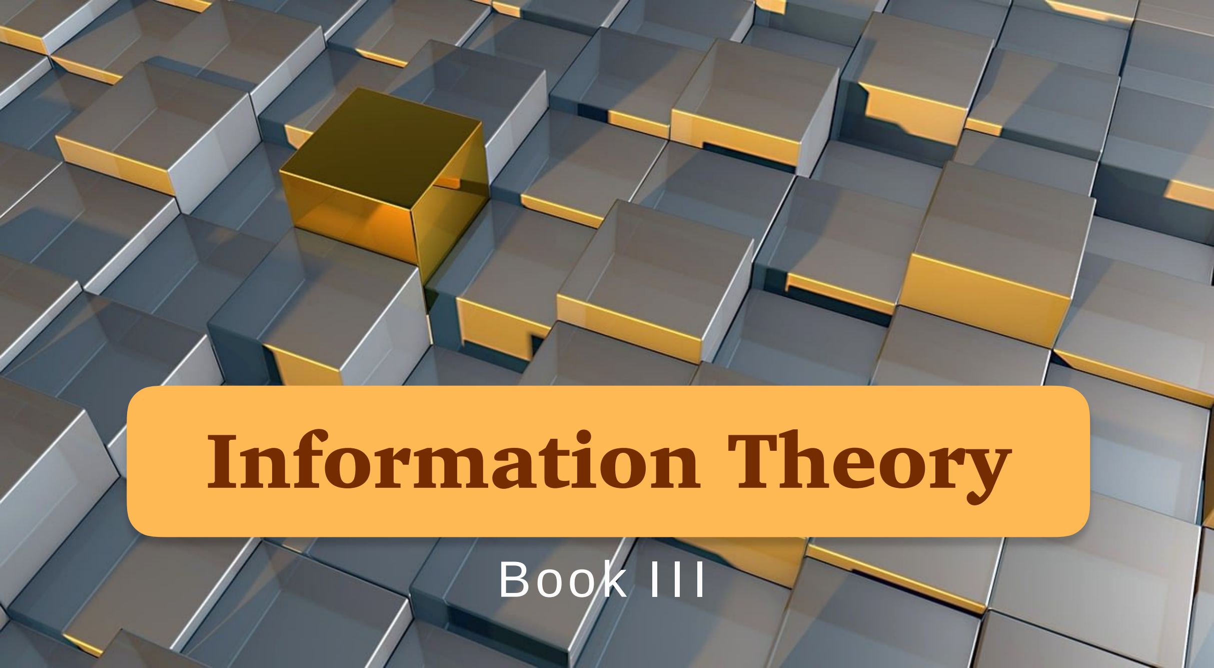

Information Theory

Book III

Spin state of an electron (disregard position and momentum) is an example of a qubit, which is a vector $|\psi\rangle \in \mathbb{C}^2 = \{ \binom{\alpha}{\beta} : \alpha, \beta \in \mathbb{C} \}$.
ket notation

Standard basis of $\mathbb{C}^2$: $|+\rangle = \binom{1}{0}$, $|-\rangle = \binom{0}{1}$
"spin up" "spin down"

$|\psi\rangle = \binom{\alpha}{\beta} = \alpha|+\rangle + \beta|-\rangle$, $|\alpha|^2 + |\beta|^2 = 1$.

An electron in this spin state is in a superposition of spin up and spin down states.

A measurement of an electron in this spin state yields a single bit of classical information:

- spin up, with probability $|\alpha|^2$;
- spin down, with probability $|\beta|^2$.

This says what happens when we measure with respect to the z-axis. (For measurement in a different direction/axis, we'll say later.)

As soon as the measurement is taken, the spin state collapses; all knowledge of α, β is then lost.

A linear functional on $\mathbb{C}^2$ is a linear transformation

$\langle\phi| : \mathbb{C}^2 \to \mathbb{C}$
bra notation

$$\langle\phi| = (\gamma\ \delta) : \binom{\alpha}{\beta} \mapsto (\gamma\ \delta)\binom{\alpha}{\beta} = \underbrace{\gamma\alpha + \delta\beta}_{\langle\phi|\psi\rangle} \in \mathbb{C}$$

Dual basis:

$\langle+| = |+\rangle^* = (1\ 0)$ (conjugate transpose)

$\langle-| = |-\rangle^* = (0\ 1)$

$$|\psi\rangle^* = \binom{\alpha}{\beta}^* = (\bar{\alpha}\ \bar{\beta}) = \bar{\alpha}\langle+| + \bar{\beta}\langle-|$$

$$\langle+|\psi\rangle = \langle+|\left(\alpha|+\rangle + \beta|-\rangle\right) = \alpha$$

$$\langle-|\psi\rangle = \beta.$$

Spin states are unit vectors in $\mathbb{C}^2$ i.e. $\binom{\alpha}{\beta}$, $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$.

i.e. in $\mathbb{R}^4$ $\qquad \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$

so $|\psi\rangle \in S^3$ = unit sphere in $\mathbb{R}^4$.

$\left.\begin{array}{l}\alpha = \alpha_1 + \alpha_2 i \\ \beta = \beta_1 + \beta_2 i\end{array}\right\} \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$

Any time we measure a spin state $|\psi\rangle \in S^3$, it collapses.
But it is possible to perform certain reversible operations $|\psi\rangle \mapsto A|\psi\rangle$ where A is a 2×2 unitary matrix $(AA^* = A^*A = I = \begin{pmatrix}1&0\\0&1\end{pmatrix})$ over $\mathbb{C}$.

Special examples of unitary matrices are scalar matrices $\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} = \lambda I$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$

These perform an operation on $|\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ whose only effect is to alter the phase of α, β

$$|\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \longmapsto A|\psi\rangle = \begin{pmatrix}\lambda\alpha\\ \lambda\beta\end{pmatrix} = \lambda\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \qquad \lambda = e^{i\theta} \quad (\theta \in [0, 2\pi))$$

which has no physical significance. For this reason the so-called density matrix

$$\underbrace{\underbrace{|\psi\rangle}_{2\times 1} \underbrace{\langle\psi|}_{1\times 2}}_{2\times 2} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\begin{pmatrix}\bar{\alpha} & \bar{\beta}\end{pmatrix} = \begin{pmatrix}\alpha\bar{\alpha} & \alpha\bar{\beta}\\ \beta\bar{\alpha} & \beta\bar{\beta}\end{pmatrix}$$

which holds all the physically significant information of ~~the~~ single qubit.

Hermitian 2×2 matrix

$H \in \mathbb{C}^{2\times 2}$ (2×2 complex matrix)

satisfying $H^* = H$

The map $\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \longmapsto \begin{pmatrix}\lambda\alpha\\ \lambda\beta\end{pmatrix}$ does not change this density matrix.

Entanglement typically occurs when we include multiple electrons in our system.

Start by reviewing statistical dependence works:

Let's say we take a random individual A from a population.

Imagine the population is 40% male, 60% female; 30% short, 70% tall.

Sampling by selecting one person gives two bits: MS, MT, FS, or FT.

Combinations of attributes: 12%, 28%, 18%, 42% if gender is independent of height.

In this example, gender and height are independent.

Height

Gender	S	T	
M	0.12	0.28	0.4
F	0.18	0.42	0.6
	0.3	0.7	1

$$\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.12 & 0.28 \\ 0.18 & 0.42 \end{bmatrix}$$

Outer product of two vectors. is a rank 1.

More typical distribution

	S	T	
M	0.1	0.3	0.4
F	0.2	0.4	0.6
	0.3	0.7	1

The matrix $\begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{bmatrix}$ has rank 2.

In this second example gender and height are (statistically) dependent.

If one electron has (spin) state $|\psi_1\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathbb{C}^2$ and a second electron has spin state $|\psi_2\rangle = \begin{pmatrix}\gamma\\ \delta\end{pmatrix} \in \mathbb{C}^2$

$|\alpha|^2 + |\beta|^2 = 1$ $\qquad |\gamma|^2 + |\delta|^2 = 1$

the pair of electrons has state $|\psi_{12}\rangle = \alpha_{11}|{+}{+}\rangle + \alpha_{12}|{+}{-}\rangle + \alpha_{21}|{-}{+}\rangle + \alpha_{22}|{-}{-}\rangle \in \mathbb{C}^4$

$\alpha_{ij} \in \mathbb{C}$, $\quad |\alpha_{11}|^2 + |\alpha_{12}|^2 + |\alpha_{21}|^2 + |\alpha_{22}|^2 = 1.$

prob. of both electrons having spin up

If the two electrons are <u>not</u> entangled then

$$\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\begin{pmatrix}\gamma & \delta\end{pmatrix} \qquad \text{rank } 1.$$

If the matrix has rank 2 then the two electrons are entangled.

$$\left.\begin{array}{ll} \text{Eg. } |\psi\rangle = \frac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle) & \text{i.e. } \begin{pmatrix}\frac{1}{\sqrt{2}} & 0\\ 0 & \frac{1}{\sqrt{2}}\end{pmatrix} \\ |\psi'\rangle = \frac{1}{\sqrt{2}}(|{+}{-}\rangle + |{-}{+}\rangle) & \text{i.e. } \begin{pmatrix}0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 0\end{pmatrix} \end{array}\right\} \text{ Examples of EPR pairs}$$

One way to talk about the spin state of a set of n electrons is

$$|\psi\rangle = \sum_{\substack{i_1\in\{0,1\}\\ i_2\in\{0,1\}\\ \vdots\\ i_n\in\{0,1\}}} \alpha_{i_1 i_2 i_3 \cdots i_n} \underbrace{|\pm\pm\pm\cdots\pm\rangle}_{\text{all } 2^n \text{ combinations of } \pm} \in \mathbb{C}^{2^n} \qquad \sum |\alpha_{i_1 i_2\cdots i_n}|^2 = 1$$

$(\alpha_{i_1 i_2 \cdots i_n} : i_1, i_2, \cdots, i_n \in \{0,1\})$ is a $\underbrace{2\times 2\times 2\times \cdots \times 2}_{n}$ array or tensor

$$\mathbb{C}^{2^n} = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}}$$ tensor product. Take basis $|+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

has basis
$$|{+}{+}{+}\cdots{+}{+}\rangle = |+\rangle \otimes |+\rangle \otimes \cdots \otimes |+\rangle$$
$$|{+}{+}{+}\cdots{+}{-}\rangle = |+\rangle \otimes |+\rangle \otimes \cdots \otimes |-\rangle$$
$$\vdots$$
$$|{-}{-}{-}\cdots{-}\rangle = |-\rangle \otimes |-\rangle \otimes \cdots \otimes |-\rangle .$$

More generally if $v_i \in \mathbb{C}^2$ $(i = 1, 2, \cdots, n)$

then $v_1 \otimes v_2 \otimes \cdots \otimes v_n \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. (pure tensors) simple

$$\mathbb{C}^2 \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^2 \qquad \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 = \mathbb{C}^{2^n}$$
$$\rotatebox{90}{$\in$} \qquad\qquad\qquad \rotatebox{90}{$\in$}$$
$$(v_1, \cdots, v_n) \longmapsto v_1 \otimes v_2 \otimes \cdots \otimes v_n$$ this map is multilinear

i.e. linear in each argument separately.

Bell's Theorem

Gleason's Theorem

Kochen-Specker Theorem