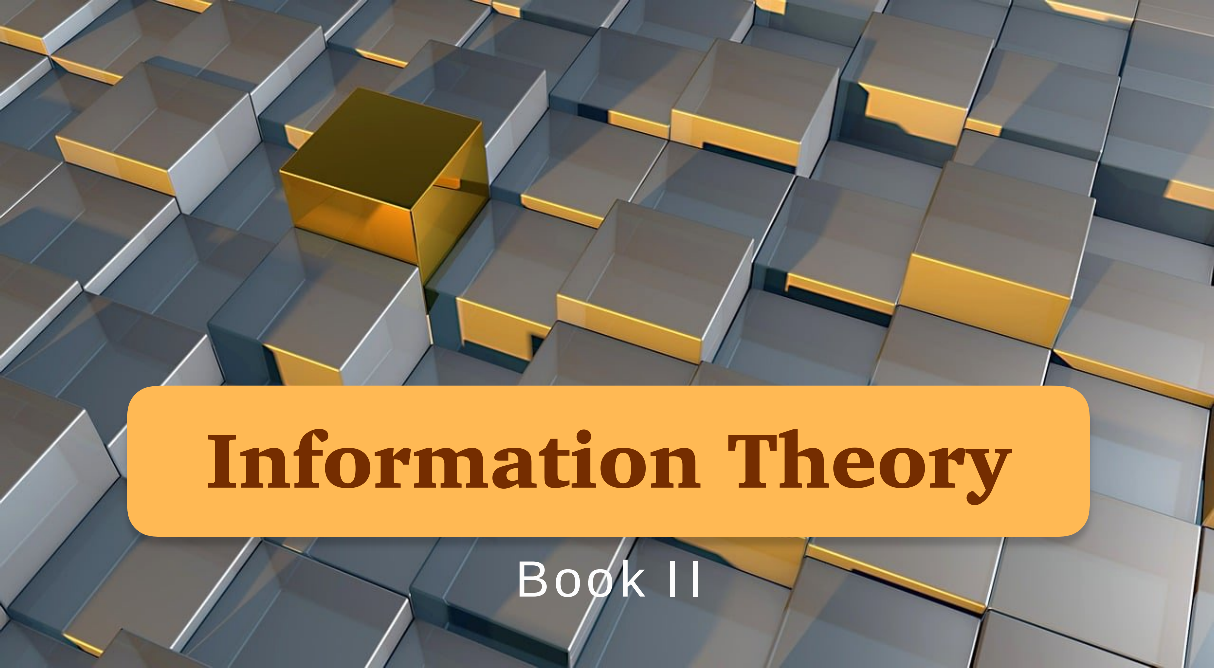
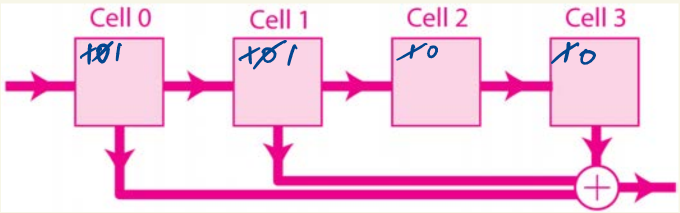


Information Theory

Book II

Eg. an infinite stream of bits $a_0a_1a_2a_3a_4\cdots$ $(a_i \in F)$ can be encoded eg. represent the plaintext bitstream as a $a_0+a_1x+a_2x^2+a_3x^3+\cdots \in \mathbb{F}_2[[x]]$

$F[[x]]$ = ring of (formal) power series in x with coefficients in F.



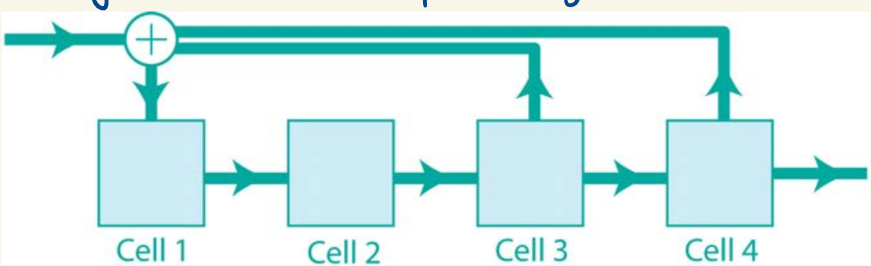
10110...

Eg. consider an input bitstream ~~110011~~0 1111 0010... which is encoded by the shift register above to obtain the output bitstream 101100101...

Compare: this is equivalent to multiplication by $1+x+x^3$:

$$(1+x+x^3)(1+x+x^4+x^5+x^7+x^8+x^9+x^{10}+x^{13}+\cdots) = 1+x^2+x^3+x^6+x^8+\cdots$$

Decoding of this data is accomplished using backward shift registers eg.



which performs division by $1+x+x^3$ in $\mathbb{F}_2((x))$

Symbolic rings and fields

$$\begin{tikzcd}
 & F((x)) \arrow[dl, dash] \arrow[dr, dash] & \\
F(x) \arrow[dr, dash] & & F[[x]] \arrow[dl, dash] \\
 & F[x] &
\end{tikzcd}$$

field of Laurent series $\to F((x))$

field of (power) series in x with coeffs in F. $\to F[[x]]$

field of rational "functions" (actually symbolic expressions) in x with coeffs in F — $F(x)$

$F[x]$ = ring of polys in x with coeffs in F

polynomials vs. polynomial functions

eg. $\mathbb{F}_3 = \{0,1,2\} = \mathbb{Z}/3\mathbb{Z}$

eg. $f(x) = 2+x+x^3 \in \mathbb{F}_3[x]$ is a polynomial of degree 3.

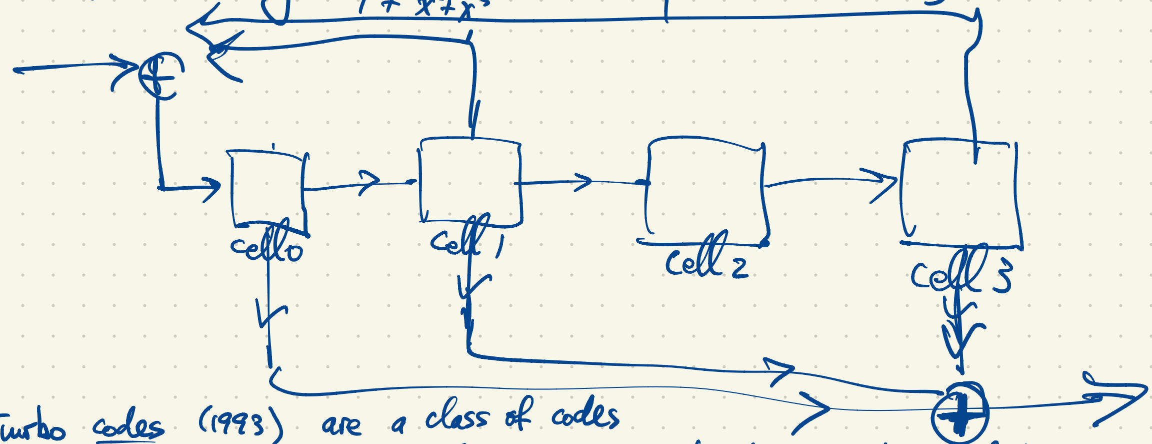
$g(x) = 2+2x \in \mathbb{F}_3[x]$ is a polynomial of degree 1.

a	$f(a)$	$g(a)$
0	2	2
1	1	1
2	0	0

$f(x), g(x)$ are distinct poly's but they represent the same function $\mathbb{F}_3 \to \mathbb{F}_3$.

eg. $f(x) = \dfrac{1+x+x^3}{x+x^2} \in \mathbb{F}_2(x)$

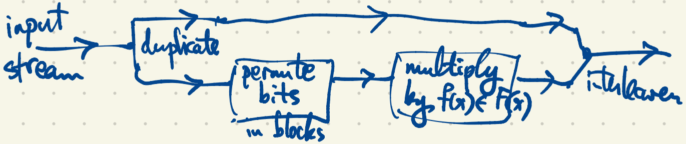
Multiplication by any rational function can be implemented using a single shift register e.g. multiplication by $\frac{1+x+x^3}{1+x^2+x^3}$ is implemented using the shift register



Turbo codes (1993) are a class of codes used for encoding streams of data using combinations of gates including

- multiplication by a rational function in $F(x)$
- splitters & interleavers
- permutations
- puncturing

eg.



$F(x) \subset F((x))$ eg. for $F = \mathbb{F}_2 = \{0,1\}$ <u>First method</u>

$$f(x) = \frac{1+x^2+x^5}{x+x^2+x^4} = \frac{1+x^2+x^5}{x(1+x+x^3)} = \frac{1}{x}\left[\frac{1+x^2+x^5}{1+x+x^3}\right] = \frac{1}{x}\left[1+x+x^3+x^5+\cdots\right] = \frac{1}{x}+1+x^2+x^4+\cdots$$

$$\frac{1+x^2+x^5}{1+x+x^3} = 1+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\cdots$$

$a_1=1$ $a_2=0$ $a_3=1$ $a_4=0$ $a_5=1$

$$1+x^2+x^5 = (1+x+x^3)(1+x+x^3+x^5+\cdots)$$

$(a+b)^2 = a^2+b^2$

$(a+b)^4 = a^4+b^4$

<u>Second method</u>   Geometric series $\frac{1}{1-u} = 1+u+u^2+u^3+u^4+\cdots$

$$\frac{1+x^2+x^5}{1+(x+x^3)} = (1+x^2+x^5)\left(1+(x+x^3)+(x+x^3)^2+(x+x^3)^3+(x+x^3)^4+(x+x^3)^5+\cdots\right)$$

$$= (1+x^2+x^5)\left(1+(x+x^3)+(x^2+x^6)+(x^3+x^5+\cdots)+(x^4+\cdots)+(x^5+\cdots)+\cdots\right)$$

$(x^3+3x^5+3x^7+x^9)$

$$= (1+x^2+x^5)(1+x+x^2+x^4+\cdots)$$

$$= 1+x+x^3+x^5+\cdots$$

$$f(x) = \frac{1}{x}(1+x+x^3+x^5+\cdots) = \frac{1}{x}+1+x^2+x^4+\cdots$$

$F = \mathbb{F}_2 = \{0,1\}$ for the time being

The irreducible (monic) polynomials in $F[x]$:

degree	irred. polys	
1	x, $x+1$	
2	x^2+x+1	
3	x^3+x+1, x^3+x^2+1	
4	x^4+x+1, x^4+x^3+1,	$x^4+x^3+x^2+x+1$
...	...	

primitive

not primitive

$x^2,\ x^2+1,\ x^2+x,\ x^2+x+1$ all poly's of degree 2.

$x^2 = x\cdot x$, $x^2+1 = (x+1)(x+1)$, $x^2+x = x(x+1)$

$x^4+x^2+1 = (x^2+x+1)^2$

See MacWilliams & Sloane, The Theory of Error-Correcting Codes for more extensive lists of irreducible polynomials.

What are all the cyclic (linear) binary codes of length 7? There are exactly 8 of them. (why?)

- subspace of F^7, $F = \mathbb{F}_2 = \{0,1\}$
- invariant under cyclic shift $(a_0,a_1,a_2,a_3,a_4,a_5,a_6) \longmapsto (a_6,a_0,a_1,\dots,a_5)$ $\quad a_i \in F$

A linear code $C \subseteq F^n$ is cyclic iff its dual code $C^\perp \subseteq F^n$ is also cyclic.

$\dim C + \dim C^\perp = n$.

eg. $\{(0000000)\}$

$\{0000000,\ 1111111\}$

$F^7 \leftarrow g(x)=1,\quad h(x)=x^7-1$

$\{\text{words in } F^7 \text{ of even weight}\} = \langle 1100000, 1010000, 1001000, 1000100, 1000010, 1000001\rangle$

$$\begin{array}{r} 1110100 \\ 0101000 \\ \hline 1011100 \end{array}$$

Hamming $[7,4,3]_2$ code $\mathcal{H} = \langle 1101000, 0110100, \dots, 1010001\rangle$ (all cyclic shifts of 1101000 span this code)

$\dim \mathcal{H} = 4$, $|\mathcal{H}| = 2^4 = 16$:

1	codeword of weight	0
7	...	3
7	...	4
1	...	7

Its dual $\mathcal{H}^\perp$, $\dim \mathcal{H}^\perp = 3$ is a $[7,3,4]_2$-code.

$\mathcal{H}^\perp$ has 1 codeword of weight 0

7 ... 4

$\mathcal{H}^\perp = \mathcal{H} \cap \langle 1111111\rangle^\perp$

$\mathcal{H}' = \langle 1011000, 0101100, \dots, 0110001\rangle$ also $[7,4,3]_2$

$\mathcal{H}'^\perp$ also $[7,3,4]_2$.

$\overset{n = \text{length}}{x^7 - 1} \in F[x]$

actually x^7+1 $\quad F = \mathbb{F}_2$

$$x^7-1 = \underset{\text{i.e. } x+1}{(x-1)}(x^6+x^5+x^4+x^3+x^2+1) = (x-1)\underbrace{(x^3+x+1)}_{(x-\alpha)(x-\alpha^2)(x-\alpha^4)}\underbrace{(x^3+x^2+1)}_{(x-\beta)(x-\beta^2)(x-\beta^4)}$$

If $E = \mathbb{F}_q$, $\quad x^q - x = \underset{x-a_0}{x}(x-1)(x-a_2)(x-a_3)\cdots(x-a_q)$

$a_0=0,\ a_1=1,\ a_2, a_3, \dots, a_q$ are the field elements.

i.e. $x^{q-1}-1$ has $q-1$ distinct roots which are the nonzero field elements.

If $\alpha \in \mathbb{F}_8$ is a root of x^3+x+1

$$\mathbb{F}_8 = \mathbb{F}_2[\alpha] = \{a_0 + a_1\alpha + a_2\alpha^2 : a_0, a_1, a_2 \in \mathbb{F}_2\}$$
$$= \{0, 1, \alpha, \alpha+1, \alpha^2, \alpha^2+1, \alpha^2+\alpha, \alpha^2+\alpha+1\}$$

$(u+v)^2 = u^2+v^2$

$(uv)^2 = u^2v^2$

Squaring is an automorphism of $\mathbb{F}_8$.

If $f(x) \in \mathbb{F}_p[x]$ is irreducible of degree d, then $\mathbb{F}_p[x]/(f(x)) \cong \mathbb{F}_{p^d} = \mathbb{F}_p[\beta]$ where β is a root of $f(x)$.

$= \{a_0 + a_1\beta + a_2\beta^2 + \cdots + a_{d-1}\beta^{d-1} : a_i \in \mathbb{F}_p\}$

(β generates $\mathbb{F}_{p^d} \supset \mathbb{F}_p$ as an algebra)

If in fact $\mathbb{F}_{p^d} = \{0, 1, \beta, \beta^2, \beta^3, \dots, \beta^{p^d-2}\}$ then we say β is a <u>primitive element</u> and we say $f(x)$ is a primitive polynomial.

If $f(x) = x^4+x^3+x^2+x+1$ and $\beta \in \mathbb{F}_{16} = \mathbb{F}_{2^4}$ is a root of $f(x)$ then $\beta^5 = 1$ since β is a root of $f(x)$

$$\beta^5 - 1 = (\beta-1)\underbrace{(\beta^4+\beta^3+\beta^2+\beta+1)}_{0} = 0$$

$0, 1, \beta, \beta^2, \beta^3, \beta^4, 1, \beta, \beta^2, \cdots$ doesn't give all of $\mathbb{F}_{16}$.

There are eight ways to factor $x^7 - 1 = g(x)h(x)$ in $\mathbb{F}_2[x]$
In each case $g(x)$ is a generator poly. and $h(x)$ is a parity check poly. for a cyclic code of length 7
over $\mathbb{F}_2 = \{0, 1\} = F$

Cyclic (linear) codes $\longleftrightarrow$ ideals in $F[x]/(x^n - 1)$

$g(x) = 1$, $h(x) = x^7 - 1$ gives F^7
$g(x) = x^7 - 1$, $h(x) = 1$ gives $\{0000000\}$
$g(x) = x + 1$, $h(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ gives all words of even weight i.e. $\langle 1100000, 1010000, \cdots, 1000001 \rangle$
$g(x) = x^6 + x^5 + \cdots + 1$, $h(x) = x + 1$ gives $\langle 1111111 \rangle = \{0000000, 1111111\}$
$g(x) = 1 + x + x^3$, $h(x) = 1 + x^2 + x^3 + x^4$ gives $\mathcal{H}$ $[7, 4, 3]_2$ code

<u>BCH bound</u> : a lower bound for performance of a cyclic code.
Consider a cyclic code of length n over F_q, i.e. an ideal in $\mathbb{F}_q[x]/(x^n - 1)$ with gen. poly. $g(x)$, parity check poly. $h(x)$, $x^n - 1 = g(x)h(x)$, $g(x)$ <u>primitive</u>, β root of $g(x)$ in $\mathbb{F}_{q^r}$, $r = \deg g(x)$,
and $\beta, \beta^2, \cdots, \beta^{s-1}$ are roots of $g(x)$, then the code has min. distance $\geq s$.

For Hamming $[7, 4, 3]_2$ code, β root of $g(x) = 1 + x + x^3 \in F[x]$, $\beta \in \mathbb{F}_8 = \mathbb{F}_2[\beta]$
Also β^2 - - - - - - - - by Freshman's Dream

$1 + \beta + \beta^3 = 0$
$(1 + \beta + \beta^3)^2 = 1 + \beta^2 + \beta^6 = 0 = 1 + \beta^2 + (\beta^2)^3 \implies \mathcal{H}$ has min. dist. ≥ 3.