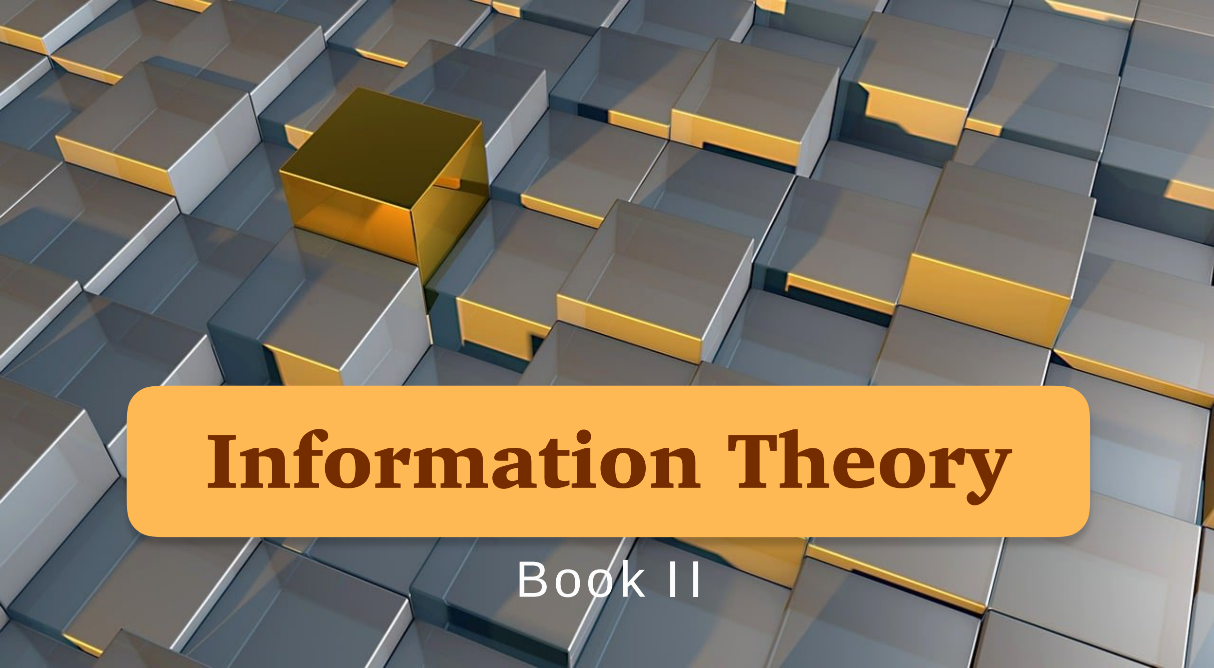
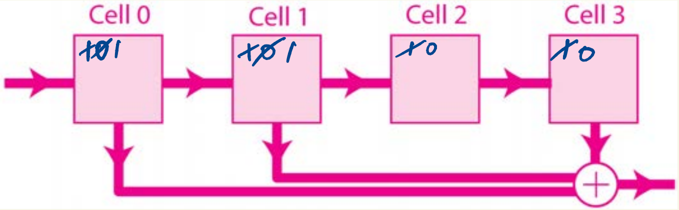


Information Theory

Book II

Eg. an infinite stream of bits $a_0 a_1 a_2 a_3 a_4 \cdots$ $(a_i \in F)$ can be encoded eg. represent the plaintext bitstream as a $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \in \mathbb{F}_2[[x]]$

$F[[x]]$ = ring of (formal) power series in x with coefficients in F.



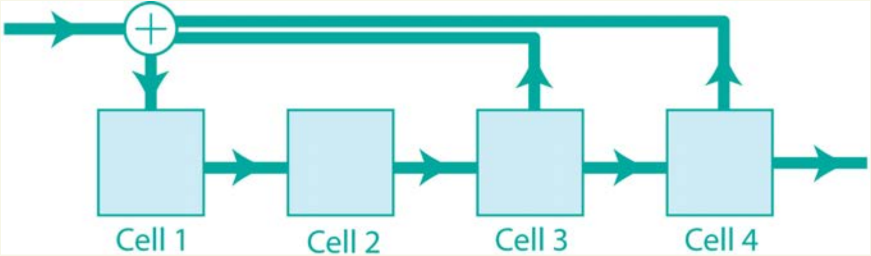
10110...

Eg. consider an input bitstream ~~110011~~011110010... which is encoded by the shift register above to obtain the output bitstream 101100101...

Compare: this is equivalent to multiplication by $1+x+x^3$:

$$(1+x+x^3)(1+x+x^4+x^5+x^7+x^8+x^9+x^{10}+x^{13}+\ldots) = 1 + x^2 + x^3 + x^6 + x^8 + \cdots$$

Decoding of this data is accomplished using backward shift registers eg.



which performs division by $1+x+x^3$ in $\mathbb{F}_2((x))$

Symbolic rings and fields

$$\begin{tikzcd}
& F((x)) \arrow[dl, dash] \arrow[dr, dash] & \\
F(x) \arrow[dr, dash] & & F[[x]] \arrow[dl, dash] \\
& F[x] &
\end{tikzcd}$$

field of Laurent series... → $F((x))$

field of (power) series in x with coeffs in F. → $F[[x]]$

field of rational "functions" (actually symbolic ~~expressions~~ expressions) in x with coeffs in F — $F(x)$

$F[x]$ = ring of polys in x with coeffs in F

polynomials vs. polynomial functions

eg. $\mathbb{F}_3 = \{0,1,2\} = \mathbb{Z}/3\mathbb{Z}$

eg. $f(x) = 2+x+x^3 \in \mathbb{F}_3[x]$ is a polynomial of degree 3.

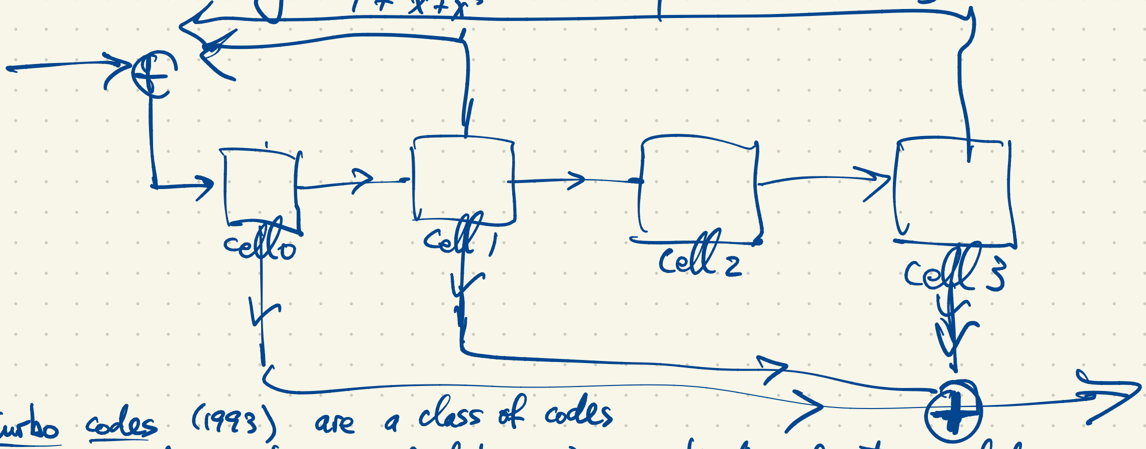
$g(x) = 2+2x \in \mathbb{F}_3[x]$ is a polynomial of degree 1.

a	$f(a)$	$g(a)$
0	2	2
1	1	1
2	0	0

$f(x), g(x)$ are distinct poly's but they epresent the same function $\mathbb{F}_3 \to \mathbb{F}_3$.

eg. $f(x) = \dfrac{1+x+x^3}{x+x^2} \in \mathbb{F}_2(x)$

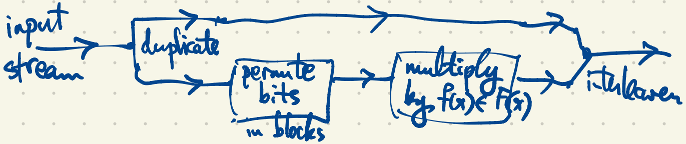
Multiplication by any rational function can be implemented using a single shift register e.g. multiplication by $\frac{1+x+x^3}{1+x^2+x^3}$ is implemented using the shift register



Turbo codes (1993) are a class of codes used for encoding streams of data using combinations of gates including

- multiplication by a rational function in F(x)
- splitters & interleavers
- permutations
- puncturing

eg.



$F(x) \subset F((x))$ eg. for $F = \mathbb{F}_2 = \{0,1\}$

First method

$\Downarrow$

$$f(x) = \frac{1+x^2+x^5}{x+x^2+x^4} = \frac{1+x^2+x^5}{x(1+x+x^3)} = \frac{1}{x}\left[\frac{1+x^2+x^5}{1+x+x^3}\right] = \frac{1}{x}\left[1+x+x^3+x^5+\cdots\right] = \frac{1}{x}+1+x^2+x^4+\cdots$$

$$(a+b)^2 = a^2+b^2$$
$$(a+b)^4 = a^4+b^4$$

$$\frac{1+x^2+x^5}{1+x+x^3} = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

$a_1=1$ $a_2=0$ $a_3=1$ $a_4=0$ $a_5=1$

$$1+x^2+x^5 = (1+x+x^3)(1+x \quad +x^3 \quad +x^5+\cdots)$$

Second method

Geometric series $\frac{1}{1-u} = 1+u+u^2+u^3+u^4+\cdots$

$$\frac{1+x^2+x^5}{1+(x+x^3)} = (1+x^2+x^5)\left(1+(x+x^3)+(x+x^3)^2+(x+x^3)^3+(x+x^3)^4+(x+x^3)^5+\cdots\right)$$
$$=(1+x^2+x^5)\left(1+(x+x^3)+(x^2+x^6)+(x^3+x^5+\cdots)+(x^4+\cdots)+(x^5+\cdots)+\cdots\right)$$
$$(x^3+3x^5+3x^7+x^9)$$
$$=(1+x^2+x^5)(1+x+x^2+x^4+\cdots)$$
$$=1+x+x^3+x^5+\cdots$$

$$f(x) = \frac{1}{x}(1+x+x^3+x^5+\cdots) = \frac{1}{x}+1+x^2+x^4+\cdots$$

$F = \mathbb{F}_2 = \{0,1\}$ for the time being

The irreducible (monic) polynomials in $F[x]$:

primitive

not primitive

degree	irred. polys
1	x, $x+1$
2	x^2+x+1
3	x^3+x+1, x^3+x^2+1
4	x^4+x+1, x^4+x^3+1, $x^4+x^3+x^2+x+1$
....	....

x^2, x^2+1, x^2+x, x^2+x+1 all poly's of degree 2.

$x^2 = x\cdot x$, $x^2+1 = (x+1)(x+1)$, $x^2+x = x(x+1)$

$x^4+x^2+1 = (x^2+x+1)^2$

See MacWilliams & Sloane, The Theory of Error-Correcting Codes for more extensive lists of irreducible polynomials.

What are all the cyclic (linear) binary codes of length 7? There are exactly 8 of them. (why?)

- subspace of F^7, $F = \mathbb{F}_2 = \{0,1\}$
- invariant under cyclic shift $(a_0,a_1,a_2,a_3,a_4,a_5,a_6) \longmapsto (a_6,a_0,a_1,\dots,a_5)$ $a_i \in F$

A linear code $\mathcal{C} \subseteq F^n$ is cyclic iff its dual code $\mathcal{C}^\perp \subseteq F^n$ is also cyclic.

$\dim \mathcal{C} + \dim \mathcal{C}^\perp = n$.

eg. $\{(0000000)\}$

$\{0000000, 1111111\}$

$F^7 \leftarrow g(x)=1$, $h(x)=x^7-1$

$\{\text{words in } F^7 \text{ of even weight}\} = \langle 1100000, 1010000, 1001000, 1000100, 1000010, 1000001 \rangle$

$$\begin{array}{r} 1110100 \\ 0101000 \\ \hline 1011100 \end{array}$$

Hamming $[7,4,3]_2$ code $\mathcal{H} = \langle 1101000, 0110100, \dots, 1010001 \rangle$ (all cyclic shifts of 1101000 span this code)

$\dim \mathcal{H} = 4$, $|\mathcal{H}| = 2^4 = 16$:
1 codeword of weight 0
7 3
7 4
1 7

Its dual $\mathcal{H}^\perp$, $\dim \mathcal{H}^\perp = 3$ is a $[7,3,4]_2$-code.
$\mathcal{H}^\perp$ has 1 codeword of weight 0
7 4

$\mathcal{H}^\perp = \mathcal{H} \cap \langle 1111111 \rangle^\perp$

$\mathcal{H}' = \langle 1011000, 0101100, \dots, 0110001 \rangle$ also $[7,4,3]_2$

$\mathcal{H}'^\perp$ also $[7,3,4]_2$.

$\overset{n = \text{length}}{x^{7}-1} \in F[x]$ $\qquad x^{7}-1 = \underset{\text{i.e. } x+1}{(x-1)}(x^{6}+x^{5}+x^{4}+x^{3}+x^{2}+1) = (x-1)\underbrace{(x^{3}+x+1)}_{(x-\alpha)(x-\alpha^{2})(x-\alpha^{4})}\underset{(x-\beta)(x-\beta^{2})(x-\beta^{4})}{(x^{3}+x^{2}+1)}$

actually $x^{7}+1$ $\qquad F=\mathbb{F}_2$

If $E=\mathbb{F}_q$, $\quad x^{q}-x = \underset{x-0}{x}(x-1)(x-a_2)(x-a_3)\cdots(x-a_q)$

$a_0=0,\ a_1=1,\ a_2, a_3, \cdots, a_q$ are the field elements.

i.e. $x^{q-1}-1$ has $q-1$ distinct roots which are the nonzero field elements.

If $\alpha \in \mathbb{F}_8$ is a root of $x^{3}+x+1$

$$\mathbb{F}_8 = \mathbb{F}_2[\alpha] = \{a_0+a_1\alpha+a_2\alpha^{2} : a_0, a_1, a_2 \in \mathbb{F}_2\}$$
$$= \{0, 1, \alpha, \alpha+1, \alpha^{2}, \alpha^{2}+1, \alpha^{2}+\alpha, \alpha^{2}+\alpha+1\}$$

Squaring is an automorphism of $\mathbb{F}_8$.

$(u+v)^{2} = u^{2}+v^{2}$

$(uv)^{2} = u^{2}v^{2}$

If $f(x) \in \mathbb{F}_p[x]$ is irreducible of degree d, then $\mathbb{F}_p[x]/(f(x)) \cong \mathbb{F}_{p^d} = \mathbb{F}_p[\beta]$ where β is a root of $f(x)$.

$$= \{a_0+a_1\beta+a_2\beta^{2}+\cdots+a_{d-1}\beta^{d-1} : a_i \in \mathbb{F}_p\}$$

(β generates $\mathbb{F}_{p^d} \supset \mathbb{F}_p$ as an algebra)

If in fact $\mathbb{F}_{p^d} = \{0, 1, \beta, \beta^{2}, \beta^{3}, \cdots, \beta^{p^d-2}\}$ then we say β is a <u>primitive element</u> and we say $f(x)$ is a primitive polynomial.

If $f(x) = x^{4}+x^{3}+x^{2}+x+1$ and $\beta \in \mathbb{F}_{16} = \mathbb{F}_{2^4}$ is a root of $f(x)$ then $\beta^{5}=1$ since β is a root of $f(x)$

$$\beta^{5}-1 = (\beta-1)\underset{0}{\cancel{(\beta^{4}+\beta^{3}+\beta^{2}+\beta+1)}} = 0$$

$0, 1, \beta, \beta^{2}, \beta^{3}, \beta^{4}, 1, \beta, \beta^{2}, \cdots$ doesn't give all of $\mathbb{F}_{16}$.

There are eight ways to factor $x^7-1 = g(x)h(x)$ in $\mathbb{F}_2[x]$.
In each case $g(x)$ is a generator poly. and $h(x)$ is a parity check poly. for a cyclic code of length 7 over $\mathbb{F}_2 = \{0,1\} = F$

Cyclic (linear) codes ⟷ ideals in $F[x]/(x^n-1)$

$g(x)=1$, $h(x)=x^7-1$ gives F^7

$g(x)=x^7-1$, $h(x)=1$ gives $\{0000000\}$

$g(x)=x+1$, $h(x)=x^6+x^5+x^4+x^3+x^2+x+1$ gives all words of even weight i.e. $\langle 1100000, 1010000, \cdots, 1000001\rangle$

$g(x)=x^6+x^5+\cdots+1$, $h(x)=x+1$ gives $\langle 1111111\rangle = \{0000000, 1111111\}$

$g(x)=1+x+x^3$, $h(x)=1+x^2+x^3+x^4$ gives $\mathcal{H}$ $[7,4,3]_2$ code

BCH bound : a lower bound for performance of a cyclic code.

Consider a cyclic code of length n over F_q, i.e. an ideal in $\mathbb{F}_q[x]/(x^n-1)$ with gen. poly. $g(x)$, parity check poly. $h(x)$, $x^n-1=g(x)h(x)$, $g(x)$ primitive, β root of $g(x)$ in $\mathbb{F}_{q^r}$, $r=\deg g(x)$, and $\beta, \beta^2, \cdots, \beta^{s-1}$ are roots of $g(x)$, then the code has min. distance $\geq s$.

For Hamming $[7,4,3]_2$ code, β root of $g(x) = 1+x+x^3 \in F[x]$, $\beta \in \mathbb{F}_8 = \mathbb{F}_2[\beta]$
Also β^2 - - - - - - - - by Freshman's Dream

$$1+\beta+\beta^3=0$$
$$(1+\beta+\beta^3)^2 = 1+\beta^2+\beta^6 = 0 = 1+\beta^2+(\beta^2)^3$$

⟹ $\mathcal{H}$ has min. dist. ≥ 3.

BCH : R.C. Bose
Dijen Ray-Chaudhuri
Hocquengham

The Gilbert-Varshamov Bound (GV-bound) : a lower bound for existence of good codes

$A_q(n,d) = \max. |\mathcal{C}|$ s.t. $\mathcal{C} \subseteq A^n$, $|A| = q$ with min. distance $\geq d$ i.e. $d(w,w') \geq d$ for all $w \neq w'$ in $\mathcal{C}$.

$e = \lfloor \frac{d-1}{2} \rfloor$ = error-correcting capability.

Ball of radius r in A^n centered at "0" $\in A^n$ has cardinality $|B_r(0)| = \sum_{k=0}^{r} \binom{n}{k} (q-1)^k$

$$1 = |B_0| < |B_1| < |B_2| < \cdots < |B_n| = |A^n| = q^n$$

Hamming bound: $A_q(n,d) \leq \frac{q^n}{|B_e|}$: balls of radius e centered at codewords $w \in \mathcal{C}$ are required to be disjoint

$$\bigsqcup_{w \in \mathcal{C}} B_e(w) \subseteq A^n. \Rightarrow |\mathcal{C}| \cdot |B_e(w)| \leq q^n$$

$$\Rightarrow |\mathcal{C}| \leq \frac{q^n}{|B_e(w)|}$$

In the other direction the GV-bound

$$A_q(n,d) \geq \frac{q^n}{|B_{d-1}(0)|}$$

so $\frac{q^n}{|B_{d-1}(0)|} \leq A_q(n,d) \leq \frac{q^n}{|B_e(0)|}$

We regard the GV bound as an existence proof only.

Proof : Let $\mathcal{C} \subseteq A^n$ be any q-ary code with $|\mathcal{C}| = A_q(n,d)$. We claim

$$\bigcup_{w \in \mathcal{C}} B_{d-1}(w) \supseteq A^n.$$

Codes satisfying this condition by greedy construction. But such codes are usually not practical because membership & decoding are not efficient.

If not, there exists $w' \in A^n$, $w' \notin \bigcup_{w \in \mathcal{C}} B_{d-1}(w)$ so $d(w', w) > d-1$ for all $w \in \mathcal{C}$.
But then $\mathcal{C} \cup \{w'\}$ has min. distance $\geq d$. This contradicts the maximality of $\mathcal{C}$ among all q-ary codes of length n having min. distance d.

So $|\mathcal{C}| \, |B_{d-1}(0)| \geq |A^n| = q^n$.

□

Recommended viewing:
YouTube videos on coding & info. theory (including alg. geom. codes) by Mary Wootters

Asymptotic version of GV-bound due to Shannon:

Fix $0 < \delta < 1$. $\quad |B_{\delta n}(0)| \approx |A^n|^{h_q(\delta)} = q^{n h_q(\delta)}, \quad 0 \le h_q(\delta) \le 1.$

$$\log_q |B_{\delta n}(0)| \approx n h_q(\delta)$$

This is a true asymptotic formula: for fixed q and $\delta \in (0,1)$,

$$\frac{\log_q |B_{\delta n}(0)|}{n h_q(\delta)} \longrightarrow 1 \quad \text{as } n \to \infty.$$

$$\log_q |B_{\delta n}(0)| \sim n h_q(\delta).$$

More precisely,

$$n h_q(\delta) - o(1) \le \log_q |B_{\delta n}(0)| \le n h_q(\delta)$$

The q-ary entropy function

binary entropy function $\quad h_2(q) = -\delta \log_2 \delta - (1-\delta) \log_q (1-\delta) = \delta \log_2 \frac{1}{\delta} + (1-\delta) \log_2 \frac{1}{1-\delta}$

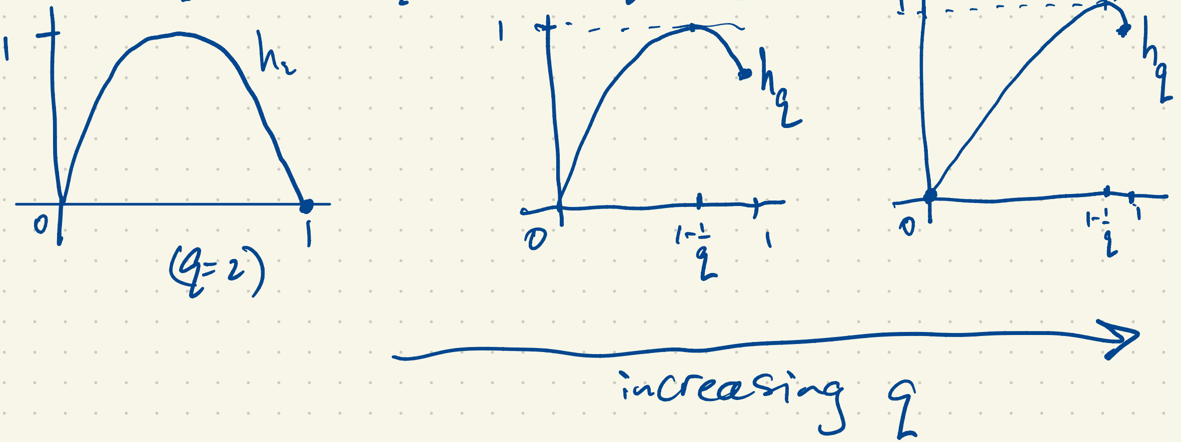
Eg. consider a random stream of information coming from letters in A, $|A| = q$, $A = \{x_1, \cdots, x_q\}$

with letter x_1 having frequency $1-p$

x_i .. .. $\frac{p}{q-1}$ $(2 \le i \le q)$ & $(1-p) + \frac{p}{q-1} + \frac{p}{q-1} + \cdots + \frac{p}{q-1} = 1.$

$$H(\overset{\text{single char. from}}{\text{this stream}}) = \sum p \log \frac{1}{p} = -\sum p \log p = -(1-p)\log(1-p) - \cancel{(q-1)}\frac{p}{\cancel{q-1}} \cdot \log \frac{p}{q-1} = p \log(q-1) - p \log p - (1-p)\log(1-p)$$

$$h_q(\delta) = \delta\log_q(q-1) - \delta\log_q\delta - (1-\delta)\log_q(1-\delta)$$



$h_q(x) = x\log_q(q-1) + \frac{\log 2}{\log q}h_2(x)$ Let $x \to 1^-$.

$h_q(x) \to \log_q(q-1)$ as $x \to 1^-$.

For long codes ($n \gg 0$) over a fixed alphabet $|A| = q$, we consider the information rate $R = \frac{\log_q|C|}{n} = \frac{k}{n}$ in the case of an $[n,k]_q$-code

relative distance $\delta = \frac{d}{n}$

relative error-correcting capability $\frac{e}{n} \sim \frac{d}{2n} = \frac{\delta}{2}$

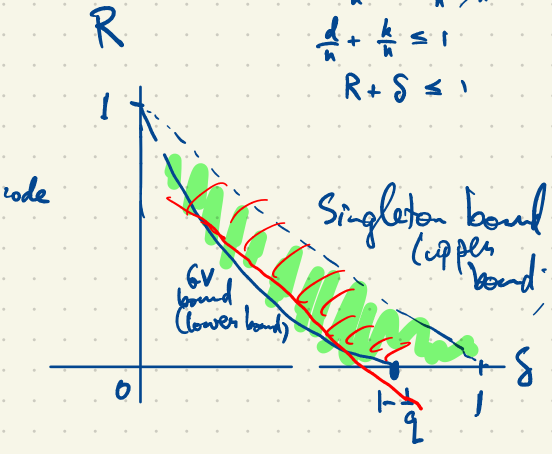
Singleton bound:

$d \leq n-k+1$

$\frac{d}{n} \leq 1 - \frac{k}{n} + \cancel{\frac{1}{n}}$

$\frac{d}{n} + \frac{k}{n} \leq 1$

$R + \delta \leq 1$



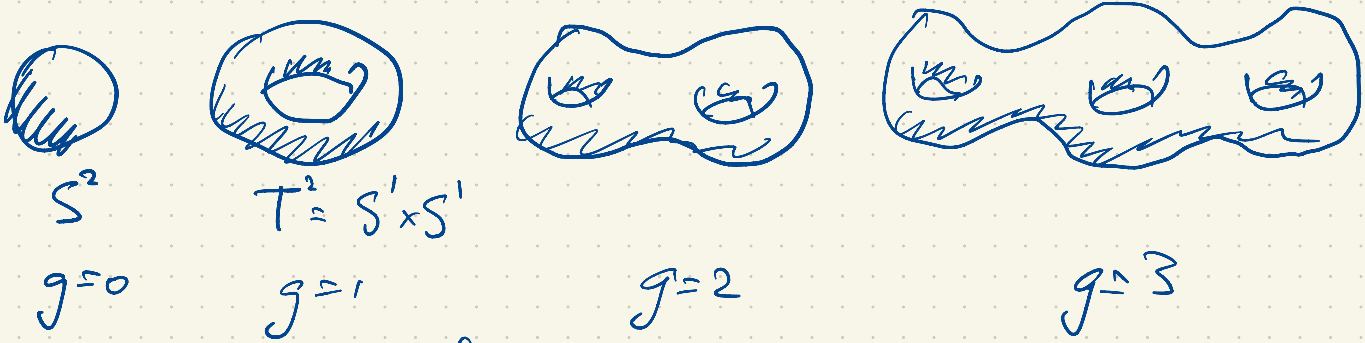
$R \geq 1 - h_q(\delta)$

For $q \geq 49$ we have a new (1982) lower bound for asymptotically good explicit codes using algebraic geometry (Tsfasman, Vladut, Zink)

The 1982 theorem literally says: There exists a family X_i of algebraic curves over $\mathbb{F}_q$ $(i=1,2,3,\dots)$ such that X_i has n_i+1 (rational) points over $\mathbb{F}_q$, genus g_i with

$$\frac{g_i}{n_i} \longrightarrow \frac{1}{\sqrt{q}-1} \quad \text{as } i\to\infty.$$

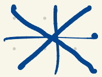
The Reed-Solomon codes come from the simplest curve of all, the projective line $\mathbb{P}^1F = F\cup\{\infty\}$ (F: field) of genus 0.



On a curve X, $\Omega_X = \{\text{smooth global differential 1-forms}\}$ is a vector space of dimension $\dim \Omega_X = g$.
The number of $\mathbb{F}_q$-points on the curve (if it's defined over $\mathbb{F}_q$), N_q, satisfies $|N_q-(q+1)|\le 2g\sqrt{q}$
Hasse-Weil bound.

Eg. $\mathbb{P}^1F$ has $N=q+1$ points, $g=0$

For a irreducible plane curve of degree d (defined by a poly. equation of degree d) has genus $g\le \binom{d-1}{2} = \frac{(d-1)(d-2)}{2}$
(equality for smooth curve; $g=\binom{d-1}{2} - \sum_{\text{singular points}}(\ )$.

$y^2=x^2 \Longleftrightarrow y=\pm x$  has $2q+1$ points

$y^2-x^2=(y+x)(y-x)=0$

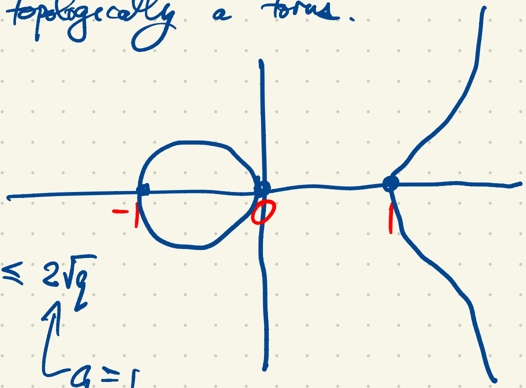
Irreducible conic:
$y=x^2$ $\quad (t,t^2)$ $\quad t\in F$ $\quad$ genus $g=0$
plus one point at infinity
$q+1$ points

Smooth curve of degree $d=3$ has genus $g = \binom{3-1}{2} = 1$ is topologically a torus.
(elliptic curve)

eg. $y^2 =$ cubic in x with no repeated roots is an elliptic curve.

$y^2 = x^3 - x = x(x+1)(x-1)$

$g=1$ (torus)



H·W bound : over $\mathbb{F}_q$ the number of points satisfies $|\underline{N} - \underline{(q+1)}| \leq 2\sqrt{q}$

$g=1$

~~$q>3$~~ $q =$ prime $p \geq 5$

$N = q+1$ if $q =$ prime $p \equiv 3 \mod 4$

$q+1 \pm \varepsilon$ if $q =$ prime $p \equiv 1 \mod 4$

$|\varepsilon| \leq 2\sqrt{q}$