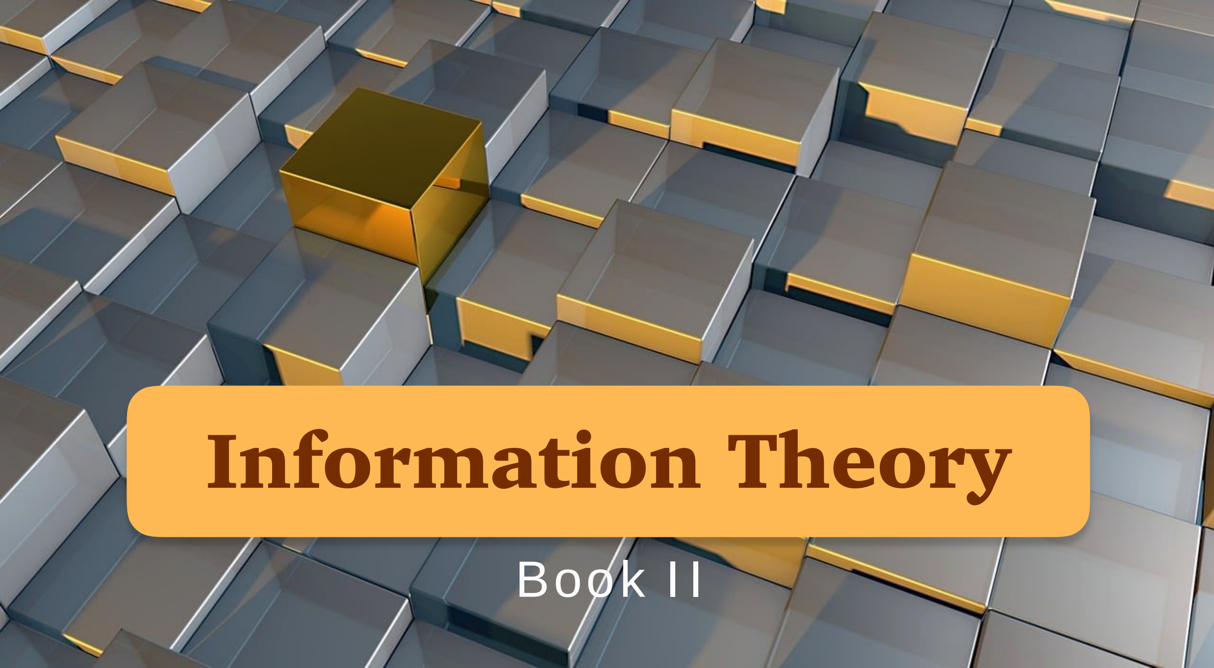
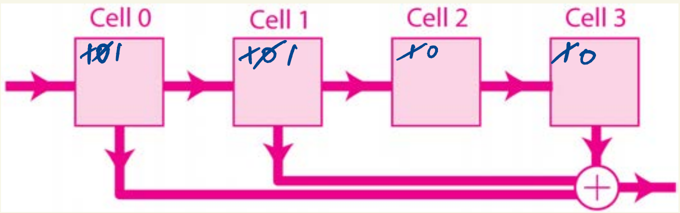


Information Theory

Book II

Eg. an infinite stream of bits $a_0 a_1 a_2 a_3 a_4 \cdots$ $(a_i \in F)$ can be encoded eg. represent the plaintext bitstream as a $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \in \mathbb{F}_2[[x]]$

$F[[x]]$ = ring of (formal) power series in x with coefficients in F.



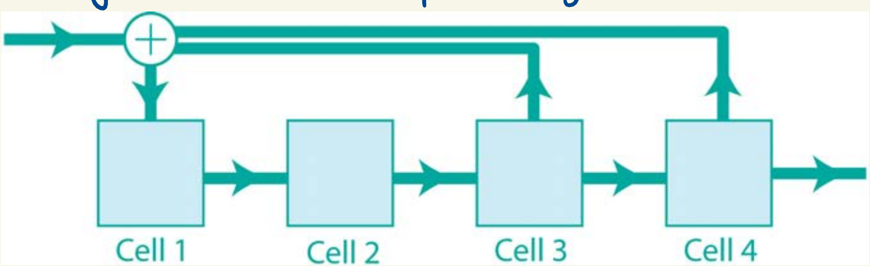
10110...

Eg. consider an input bitstream $\cancel{1}\cancel{1}\cancel{0}\cancel{0}\cancel{1}\cancel{1}0111\,10010\ldots$ which is encoded by the shift register above to obtain the output bitstream $101100101\ldots$

Compare: this is equivalent to multiplication by $1+x+x^3$:

$$(1+x+x^3)(1+x+x^4+x^5+x^7+x^8+x^9+x^{10}+x^{13}+\cdots) = 1 + x^2 + x^3 + x^6 + x^8 + \cdots$$

Decoding of this data is accomplished using backward shift registers eg.



which performs division by $1+x+x^3$ in $\mathbb{F}_2((x))$

Symbolic rings and fields

$$\begin{tikzcd}
 & F((x)) \arrow[dl, dash] \arrow[dr, dash] & \\
F(x) \arrow[dr, dash] & & F[[x]] \arrow[dl, dash] \\
 & F[x] &
\end{tikzcd}$$

field of Laurent series $\to F((x))$

field of (power) series in x with coeffs in F. $\to F[[x]]$

field of rational "functions" (actually symbolic expressions) in x with coeffs in F: $F(x)$

$F[x]$ = ring of polys in x with coeffs in F

polynomials vs. polynomial functions

eg. $\mathbb{F}_3 = \{0,1,2\} = \mathbb{Z}/3\mathbb{Z}$

eg. $f(x) = 2 + x + x^3 \in \mathbb{F}_3[x]$ is a polynomial of degree 3.

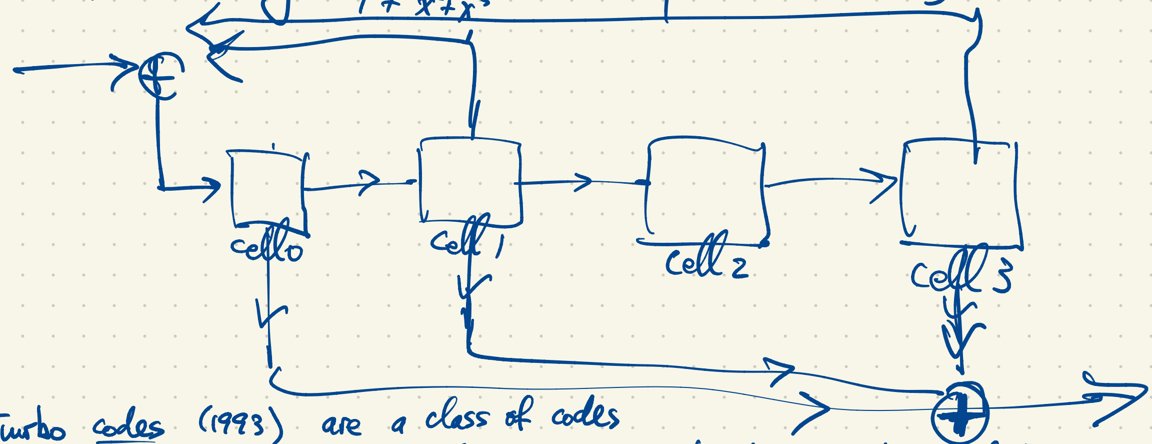
$g(x) = 2 + 2x \in \mathbb{F}_3[x]$ is a polynomial of degree 1.

a	$f(a)$	$g(a)$
0	2	2
1	1	1
2	0	0

$f(x), g(x)$ are distinct poly's but they represent the same function $\mathbb{F}_3 \to \mathbb{F}_3$.

eg. $f(x) = \dfrac{1+x+x^3}{x+x^2} \in \mathbb{F}_2(x)$

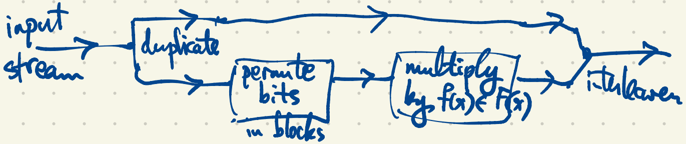
Multiplication by any rational function can be implemented using a single shift register e.g. multiplication by $\frac{1+x+x^3}{1+x^2+x^3}$ is implemented using the shift register



<u>Turbo</u> <u>codes</u> (1993) are a class of codes used for encoding streams of data using combinations of gates including

- multiplication by a rational function in $F(x)$
- splitters & interleavers
- permutations
- puncturing

eg.



$F(x) \subset F((x))$ eg. for $F = \mathbb{F}_2 = \{0,1\}$

First method

$$f(x) = \frac{1+x^2+x^5}{x+x^2+x^4} = \frac{1+x^2+x^5}{x(1+x+x^3)} = \frac{1}{x}\left[\frac{1+x^2+x^5}{1+x+x^3}\right] = \frac{1}{x}\left[1+x+x^3+x^5+\cdots\right] = \frac{1}{x}+1+x^2+x^4+\cdots$$

$$(a+b)^2 = a^2+b^2$$
$$(a+b)^4 = a^4+b^4$$

$$\frac{1+x^2+x^5}{1+x+x^3} = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

$a_1=1$ $a_2=0$ $a_3=1$ $a_4=0$ $a_5=1$

$$1+x^2+x^5 = (1+x+x^3)(1+x+x^3+x^5+\cdots)$$

Second method Geometric series $\frac{1}{1-u} = 1+u+u^2+u^3+u^4+\cdots$

$$\frac{1+x^2+x^5}{1+(x+x^3)} = (1+x^2+x^5)\left(1+(x+x^3)+(x+x^3)^2+(x+x^3)^3+(x+x^3)^4+(x+x^3)^5+\cdots\right)$$

$$= (1+x^2+x^5)\left(1+(x+x^3)+(x^2+x^6)+(x^3+x^5+\cdots)+(x^4+\cdots)+(x^5+\cdots)+\cdots\right)$$

$(x^3+3x^5+3x^7+x^9)$

$$= (1+x^2+x^5)(1+x+x^2+x^4+\cdots)$$
$$= 1+x+x^3+x^5+\cdots$$

$$f(x) = \frac{1}{x}(1+x+x^3+x^5+\cdots) = \frac{1}{x}+1+x^2+x^4+\cdots$$

$F = \mathbb{F}_2 = \{0,1\}$ for the time being

The irreducible (monic) polynomials in $F[x]$:

primitive

not primitive

degree	irred. polys
1	x, $x+1$
2	x^2+x+1
3	x^3+x+1, x^3+x^2+1
4	x^4+x+1, x^4+x^3+1, $x^4+x^3+x^2+x+1$
....	....

$x^2,\ x^2+1,\ x^2+x,\ x^2+x+1$ all poly's of degree 2.

$x^2 = x\cdot x$, $x^2+1 = (x+1)(x+1)$, $x^2+x = x(x+1)$

$x^4+x^2+1 = (x^2+x+1)^2$

See MacWilliams & Sloane, The Theory of Error-Correcting Codes for more extensive lists of irreducible polynomials.

What are all the cyclic (linear) binary codes of length 7? There are exactly 8 of them. (why?)

- subspace of F^7, $F = \mathbb{F}_2 = \{0,1\}$
- invariant under cyclic shift $(a_0,a_1,a_2,a_3,a_4,a_5,a_6) \longmapsto (a_6,a_0,a_1,\dots,a_5)$ $\quad a_i \in F$

A linear code $\mathcal{C} \subseteq F^n$ is cyclic iff its dual code $\mathcal{C}^\perp \subseteq F^n$ is also cyclic.

$\dim \mathcal{C} + \dim \mathcal{C}^\perp = n.$

eg. $\{(0000000)\}$

$\{0000000,\ 1111111\}$

$F^7 \leftarrow g(x)=1,\quad h(x) = x^7-1$

$\{\text{words in } F^7 \text{ of even weight}\} = \langle 1100000,\ 1010000,\ 1001000,\ 1000100,\ 1000010,\ 1000001\rangle$

$$\begin{array}{r} 1110100 \\ 0101000 \\ \hline 1011100 \end{array}$$

Hamming $[7,4,3]_2$ code $\mathcal{H} = \langle 1101000,\ 0110100,\ \dots,\ 1010001\rangle$ (all cyclic shifts of 1101000 span this code)

$\dim \mathcal{H} = 4$, $|\mathcal{H}| = 2^4 = 16$:

1	codeword of weight	0
7	...	3
7	...	4
1	...	7

Its dual $\mathcal{H}^\perp$, $\dim \mathcal{H}^\perp = 3$ is a $[7,3,4]_2$-code.

$\mathcal{H}^\perp$ has 1 codeword of weight 0
7 ... 4

$\mathcal{H}^\perp = \mathcal{H} \cap \langle 1111111\rangle^\perp$

$\mathcal{H}' = \langle 1011000,\ 0101100,\ \dots,\ 0110001\rangle$ also $[7,4,3]_2$

$\mathcal{H}'^\perp$ also $[7,3,4]_2$.

$\overset{n = \text{length}}{x^{7}} - 1 \in F[x]$ $\quad x^7 - 1 = \underset{\text{i.e. } x+1}{(x-1)}(x^6 + x^5 + x^4 + x^3 + x^2 + 1) = (x-1)\underbrace{(x^3 + x + 1)}_{(x-\alpha)(x-\alpha^2)(x-\alpha^4)}\underbrace{(x^3 + x^2 + 1)}_{(x-\beta)(x-\beta^2)(x-\beta^4)}$

actually $x^7 + 1$ $\quad F = \mathbb{F}_2$

If $E = \mathbb{F}_q$, $\quad x^q - x = \underset{x-0}{x}(x-1)(x-a_2)(x-a_3)\cdots(x-a_q)$

$a_0 = 0,\ a_1 = 1,\ a_2, a_3, \cdots, a_q$ are the field elements.

i.e. $x^{q-1} - 1$ has $q-1$ distinct roots which are the nonzero field elements.

If $\alpha \in \mathbb{F}_8$ is a root of $x^3 + x + 1$

$$\mathbb{F}_8 = \mathbb{F}_2[\alpha] = \{a_0 + a_1\alpha + a_2\alpha^2 : a_0, a_1, a_2 \in \mathbb{F}_2\}$$
$$= \{0, 1, \alpha, \alpha+1, \alpha^2, \alpha^2+1, \alpha^2+\alpha, \alpha^2+\alpha+1\}$$

$(u+v)^2 = u^2 + v^2$

$(uv)^2 = u^2 v^2$

Squaring is an automorphism of $\mathbb{F}_8$.

If $f(x) \in \mathbb{F}_p[x]$ is irreducible of degree d, then $\mathbb{F}_p[x]/(f(x)) \cong \mathbb{F}_{p^d} = \mathbb{F}_p[\beta]$ where β is a root of $f(x)$.

$= \{a_0 + a_1\beta + a_2\beta^2 + \cdots + a_{d-1}\beta^{d-1} : a_i \in \mathbb{F}_p\}$

(β generates $\mathbb{F}_{p^d} \supset \mathbb{F}_p$ as an algebra)

If in fact $\mathbb{F}_{p^d} = \{0, 1, \beta, \beta^2, \beta^3, \cdots, \beta^{p^d-2}\}$ then we say β is a <u>primitive element</u> and we say $f(x)$ is a primitive polynomial.

If $f(x) = x^4 + x^3 + x^2 + x + 1$ and $\beta \in \mathbb{F}_{16} = \mathbb{F}_{2^4}$ is a root of $f(x)$ then $\beta^5 = 1$ since β is a root of $f(x)$

$$\beta^5 - 1 = (\beta - 1)\underbrace{(\beta^4 + \beta^3 + \beta^2 + \beta + 1)}_{0} = 0$$

$0, 1, \beta, \beta^2, \beta^3, \beta^4, 1, \beta, \beta^2, \cdots$ doesn't give all of $\mathbb{F}_{16}$.

There are eight ways to factor $x^7-1 = g(x)h(x)$ in $\mathbb{F}_2[x]$.
In each case $g(x)$ is a generator poly. and $h(x)$ is a parity check poly. for a cyclic code of length 7 over $\mathbb{F}_2 = \{0,1\} = F$

Cyclic (linear) codes $\longleftrightarrow$ ideals in $F[x]/(x^n-1)$

$g(x) = 1$, $h(x) = x^7-1$ gives F^7

$g(x) = x^7-1$, $h(x) = 1$ gives $\{0000000\}$

$g(x) = x+1$, $h(x) = x^6+x^5+x^4+x^3+x^2+x+1$ gives all words of even weight i.e. $\langle 1100000, 1010000, \dots, 1000001 \rangle$

$g(x) = x^6+x^5+\dots+1$, $h(x) = x+1$ gives $\langle 1111111 \rangle = \{0000000, 1111111\}$

$g(x) = 1+x+x^3$, $h(x) = 1+x^2+x^3+x^4$ gives $\mathcal{H}$ $[7,4,3]_2$ code

BCH bound : a lower bound for performance of a cyclic code.

Consider a cyclic code of length n over $\mathbb{F}_q$, i.e. an ideal in $\mathbb{F}_q[x]/(x^n-1)$ with gen. poly. $g(x)$, parity check poly. $h(x)$, $x^n-1 = g(x)h(x)$, $g(x)$ primitive, β root of $g(x)$ in $\mathbb{F}_{q^r}$, $r = \deg g(x)$, and $\beta, \beta^2, \dots, \beta^{s-1}$ are roots of $g(x)$, then the code has min. distance $\geq s$.

For Hamming $[7,4,3]_2$ code, β root of $g(x) = 1+x+x^3 \in F[x]$, $\beta \in \mathbb{F}_8 = \mathbb{F}_2[\beta]$
Also β^2 - - - - - - - - by Freshman's Dream

$$1+\beta+\beta^3 = 0$$
$$(1+\beta+\beta^3)^2 = 1+\beta^2+\beta^6 = 0 = 1+\beta^2+(\beta^2)^3$$

$\Rightarrow$ $\mathcal{H}$ has min. dist. ≥ 3.

BCH : R.C. Bose
Dijen Ray-Chaudhuri
Hocquenghem

The Gilbert-Varshamov Bound (GV-bound): a lower bound for existence of good codes

$A_q(n,d) = \max. |\mathcal{C}|$ s.t. $\mathcal{C} \subseteq A^n$, $|A|=q$ with min. distance $\geq d$ i.e. $d(w,w') \geq d$ for all $w \neq w'$ in $\mathcal{C}$.

$e = \lfloor \frac{d-1}{2} \rfloor$ = error-correcting capability.

Ball of radius r in A^n centered at "0" $\in A^n$ has cardinality $|B_r(0)| = \sum_{k=0}^{r} \binom{n}{k} (q-1)^k$

$1 = |B_0| < |B_1| < |B_2| < \cdots < |B_n| = |A^n| = q^n$

Hamming bound: $A_q(n,d) \leq \frac{q^n}{|B_e|}$: balls of radius e centered at codewords $w \in \mathcal{C}$ are required to be disjoint

$$\bigsqcup_{w \in \mathcal{C}} B_e(w) \subseteq A^n. \Rightarrow |\mathcal{C}| \cdot |B_e(w)| \leq q^n$$

$$\Rightarrow |\mathcal{C}| \leq \frac{q^n}{|B_e(w)|}$$

In the other direction the GV-bound

$$A_q(n,d) \geq \frac{q^n}{|B_{d-1}(0)|}$$

so $\frac{q^n}{|B_{d-1}(0)|} \leq A_q(n,d) \leq \frac{q^n}{|B_e(0)|}$

We regard the GV bound as an existence proof only.

<u>Proof</u>: Let $\mathcal{C} \subseteq A^n$ be any q-ary code with $|\mathcal{C}| = A_q(n,d)$. We claim

$$\bigcup_{w \in \mathcal{C}} B_{d-1}(w) \supseteq A^n.$$

Codes satisfying this condition by greedy construction. But such codes are usually not practical because membership & decoding are not efficient.

If not, there exists $w' \in A^n$, $w' \notin \bigcup_{w \in \mathcal{C}} B_{d-1}(w)$ so $d(w', w) > d-1$ for all $w \in \mathcal{C}$.

But then $\mathcal{C} \cup \{w'\}$ has min. distance $\geq d$. This contradicts the maximality of $\mathcal{C}$ among all q-ary codes of length n having min. distance d.

So $|\mathcal{C}| \, |B_{d-1}(0)| \geq |A^n| = q^n$. $\square$

Recommended viewing:
YouTube videos on coding & info. theory (including alg. geom. codes) by Mary Wootters

Asymptotic version of GV-bound due to Shannon:

Fix $0 < \delta < 1$. $\quad |B_{\delta n}(0)| \approx |A^n|^{h_q(\delta)} = q^{n h_q(\delta)}, \quad 0 \le h_q(\delta) \le 1.$

$$\log_q |B_{\delta n}(0)| \approx n h_q(\delta)$$

This is a true asymptotic formula: for fixed q and $\delta \in (0,1)$,

$$\frac{\log_q |B_{\delta n}(0)|}{n h_q(\delta)} \longrightarrow 1 \quad \text{as } n \to \infty.$$

$$\log_q |B_{\delta n}(0)| \sim n h_q(\delta).$$

More precisely,

$$n h_q(\delta) - o(1) \le \log_q |B_{\delta n}(0)| \le n h_q(\delta)$$

The q-ary entropy function

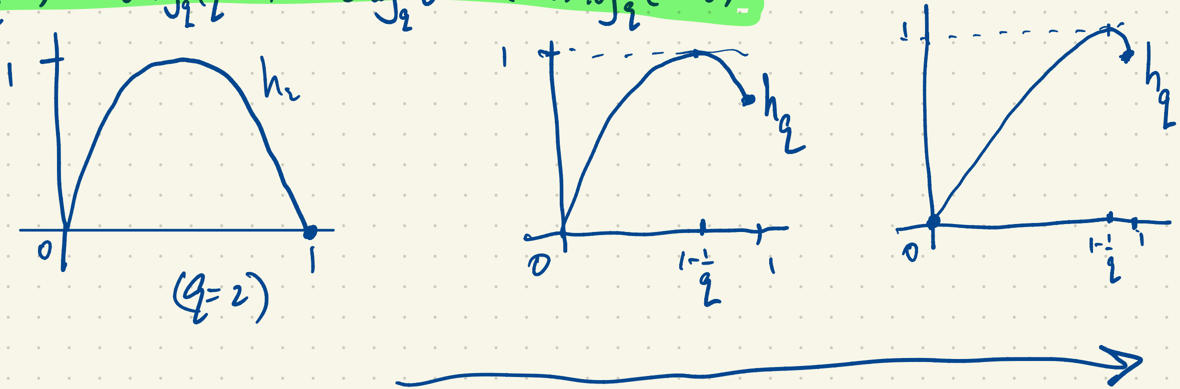
binary entropy function $\quad h_2(q) = -\delta \log_2 \delta - (1-\delta)\log_q(1-\delta) = \delta \log_2 \frac{1}{\delta} + (1-\delta)\log_2 \frac{1}{1-\delta}$

Eg. consider a random stream of information coming from letters in A, $|A| = q$, $A = \{x_1, \dots, x_q\}$

with letter x_1 having frequency $1-p$
x_i .. .. $\frac{p}{q-1}$ $(2 \le i \le q)$ & $(1-p) + \frac{p}{q-1} + \frac{p}{q-1} + \cdots + \frac{p}{q-1} = 1.$

$H(\text{this stream})$ (single char. from) $= \sum p \log \frac{1}{p} = -\sum p \log p = -(1-p)\log(1-p) - \cancel{(q-1)}\frac{p}{\cancel{q-1}} \cdot \log \frac{p}{q-1} = p\log(q-1) - p\log p - (1-p)\log(1-p)$

$$h_q(\delta) = \delta \log_q(q-1) - \delta\log_q\delta - (1-\delta)\log_q(1-\delta)$$



increasing q

$h_q(x) = x\log_q(q-1) + \frac{\log 2}{\log q} h_2(x)$ Let $x \to 1^-$.

$h_q(x) \longrightarrow \log_q(q-1)$ as $x \to 1^-$.

For long codes ($n \gg 0$) over a fixed alphabet $|A| = q$, we consider the information rate $R = \frac{\log_q|C|}{n} = \frac{k}{n}$ in the case of an $[n,k]_q$-code

relative distance $\delta = \frac{d}{n}$

relative error-correcting capability $\frac{e}{n} \approx \frac{d}{2n} = \frac{\delta}{2}$

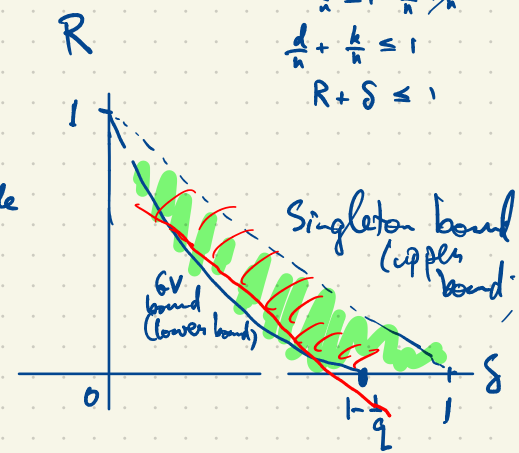
Singleton bound:

$d \le n-k+1$

$\frac{d}{n} \le 1 - \frac{k}{n} + \cancel{\frac{1}{n}}$

$\frac{d}{n} + \frac{k}{n} \le 1$

$R + \delta \le 1$



$R \ge 1 - h_q(\delta)$

For $q \ge 49$ we have a new (1982) lower bound for asymptotically good explicit codes using algebraic geometry (Tsfasman, Vladut, Zink)

The 1982 theorem literally says: There exists a family X_i of algebraic curves over $\mathbb{F}_q$ $(i=1,2,3,\dots)$ such that X_i has n_i+1 (rational) points over $\mathbb{F}_q$, genus g_i with

$$\frac{g_i}{n_i} \to \frac{1}{\sqrt{q}-1} \quad \text{as } i\to\infty.$$

The Reed-Solomon codes come from the simplest curve of all, the projective line $\mathbb{P}^1F = F \cup \{\infty\}$ (F: field) of genus 0.

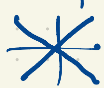
S^2 $\qquad$ $T^2 = S^1\times S^1$

$g=0$ $\qquad$ $g=1$ $\qquad$ $g=2$ $\qquad$ $g=3$

On a curve X, $\Omega_X = \{\text{smooth global differential 1-forms}\}$ is a vector space of dimension $\dim \Omega_X = g$.
The number of $\mathbb{F}_q$-points on the curve (if it's defined over $\mathbb{F}_q$), N_q, satisfies $|N_q-(q+1)| \le 2g\sqrt{q}$
Hasse-Weil bound.

Eg. $\mathbb{P}^1F$ has $N=q+1$ points, $g=0$

For a irreducible plane curve of degree d (defined by a poly. equation of degree d) has genus $g \le \binom{d-1}{2} = \frac{(d-1)(d-2)}{2}$
(equality for smooth curve; $g = \binom{d-1}{2} - \sum_{\text{singular points}} (\ \ )$.

$y^2=x^2 \iff y=\pm x$  has $2q+1$ points

$y^2-x^2=(y+x)(y-x)=0$

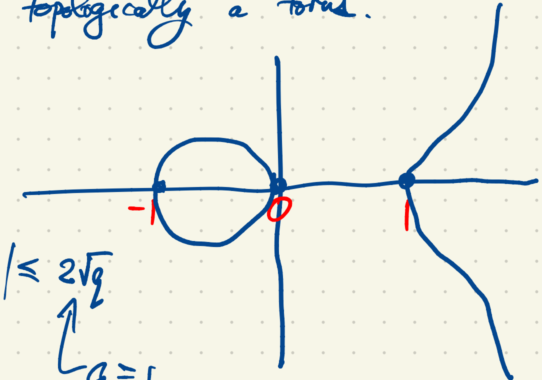
Irreducible conic:
$y=x^2$ $\qquad (t,t^2)$ $\qquad t\in F$ $\qquad$ genus $g=0$
plus one point at infinity
$q+1$ points

Smooth curve of degree $d=3$ (elliptic curve) has genus $g = \binom{3-1}{2} = 1$ is topologically a torus.

eg. $y^2 =$ cubic in x with no repeated roots is an elliptic curve.

$y^2 = x^3 - x = x(x+1)(x-1)$

$g=1$ (torus)



H·W bound : over $\mathbb{F}_q$ the number of points satisfies $|\underline{N} - (\underline{q}+1)| \le 2\sqrt{q}$

$\cancel{q>3}$ $\quad q =$ prime $p \geq 5$ $\qquad g=1$

$N = q+1$ if $q =$ prime $p \equiv 3 \mod 4$

$q+1 \pm \varepsilon$ if $q =$ prime $p \equiv 1 \mod 4$

$|\varepsilon| \le 2\sqrt{q}$

Projective line $\mathbb{P}^1 F = F \cup \{\infty\} = X$

We consider rational functions $f(x) \in F(X)$ defined on a curve X (e.g. $X = \mathbb{P}^1 F$)

Eg. $F = \mathbb{F}_7 = \{0,1,2,3,4,5,6\}$

A B C D E F G ∞ $= X$
0 1 2 3 4 5 6

Formal integer-linear combinations of points A, B, C, D, E, F, G, ∞ on X are called <u>divisors</u> as a book-keeping device for keeping track of zeroes and poles of functions on X.

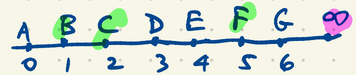
eg. $f(x) = (x-1)(x-2)(x-5) = x^3+3x^2-x-3 = x^3+3x^2+6x+4$ has simple zeroes at B, C, F and a triple pole at ∞

Near ∞, $z = \frac{1}{x}$; $f(x) = f(\frac{1}{z}) = \frac{1}{z^3} + \frac{3}{z^2} + \frac{6}{z} + 4 = \frac{1+3z+6z^2+4z^3}{z^3}$ so f has a triple pole at $z=0$ (i.e. at $x=\infty$).

The divisor of $f(x)$ is $B+C+F-3\infty =: \mathrm{Div}(f)$ (Sometimes abbreviated (f)).

More complicated: $f(x) = \dfrac{(x-1)^2(x+3)^4(x+5)}{(x+2)(x+1)^3}$

$$\mathrm{Div}(f) = 2B + 4E + C - F - 3G - 3\infty$$



$= X$

$z = \frac{1}{x}$

$$f(x) = f\left(\tfrac{1}{z}\right) = \frac{(\frac{1}{z}-1)^2(\frac{1}{z}+3)^4(\frac{1}{z}+5)}{(\frac{1}{z}+2)(\frac{1}{z}+1)^3} \cdot \frac{z^7}{z^7} = \frac{(1-z)^2(1+3z)^4(1+5z)}{(1+2z)(1+z)^3 \cdot z^3}$$

triple pole at $z=0$ (i.e. at $x=\infty$)

The degree of $D = \sum_i m_i P_i$ is $\deg D = \sum_i m_i$, $(m_i \in \mathbb{Z})$

For any $f(x) \in F(X)$, $\deg(\mathrm{Div}\, f) = 0$. (equally many poles as zeroes)

Given a divisor $D = \sum_i m_i P_i - \sum_j n_j Q_j$ $(m_i, n_j \geq 1)$,

we consider the vector space $\mathcal{L}(D) = \{ f(x) \in F(X) :$ f has a zero of multiplicity at least m_i at P_i,
f has a pole of order at most n_j at Q_j,
and possibly other zeroes but no other poles$\}$.

In the case of $\mathbb{P}^1 F = F \cup \{\infty\}$ consider $D = k\infty$, $\deg D = k$.

$\mathcal{L}(k\infty) = \{ f(x) \in F(x) :$ f has a pole of order at most k at ∞; no other poles$\}$
(there can be as many zeroes as you like)

$\dim \mathcal{L}(k\infty) = k+1$. $\mathcal{L}(D) = \{ f : \mathrm{Div} f + D \geq 0 \}$

$\mathcal{L}(-k\infty) = \{$polynomials in x of degree at most $k\}$ $= \{ a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k : a_i \in F \}$.
has basis $\{1, x, x^2, \cdots, x^k\}$.

The Riemann-Roch theorem gives a relation for determining $\ell(D) = \dim \mathcal{L}(D)$.

$\underbrace{\ell(D)} - \underbrace{\ell(K-D)}_{\text{non-negative integers}} = \deg D - g + 1$. where K is a "canonical divisor"

$\ell(D) \geq \deg D - g + 1$ is Riemann's bound