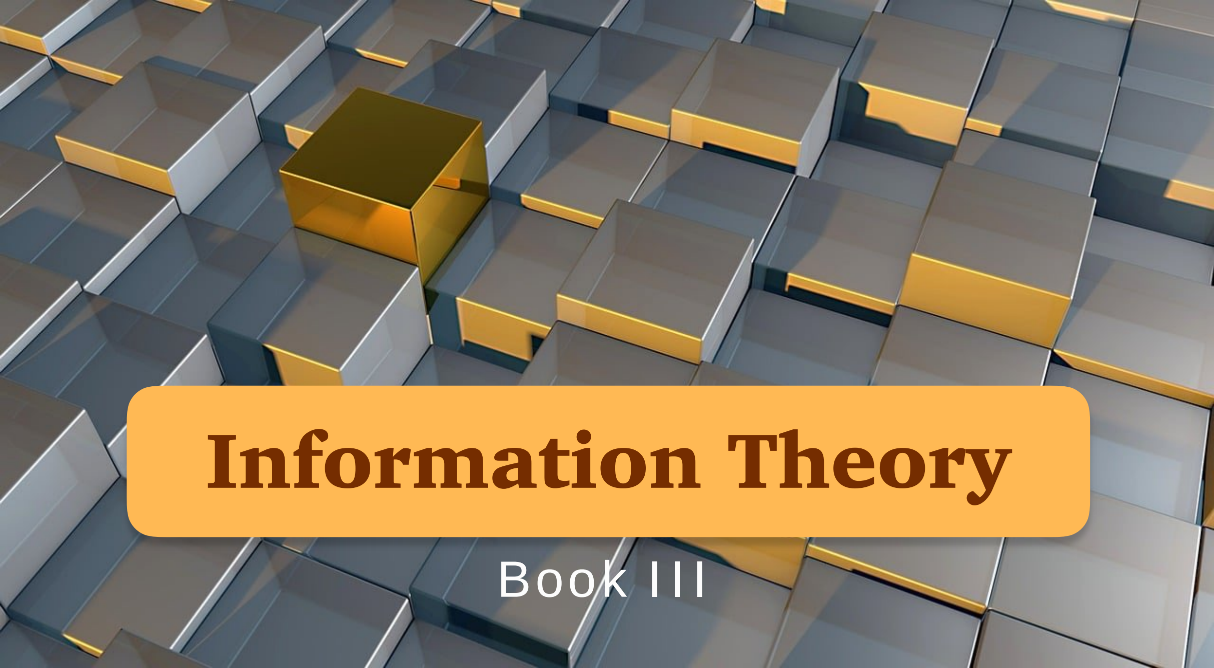


Information Theory

Book III

Spin state of an electron (disregard position and momentum) is an example of a qubit, which is a vector $|\psi\rangle \in \mathbb{C}^2 = \left\{ \begin{pmatrix}\alpha\\ \beta\end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$.

ket notation

Standard basis of $\mathbb{C}^2$: $|+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$
"spin up" "spin down"

$|\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \alpha|+\rangle + \beta|-\rangle$, $|\alpha|^2 + |\beta|^2 = 1$.

An electron in this spin state is in a superposition of spin up and spin down states

A measurement of an electron in this spin state yields a single bit of classical information:

- spin up, with probability $|\alpha|^2$;
- spin down, with probability $|\beta|^2$.

This says what happens when we measure with respect to the z-axis. (For measurement in a different direction/axis, we'll say later.)

As soon as the measurement is taken, the spin state collapses; all knowledge of α, β is then lost.

A linear functional on $\mathbb{C}^2$ is a linear transformation
$\langle\phi| : \mathbb{C}^2 \to \mathbb{C}$
bra notation

$$\langle\phi| = (\gamma\ \delta) : \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \mapsto (\gamma\ \delta)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \underbrace{\gamma\alpha + \delta\beta}_{= \langle\phi|\psi\rangle} \in \mathbb{C}.$$

Dual basis:

$\langle +| = |+\rangle^* = (1\ 0)$ (conjugate transpose)

$\langle -| = |-\rangle^* = (0\ 1)$

$$|\psi\rangle^* = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}^* = (\bar{\alpha}\ \bar{\beta}) = \bar{\alpha}\langle +| + \bar{\beta}\langle -|$$

$$\langle +|\psi\rangle = \langle +|\left(\alpha|+\rangle + \beta|-\rangle\right) = \alpha$$

$$\langle -|\psi\rangle = \beta.$$

Spin states are unit vectors in $\mathbb{C}^2$ i.e. $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$.

i.e. in $\mathbb{R}^4$

$\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$

so $|\psi\rangle \in S^3$ = unit sphere in $\mathbb{R}^4$.

$\left.\begin{array}{l}\alpha = \alpha_1 + \alpha_2 i\\ \beta = \beta_1 + \beta_2 i\end{array}\right\} \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$

Any time we measure a spin state $|\psi\rangle \in S^3$, it collapses.

But it is possible to perform certain reversible operations $|\psi\rangle \mapsto A|\psi\rangle$ where A is a 2×2 unitary matrix $\left(AA^* = A^*A = I = \begin{pmatrix}1&0\\0&1\end{pmatrix}\right)$ over $\mathbb{C}$.

Special examples of unitary matrices are scalar matrices $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$

These perform an operation on $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ whose only effect is to alter the phase of α, β

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longmapsto A|\psi\rangle = \begin{pmatrix} \lambda\alpha \\ \lambda\beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$\lambda = e^{i\theta}$ $\quad (\theta \in [0, 2\pi))$

which has no physical significance. For this reason the so-called density matrix

$$\underbrace{\underbrace{|\psi\rangle}_{2\times 1} \underbrace{\langle\psi|}_{1\times 2}}_{2\times 2} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \end{pmatrix} = \begin{pmatrix} \alpha\bar{\alpha} & \alpha\bar{\beta} \\ \beta\bar{\alpha} & \beta\bar{\beta} \end{pmatrix}$$

which holds all the physically significant information of ~~the~~ the single qubit.

Hermitian 2×2 matrix

$H \in \mathbb{C}^{2\times 2}$ $\quad$ (2×2 complex matrix)

satisfying $H^* = H$

The map $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longmapsto \begin{pmatrix} \lambda\alpha \\ \lambda\beta \end{pmatrix}$ does not change this density matrix.

Entanglement typically occurs when we include multiple electrons in our system.

Start by reviewing statistical dependence works:

Let's say we take a random individual A from a population.

Imagine the population is 40% male, 60% female; 30% short, 70% tall.

Sampling by selecting one person gives two bits: MS, MT, FS, or FT.

Combinations of attributes: 12%, 28%, 18%, 42% if gender is independent of height.

		Height		
		S	T	
Gender	M	0.12	0.28	0.4
	F	0.18	0.42	0.6
		0.3	0.7	1

In this example, gender and height are independent.

$$\begin{bmatrix}0.4\\0.6\end{bmatrix}\begin{bmatrix}0.3 & 0.7\end{bmatrix} = \begin{bmatrix}0.12 & 0.28\\0.18 & 0.42\end{bmatrix}$$

Outer product of two vectors. is a rank 1.

More typical distribution

	S	T	
M	0.1	0.3	0.4
F	0.2	0.4	0.6
	0.3	0.7	1

The matrix $\begin{bmatrix}0.1 & 0.3\\0.2 & 0.4\end{bmatrix}$ has rank 2.

In this second example gender and height are (statistically) dependent.

If one electron has (spin) state $|\psi_1\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathbb{C}^2$ and a second electron has spin state $|\psi_2\rangle = \begin{pmatrix}\gamma\\ \delta\end{pmatrix} \in \mathbb{C}^2$

$|\alpha|^2 + |\beta|^2 = 1$ $\qquad$ $|\gamma|^2 + |\delta|^2 = 1$

the pair of electrons has state $|\psi_{12}\rangle = \alpha_{11}|{+}{+}\rangle + \alpha_{12}|{+}{-}\rangle + \alpha_{21}|{-}{+}\rangle + \alpha_{22}|{-}{-}\rangle \in \mathbb{C}^4$

$\alpha_{ij} \in \mathbb{C}$, $\quad |\alpha_{11}|^2 + |\alpha_{12}|^2 + |\alpha_{21}|^2 + |\alpha_{22}|^2 = 1.$

prob. of both electrons having spin up

If the two electrons are not entangled then

$$\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\begin{pmatrix}\gamma & \delta\end{pmatrix} \qquad \text{rank } 1.$$

If the matrix has rank 2 then the two electrons are entangled.

Eg. $|\psi\rangle = \frac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle)$ i.e. $\begin{pmatrix}\frac{1}{\sqrt{2}} & 0\\ 0 & \frac{1}{\sqrt{2}}\end{pmatrix}$

$|\psi'\rangle = \frac{1}{\sqrt{2}}(|{+}{-}\rangle + |{-}{+}\rangle)$ i.e. $\begin{pmatrix}0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 0\end{pmatrix}$

} Examples of EPR pairs

One way to talk about the spin state of a set of n electrons is

$$|\psi\rangle = \sum_{\substack{i_1\in\{0,1\}\\ i_2\in\{0,1\}\\ \vdots\\ i_n\in\{0,1\}}} \alpha_{i_1 i_2 i_3\cdots i_n} |\underbrace{\pm \pm \pm \cdots \pm}_{\text{all } 2^n \text{ combinations of } \pm}\rangle \in \mathbb{C}^{2^n} \qquad \sum |\alpha_{i_1 i_2 \cdots i_n}|^2 = 1$$

$(\alpha_{i_1 i_2 \cdots i_n} : i_1, i_2, \dots, i_n \in \{0,1\})$ is a $\underbrace{2\times 2\times 2\times \cdots \times 2}_{n}$ array or tensor

$\mathbb{C}^{2^n} = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}}$ tensor product. Take basis $|+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

has basis
$$|++ \cdots ++\rangle = |+\rangle \otimes |+\rangle \otimes \cdots \otimes |+\rangle$$
$$|++ \cdots +-\rangle = |+\rangle \otimes |+\rangle \otimes \cdots \otimes |-\rangle$$
$$\vdots$$
$$|-- \cdots -\rangle = |-\rangle \otimes |-\rangle \otimes \cdots \otimes |-\rangle.$$

In $\mathbb{C}^m \otimes \mathbb{C}^n \cong \mathbb{C}^{mn}$ every vector is a sum of at most $\min\{m,n\}$ pure tensors. The corresponding result for $\mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2} \otimes \cdots \otimes \mathbb{C}^{m_k}$ is not known and extremely hard. (In Algebraic Geometry look up Higher Secant varieties of Segre Varieties)

More generally if $v_i \in \mathbb{C}^2$ $(i = 1, 2, \cdots, n)$ then $v_1 \otimes v_2 \otimes \cdots \otimes v_n \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. (pure tensors) simple

$$\mathbb{C}^2 \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^2 \qquad \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 = \mathbb{C}^{2^n}$$
$$(v_1, \cdots, v_n) \longmapsto v_1 \otimes v_2 \otimes \cdots \otimes v_n$$

this map is multilinear i.e. linear in each argument separately.

Bell's Theorem
Gleason's Theorem
Kochen-Specker Theorem

Recall: the spin state of a single electron is a cubit $|\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathbb{C}^2$, $|\alpha|^2 + |\beta|^2 = 1$.

Standard basis $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

spin up/down with respect to the z-axis

How do we measure the spin in an arbitrary direction?

In the vertical direction we make use of basis $|+_z\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|-_z\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ basis of eigenvectors for the Pauli spin operator $\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.

$$\sigma_z |+_z\rangle = |+_z\rangle$$

$$\sigma_z |-_z\rangle = -|-_z\rangle$$

Any electron with spin state $|\psi\rangle = \alpha|+_z\rangle + \beta|-_z\rangle$ can be measured in the vertical direction

$$\sigma_z|\psi\rangle = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha\\ -\beta\end{pmatrix}$$

Follow this by a linear functional eg. $|+_z\rangle^* = \langle +_z|$

$\langle +_z|\sigma_z|\psi\rangle = (1\ \ 0)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = (1\ \ 0)\begin{pmatrix}\alpha\\ -\beta\end{pmatrix} = \alpha$, the amplitude for the electron to be spin up.

Once the measurement is performed, the state collapses into that spin state $|\psi\rangle \longmapsto |+_z\rangle$.

$$\langle +_z|\underbrace{\sigma_z|+_z\rangle}_{|+_z\rangle} = 1.$$

If we measure $|\psi\rangle \longmapsto \langle +_z|\sigma_z|\psi\rangle$ and find spin down, the state collapses to spin down $|-_z\rangle$

$\langle +_z|\sigma_z|\psi\rangle = -\beta$, $|-\beta|^2 = |\beta|^2$

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Eigenvectors: $|+_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $|-_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $|+_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$, $|-_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $|+_z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|-_z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ eigenvalues $+1, -1$

Hermitian: $\sigma^* = \sigma$

eg. $\sigma_y \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} = |+_y\rangle$

$\sigma_y \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ i \end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = -|-_y\rangle$

If we measure an electron having spin $|+_y\rangle$ (in the pos. y-direction) with respect to the x-axis

$\langle +_x|+_y\rangle = \frac{1}{\sqrt{2}}(1 \;\; 1)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+i \\ 1+i \end{pmatrix}$ $\left|\frac{1+i}{2}\right|^2 = \frac{2}{4} = \frac{1}{2}$ $|a+bi|^2 = a^2+b^2$

Density matrix of $|\psi\rangle$ is $|\psi\rangle\langle\psi| = |\psi\rangle|\psi\rangle^* = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}(\bar\alpha \;\; \bar\beta) = \begin{pmatrix} \alpha\bar\alpha & \alpha\bar\beta \\ \bar\alpha\beta & \beta\bar\beta \end{pmatrix}$ $|\alpha|^2+|\beta|^2=1$

is Hermitian having eigenvalues 1, 0 ; corresponding eigenvectors $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $|\psi'\rangle = \begin{pmatrix} \bar\beta \\ -\bar\alpha \end{pmatrix}$

$|\psi\rangle\langle\psi|\psi\rangle = |\psi\rangle$ since $\langle\psi|\psi\rangle = 1$ $\langle\psi'|\psi\rangle = (\beta \; -\alpha)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha\beta - \alpha\beta = 0.$

$|\psi\rangle\langle\psi|\psi'\rangle = 0. = 0|\psi'\rangle.$ $\langle\psi|\psi\rangle = 1 = \operatorname{tr} |\psi\rangle\langle\psi|$

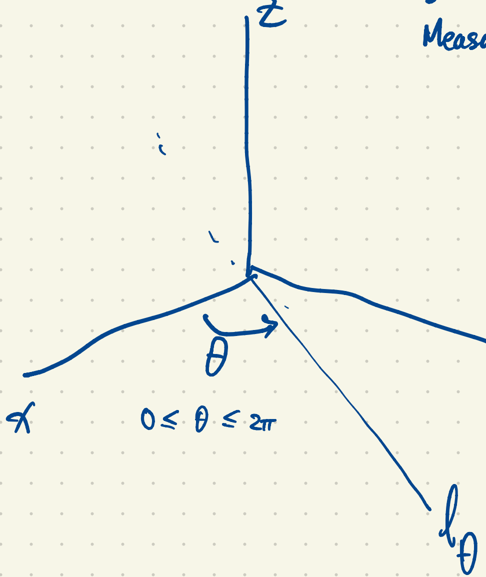
$\operatorname{tr} AB = \operatorname{tr} BA.$

What is the corresponding Pauli spin operator in an arbitrary direction $n = (n_x, n_y, n_z) \in \mathbb{R}^3$

$n_x^2 + n_y^2 + n_z^2 = 1.$

$\sigma_n = n \cdot \sigma = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z$

$\sigma = (\sigma_x, \sigma_y, \sigma_z)$



Measure spin wrt line l_θ in x-y plane at angle θ as shown.

$n = (\cos\theta, \sin\theta, 0)$

σ_θ : Pauli spin operator for the direction n

$$\sigma_\theta = n \cdot (\sigma_x, \sigma_y, \sigma_z) = \cos\theta \underbrace{\sigma_x}_{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} + \sin\theta \underbrace{\sigma_y}_{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}} = \begin{bmatrix} 0 & \cos\theta - i\sin\theta \\ \cos\theta + i\sin\theta & 0 \end{bmatrix} = \begin{bmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{bmatrix}$$

using de Moivre's formula $e^{i\theta} = \cos\theta + i\sin\theta$

Eigenvectors $|+_\theta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix}$

$|-_\theta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\theta/2} \\ -e^{i\theta/2} \end{bmatrix}$

Check: $\sigma_\theta |+_\theta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{bmatrix} \begin{bmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix} = |+_\theta\rangle$ eigenvector with eigenvalue +1

$\sigma_\theta |-_\theta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{bmatrix} \begin{bmatrix} e^{-i\theta/2} \\ -e^{i\theta/2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix} = -|-_\theta\rangle$ - - - - - -1.

The map $l_\theta \longmapsto \left\{ \begin{array}{l} |+_\theta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix} \\ |-_\theta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\theta/2} \\ -e^{i\theta/2} \end{bmatrix} \end{array} \right.$

is 2-to-1.

Spin vectors go around "full circle" in $\mathbb{C}^2$ as θ goes from 0 to 4π; the "+" direction of l_θ goes twice around a circle in this same θ-interval.

If we measure an electron in spin state $|\psi\rangle = \alpha|+_\theta\rangle + \beta|-_\theta\rangle$ with respect to the direction l_θ, we get $|+_\theta\rangle$ with prob. $|\alpha|^2$, spin $|-_\theta\rangle$ with prob. $|\beta|^2$.

Spin states actually lie in S^3 = unit vector in $\mathbb{C}^2$ which is a double cover of of $SO_3(\mathbb{R})$ = {rotations of $\mathbb{R}^3$ about the origin} = $\{3\times 3$ real matrices A : $AA^T = I$, $\det A = 1\}$.

Bob has an electron in spin state $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle = \binom{\alpha}{\beta} \in \mathbb{C}^2$, $|\alpha|^2 + |\beta|^2 = 1$.
He wants to send this to Alice. Bob doesn't know α, β and he cannot directly measure them.

Analogy: transporting Captain Kirk from enterprise to planet's surface.

In advance of this teleportation process, Alice and Bob have stockpiled some EPR pairs