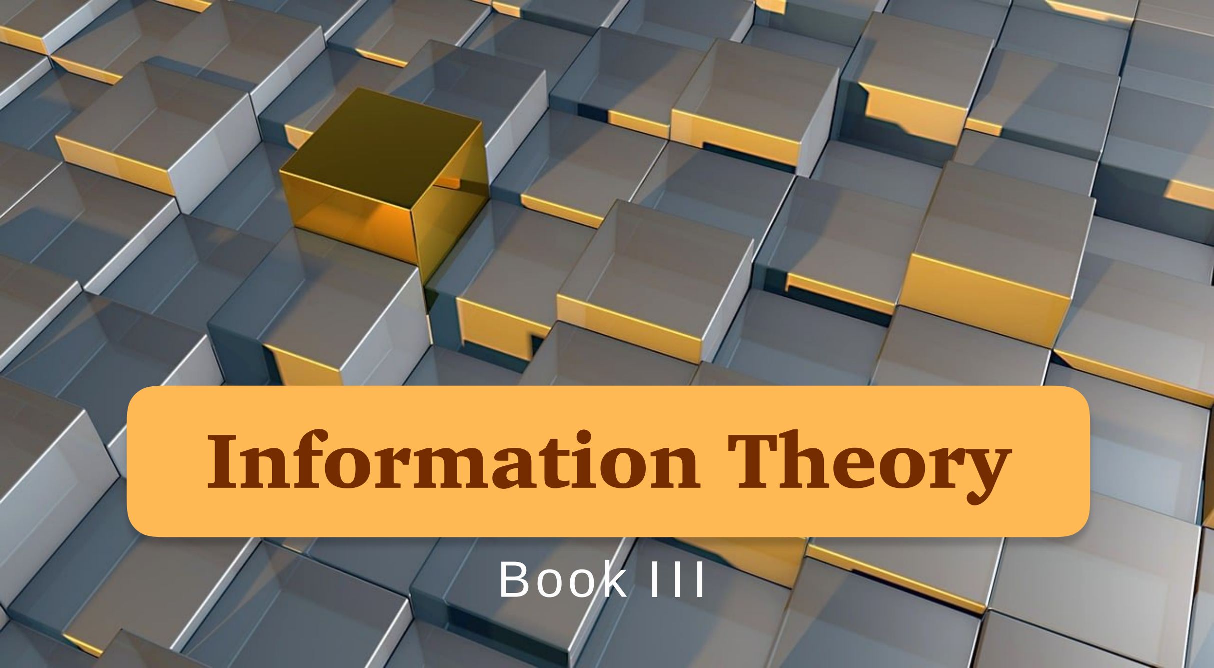

Information Theory

Book III

Spin state of an electron (disregard position and momentum) is an example of a qubit, which is a vector $|\psi\rangle \in \mathbb{C}^2 = \left\{ \begin{pmatrix}\alpha\\ \beta\end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$.
ket notation

$|\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \alpha|+\rangle + \beta|-\rangle$, $|\alpha|^2 + |\beta|^2 = 1$.

Standard basis of $\mathbb{C}^2$: $|+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$
"spin up" "spin down"

An electron in this spin state is in a superposition of spin up and spin down states

A linear functional on $\mathbb{C}^2$ is a linear transformation
$\langle\phi| : \mathbb{C}^2 \to \mathbb{C}$
bra notation

$\langle\phi| = (\gamma\ \delta) : \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \longmapsto (\gamma\ \delta)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \underbrace{\gamma\alpha + \delta\beta}_{= \langle\phi|\psi\rangle} \in \mathbb{C}$.

Dual basis:
$\langle+| = |+\rangle^* = (1\ 0)$ (conjugate transpose)
$\langle-| = |-\rangle^* = (0\ 1)$

$|\psi\rangle^* = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}^* = (\bar{\alpha}\ \bar{\beta}) = \bar{\alpha}\langle+| + \bar{\beta}\langle-|$

$\langle+|\psi\rangle = \langle+|\left(\alpha|+\rangle + \beta|-\rangle\right) = \alpha$
$\langle-|\psi\rangle = \beta$.

A measurement of an electron in this spin state yields a single bit of classical information:
- spin up, with probability $|\alpha|^2$;
- spin down, with probability $|\beta|^2$.

This says what happens when we measure with respect to the z-axis. (For measurement in a different direction/axis, we'll say later.)
As soon as the measurement is taken, the spin state collapses; all knowledge of α, β is then lost.

Spin states are unit vectors in $\mathbb{C}^2$ i.e. $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$.
i.e. in $\mathbb{R}^4$
so $|\psi\rangle \in S^3$ = unit sphere in $\mathbb{R}^4$.

$\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$

$\left.\begin{array}{l}\alpha = \alpha_1 + \alpha_2 i\\ \beta = \beta_1 + \beta_2 i\end{array}\right\} \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$

Any time we measure a spin state $|\psi\rangle \in S^3$, it collapses.
But it is possible to perform certain reversible operations $|\psi\rangle \longmapsto A|\psi\rangle$ where A is a 2×2 unitary matrix $\left(AA^* = A^*A = I = \begin{pmatrix}1&0\\0&1\end{pmatrix}\right)$ over $\mathbb{C}$.

Special examples of unitary matrices are scalar matrices $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$

These perform an operation on $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ whose only effect is to alter the phase of α, β

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longmapsto A|\psi\rangle = \begin{pmatrix} \lambda\alpha \\ \lambda\beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$\lambda = e^{i\theta}$ $(\theta \in [0, 2\pi))$

which has no physical significance. For this reason the so-called density matrix

$$\underbrace{\underbrace{|\psi\rangle}_{2\times 1} \underbrace{\langle\psi|}_{1\times 2}}_{2\times 2} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \end{pmatrix} = \begin{pmatrix} \alpha\bar{\alpha} & \alpha\bar{\beta} \\ \beta\bar{\alpha} & \beta\bar{\beta} \end{pmatrix}$$

which holds all the physically significant information of the single qubit.

Hermitian 2×2 matrix

$H \in \mathbb{C}^{2\times 2}$ (2×2 complex matrix)

satisfying $H^* = H$

The map $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longmapsto \begin{pmatrix} \lambda\alpha \\ \lambda\beta \end{pmatrix}$ does not change this density matrix.

Entanglement typically occurs when we include multiple electrons in our system.

Start by reviewing statistical dependence works:

Let's say we take a random individual A from a population.

Imagine the population is 40% male, 60% female; 30% short, 70% tall.

Sampling by selecting one person gives two bits: MS, MT, FS, or FT.

Combinations of attributes: 12%, 28%, 18%, 42% if gender is independent of height.

Height

Gender	S	T	
M	0.12	0.28	0.4
F	0.18	0.42	0.6
	0.3	0.7	1

In this example, gender and height are independent.

$$\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.12 & 0.28 \\ 0.18 & 0.42 \end{bmatrix}$$

Outer product of two vectors. is a rank 1.

More typical distribution

	S	T	
M	0.1	0.3	0.4
F	0.2	0.4	0.6
	0.3	0.7	1

The matrix $\begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{bmatrix}$ has rank 2.

In this second example gender and height are (statistically) dependent.

If one electron has (spin) state $|\psi_1\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathbb{C}^2$ and a second electron has spin state $|\psi_2\rangle = \begin{pmatrix}\gamma\\ \delta\end{pmatrix} \in \mathbb{C}^2$

$|\alpha|^2+|\beta|^2=1$ $\qquad$ $|\gamma|^2+|\delta|^2=1$

the pair of electrons has state $|\psi_{12}\rangle = \alpha_{11}|{+}{+}\rangle + \alpha_{12}|{+}{-}\rangle + \alpha_{21}|{-}{+}\rangle + \alpha_{22}|{-}{-}\rangle \in \mathbb{C}^4$

$\alpha_{ij} \in \mathbb{C}$, $\quad |\alpha_{11}|^2 + |\alpha_{12}|^2 + |\alpha_{21}|^2 + |\alpha_{22}|^2 = 1.$

prob. of both electrons having spin up

If the two electrons are <u>not</u> entangled then

$$\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\begin{pmatrix}\gamma & \delta\end{pmatrix} \qquad \text{rank } 1.$$

If the matrix has rank 2 then the two electrons are entangled.

$$\left.\begin{array}{l} \text{Eg. } |\psi\rangle = \frac{1}{\sqrt{2}}\left(|{+}{+}\rangle + |{-}{-}\rangle\right) \quad \text{i.e. } \begin{pmatrix}\frac{1}{\sqrt{2}} & 0\\ 0 & \frac{1}{\sqrt{2}}\end{pmatrix} \\ |\psi'\rangle = \frac{1}{\sqrt{2}}\left(|{+}{-}\rangle + |{-}{+}\rangle\right) \quad \text{i.e. } \begin{pmatrix}0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 0\end{pmatrix} \end{array}\right\} \text{Examples of EPR pairs}$$

One way to talk about the spin state of a set of n electrons is

$$|\psi\rangle = \sum_{\substack{i_1\in\{0,1\}\\ i_2\in\{0,1\}\\ \vdots\\ i_n\in\{0,1\}}} \alpha_{i_1 i_2 i_3\cdots i_n} |\underbrace{\pm\pm\pm\cdots\pm}_{\text{all } 2^n \text{ combinations of } \pm}\rangle \in \mathbb{C}^{2^n} \qquad \sum |\alpha_{i_1 i_2\cdots i_n}|^2 = 1$$

$\left(\alpha_{i_1 i_2 \cdots i_n} : i_1, i_2, \cdots, i_n \in \{0,1\}\right)$ is a $\underbrace{2\times2\times2\times\cdots\times2}_{n}$ array or tensor

$\mathbb{C}^{2^n} = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}}$ tensor product. Take basis $|+\rangle = \binom{1}{0}$, $|-\rangle = \binom{0}{1}$

has basis $|++ + \cdots ++\rangle = |+\rangle \otimes |+\rangle \otimes \cdots \otimes |+\rangle$
$|+++\cdots +-\rangle = |+\rangle \otimes |+\rangle \otimes \cdots \otimes |-\rangle$
$\vdots$
$|---\cdots -\rangle = |-\rangle \otimes |-\rangle \otimes \cdots \otimes |-\rangle$.

More generally if $v_i \in \mathbb{C}^2$ $(i = 1, 2, \cdots, n)$
then $v_1 \otimes v_2 \otimes \cdots \otimes v_n \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. (pure tensors)
simple

$$\begin{array}{ccc} \mathbb{C}^2 \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^2 & & \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 = \mathbb{C}^{2^n} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (v_1, \cdots, v_n) & \longmapsto & v_1 \otimes v_2 \otimes \cdots \otimes v_n \end{array}$$

this map is multilinear
i.e. linear in each argument separately.

In $\mathbb{C}^m \otimes \mathbb{C}^n \cong \mathbb{C}^{mn}$
every vector is a sum of at most $\min\{m,n\}$ pure tensors.
The corresponding result for $\mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2} \otimes \cdots \otimes \mathbb{C}^{m_k}$
is not known and extremely hard.
(In Algebraic Geometry look up Higher Secant varieties of Segre Varieties)

Bell's Theorem
Gleason's Theorem
Kochen-Specker Theorem

Recall: the spin state of a single electron is a cubit $|\psi\rangle = \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \in \mathbb{C}^2$, $|\alpha|^2 + |\beta|^2 = 1$.

Standard basis $|0\rangle = \begin{pmatrix}1 \\ 0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0 \\ 1\end{pmatrix}$

spin up/down with respect to the z-axis

How do we measure the spin in an arbitrary direction?

In the vertical direction we make use of basis $|+_z\rangle = \begin{pmatrix}1 \\ 0\end{pmatrix}$, $|-_z\rangle = \begin{pmatrix}0 \\ 1\end{pmatrix}$ basis of eigenvectors for the Pauli spin operator $\sigma_z = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$.

$$\sigma_z |+_z\rangle = |+_z\rangle$$

$$\sigma_z |-_z\rangle = -|-_z\rangle$$

Any electron with spin state $|\psi\rangle = \alpha|+_z\rangle + \beta|-_z\rangle$ can be measured in the vertical direction

$$\sigma_z|\psi\rangle = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix}\alpha \\ -\beta\end{pmatrix}$$

Follow this by a linear functional eg. $|+_z\rangle^* = \langle +_z|$

$$\langle +_z|\sigma_z|\psi\rangle = (1\ 0)\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = (1\ 0)\begin{pmatrix}\alpha \\ -\beta\end{pmatrix} = \alpha,$$ the amplitude for the electron to be spin up.

Once the measurement is performed, the state collapses into that spin state $|\psi\rangle \longmapsto |+_z\rangle$.

$$\langle +_z|\underbrace{\sigma_z|+_z\rangle}_{|+_z\rangle} = 1.$$

If we measure $|\psi\rangle \longmapsto \langle +_z|\sigma_z|\psi\rangle$ and find spin down, the state collapses to spin down $|-_z\rangle$

$$\langle +_z|\sigma_z|\psi\rangle = -\beta, \quad |-\beta|^2 = |\beta|^2$$

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Eigenvectors: $|+_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, $|-_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $|+_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$, $|-_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $|+_z\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|-_z\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ eigenvalues $+1, -1$

Hermitian: $\sigma^* = \sigma$

eg. $\sigma_y \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\i\end{pmatrix} = |+_y\rangle$

$\sigma_y \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\i\end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = -|-_y\rangle$

If we measure an electron having spin $|+_y\rangle$ (in the pos. y-direction) with respect to the x-axis

$\langle +_x | +_y\rangle = \frac{1}{\sqrt{2}}(1 \;\; 1)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1+i\\1+i\end{pmatrix}$ $\left|\frac{1+i}{2}\right|^2 = \frac{2}{4} = \frac{1}{2}$ $|a+bi|^2 = a^2+b^2$

Density matrix of $|\psi\rangle$ is $|\psi\rangle\langle\psi| = |\psi\rangle|\psi\rangle^* = \begin{pmatrix}\alpha\\\beta\end{pmatrix}(\bar\alpha \;\; \bar\beta) = \begin{pmatrix}\alpha\bar\alpha & \alpha\bar\beta\\ \bar\alpha\beta & \beta\bar\beta\end{pmatrix}$ $|\alpha|^2+|\beta|^2=1$

is Hermitian having eigenvalues $1, 0$; corresponding eigenvectors $|\psi\rangle = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$, $|\psi'\rangle = \begin{pmatrix}\bar\beta\\-\bar\alpha\end{pmatrix}$

$|\psi\rangle\langle\psi|\psi\rangle = |\psi\rangle$ since $\langle\psi|\psi\rangle = 1$

$\langle\psi'|\psi\rangle = (\beta \; -\alpha)\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \alpha\beta - \alpha\beta = 0.$

$|\psi\rangle\langle\psi|\psi'\rangle = 0. = 0|\psi'\rangle.$

$\langle\psi|\psi\rangle = 1 = \operatorname{tr}|\psi\rangle\langle\psi|$

$\operatorname{tr} AB = \operatorname{tr} BA.$

What is the corresponding Pauli spin operator in an arbitrary direction $n = (n_x, n_y, n_z) \in \mathbb{R}^3$

$n_x^2 + n_y^2 + n_z^2 = 1.$

$\sigma_n = n \cdot \sigma = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z$

$\sigma = (\sigma_x, \sigma_y, \sigma_z)$