1}{\sqrt{2}}(\alpha|++\rangle + \beta|+-\rangle + \alpha|-+\rangle + \beta|--\rangle)$$

Bob performs a reversible (unitary) transformation with respect to  $e_2, e_3$  defined by

$$\begin{aligned} |++\rangle &\mapsto \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle) \\ |--\rangle &\mapsto \frac{1}{\sqrt{2}}(|++\rangle - |--\rangle) \\ |+-\rangle &\mapsto \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) \\ |-+\rangle &\mapsto \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \end{aligned}$$

This transforms  $|\psi_{123}\rangle$  to

$$\begin{aligned} |\psi_{123}\rangle &\mapsto \frac{1}{2} \left[ (\alpha|++\rangle + \alpha|+-\rangle) + (\beta|++\rangle + \beta|+-\rangle) + (\alpha|+-\rangle - \alpha|-+\rangle) + (\beta|+-\rangle - \beta|--\rangle) \right] \\ &= (\alpha|+\rangle + \beta|-\rangle) \otimes \frac{1}{2}|++\rangle + (\alpha|+\rangle - \beta|-\rangle) \otimes \frac{1}{2}|--\rangle + (\beta|+\rangle + \alpha|-\rangle) \otimes \frac{1}{2}|+\rangle + (\beta|+\rangle - \alpha|-\rangle) \otimes \frac{1}{2}|-\rangle \end{aligned}$$

Now Bob measures  $e_2, e_3$  with respect to the basis  $|+\rangle, |-\rangle, |+\rangle, |-\rangle$  of  $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$ .

$e_2, e_3$  collapse into one of these four states. At this moment we know  $e_1$  is in one of the four states. Bob sends this classical information (2 classical bits) to Alice.

Alice applies the appropriate unitary  $2 \times 2$  matrix to  $e_1$  which transforms  $e_1$  into the correct state.

Note: Alice's operations on  $e_1$  can be described using Pauli spin matrices.

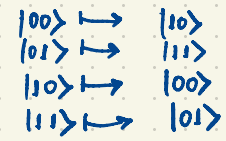
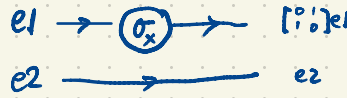
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ identity } |0\rangle \mapsto |0\rangle, |1\rangle \mapsto |1\rangle \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}: \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \text{ i.e. } |0\rangle \leftrightarrow |1\rangle \text{ 'bit flip' or 'NOT'}$$

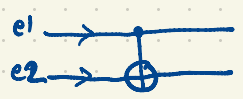
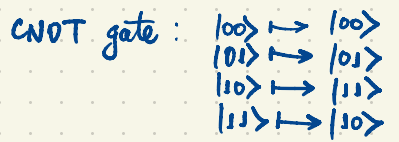
$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}: \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} \text{ i.e. } |0\rangle \mapsto |0\rangle, |1\rangle \mapsto -|1\rangle \text{ 'phase shift'}$$

$$\sigma_y = \begin{bmatrix} 0 & 1 \\ -i & 0 \end{bmatrix}: \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} i\beta \\ \alpha \end{pmatrix} = -i \begin{pmatrix} \beta \\ -\alpha \end{pmatrix}$$

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &\mapsto \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha|+\rangle + \beta|-\rangle \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &\mapsto \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \alpha|-\rangle + \beta|+\rangle \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &\mapsto \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \beta|+\rangle + \alpha|-\rangle \\ \begin{bmatrix} 0 & 1 \\ -i & 0 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &\mapsto \begin{pmatrix} i\beta \\ \alpha \end{pmatrix} = \beta|+\rangle - \alpha|-\rangle \end{aligned}$$



$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \sigma_x \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$   
 wrt basis  $|00\rangle, |10\rangle, |01\rangle, |11\rangle$



wrt basis  $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

In quantum computation, quantum information is often modeled as qubits.

An ensemble of  $n$  electrons has spin state  $|\psi\rangle \in \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_n \cong \mathbb{C}^{2^n}$  (unit vector).

Can initialize  $|\psi\rangle$  in a particular state, usually  $|000\dots 0\rangle = |0\rangle \otimes \dots \otimes |0\rangle$  (all electrons spin up).

Cannot clone a qubit. Measurement of a qubit yields at most  $n$  classical bits of information.

Can perform reversible processes  $|\psi\rangle \mapsto U|\psi\rangle$ ,  $U$  unitary  $2^n \times 2^n$  matrix.

Can measure  $|\psi\rangle$ , typically by measuring spin of each electron separately.

Kochen-Specker Theorem 1967, as simplified by Peres shortly after.

Electrons vs. Photons  
 Spin  $\in \mathbb{C}^2$  (unit vector)  
 Measurement yields one classical bit  
 up or down

Spin  $\in \mathbb{C}^3$   
 Measurement wrt orthonormal frame  $(x, y, z)$   
 Spin (when measured) gives 0,1 or 1,0,1 or 1,1,0  
 classical trit (ternary digit)



We designate 33 choices of axis in  $\mathbb{R}^3$  prescribed by the Peres configuration containing 24 different orthogonal frames (an orthogonal frame is a choice of  $(x, y, z)$  coordinate system with perpendicular axes).

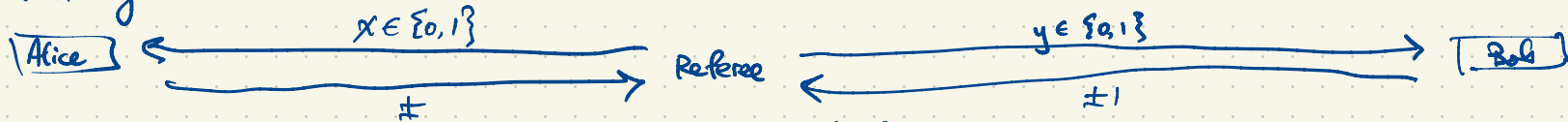
Take an EPR pair of photons (photons 1 and 2) separated by a vast distance.

Spin is actually measured in units of  $\frac{\hbar}{2}$ ,  $\hbar = h/2\pi$

Electron spin:  $-\frac{\hbar}{2}$  or  $\frac{\hbar}{2}$  spin  $\frac{1}{2}$  particle (fermion)

Photon spin:  $-\hbar, 0, \hbar$  spin 1 particle (boson)

CHSH game: Alice and Bob.



when  $(x, y) = \{(0, 0), (1, 0), (0, 1)\}$ , Alice and Bob win the round when their guesses agree

when  $(x, y) = (1, 1)$ , Alice and Bob win the round when their guesses disagree.

Strategy for winning 75% of the time: Both Alice and Bob guess '+' every round.

Can't beat this (apparently) without EPR pairs.

Consider an electron pair  $e_1, e_2$  in spin state  $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle$   
 Alice has  $e_1$ , Bob has  $e_2$ . Alice chooses a pair of states  $|+\rangle_A, |-\rangle_A$  which is an orthonormal basis for  $\mathbb{C}^2$ . Bob has  $|+\rangle_B, |-\rangle_B$ .

Probability that Alice and Bob agree in their spin measurement is  $|\langle +_A | +_B \rangle|^2$ .

$$\begin{aligned}
 \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & = & |x|^2 + |y|^2 = 1, & |r|^2 + |s|^2 = 1 \\
 \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & = & & \\
 \langle +_A | +_A \rangle & = & 1 & & \vec{a} \cdot \vec{b} = 0 \\
 & & & & \langle +_A | -_A \rangle = 0
 \end{aligned}$$

Born's Rule: If Alice measures an electron in state  $|\psi\rangle \in \mathbb{C}^2$  with respect to her choice of basis, the prob. of spin up is  $|\langle +_A | \psi \rangle|^2$   
 down is  $|\langle -_A | \psi \rangle|^2$  So  $|\langle +_A | \psi \rangle|^2 + |\langle -_A | \psi \rangle|^2 = 1. \leftarrow \|\psi\|^2 = \langle \psi | \psi \rangle$

In particular  $|\langle +_A | +_B \rangle|^2 = 1 - |\langle -_A | +_B \rangle|^2 = |\langle -_A | -_B \rangle|^2$

If Alice and Bob measure their electrons, what is the probability they agree on spin direction?  
 Assume Alice measures first:

- Suppose Alice measures  $e_1$  to be spin up. This occurs with probability  $|\langle +_A | \psi \rangle|^2$ .  
 In this case  $e_1$  collapses into state  $|+_A\rangle$ . Instantly we know  $e_2$  is also in this state  $|+_A\rangle$ . The prob. that Bob also measures  $e_2$  to be 'up' is  $|\langle +_B | +_A \rangle|^2$ .  
 The prob. that all this occurs is  $|\langle +_A | \psi \rangle|^2 |\langle +_B | +_A \rangle|^2$ .

- Suppose Alice measures  $e_1$  to be spin down. Prob. (this) =  $|\langle -_A | \psi \rangle|^2$ .  
 In this case  $e_1$  and  $e_2$  are in state  $|-_A\rangle$ . The prob. that Bob also finds  $e_2$  to be in 'down' state is  $|\langle -_B | -_A \rangle|^2$ . All this occurs with prob.  $|\langle -_A | \psi \rangle|^2 |\langle -_B | -_A \rangle|^2$ .

Total prob. that Alice and Bob 'agree' is

$$|\langle +_A | \psi \rangle|^2 |\langle +_B | +_A \rangle|^2 + |\langle -_A | \psi \rangle|^2 |\langle +_B | -_A \rangle|^2 = \underbrace{(|\langle +_A | \psi \rangle|^2 + |\langle -_A | \psi \rangle|^2)}_1 |\langle +_B | +_A \rangle|^2 = |\langle +_B | +_A \rangle|^2$$