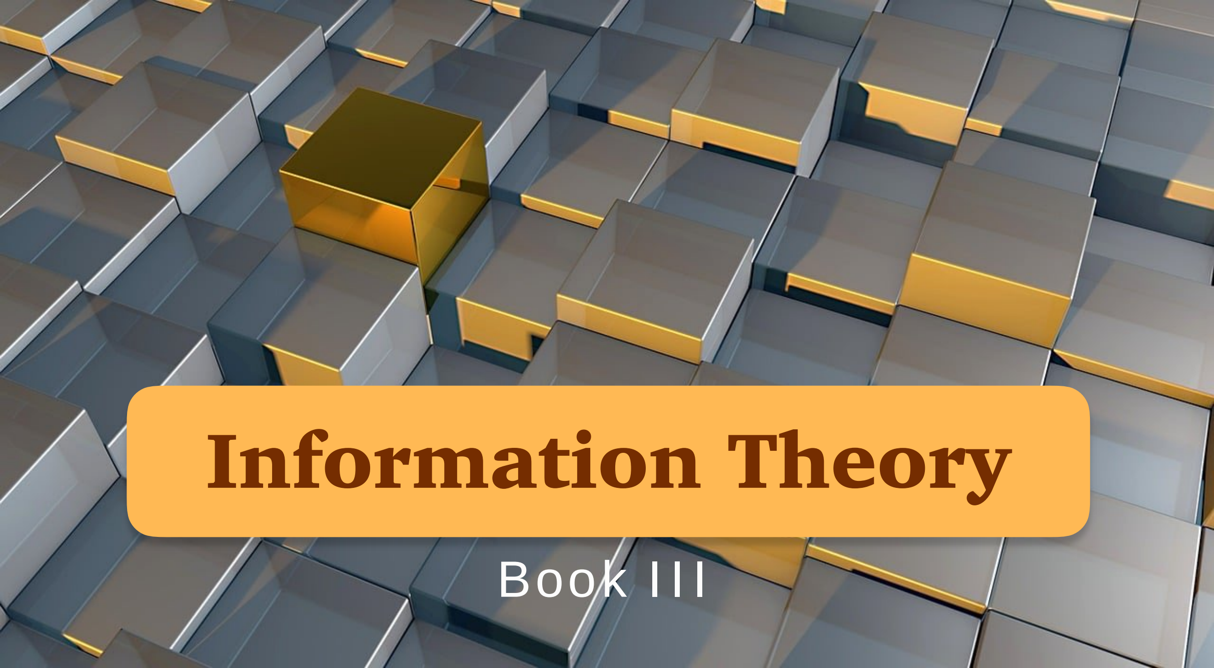

Information Theory

Book III

Spin state of an electron (disregard position and momentum) is an example of a qubit, which is a vector $|\psi\rangle \in \mathbb{C}^2 = \left\{ \begin{pmatrix}\alpha\\ \beta\end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$.
ket notation

$|\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \alpha|+\rangle + \beta|-\rangle$, $|\alpha|^2 + |\beta|^2 = 1$.

Standard basis of $\mathbb{C}^2$: $|+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$
"spin up" "spin down"

An electron in this spin state is in a superposition of spin up and spin down states.

A measurement of an electron in this spin state yields a single bit of classical information:

- spin up, with probability $|\alpha|^2$;
- spin down, with probability $|\beta|^2$.

This says what happens when we measure with respect to the z-axis. (For measurement in a different direction/axis, we'll say later.)

As soon as the measurement is taken, the spin state collapses; all knowledge of α, β is then lost.

A linear functional on $\mathbb{C}^2$ is a linear transformation
$\langle\phi| : \mathbb{C}^2 \to \mathbb{C}$
bra notation

$$\langle\phi| = (\gamma\ \delta) : \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \mapsto (\gamma\ \delta)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \gamma\alpha + \delta\beta \in \mathbb{C}$$
$$\gamma\alpha + \delta\beta = \langle\phi|\psi\rangle$$

Dual basis:
$\langle +| = |+\rangle^* = (1\ 0)$ (conjugate transpose)
$\langle -| = |-\rangle^* = (0\ 1)$

$$|\psi\rangle^* = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}^* = (\bar\alpha\ \bar\beta) = \bar\alpha\langle +| + \bar\beta\langle -|$$

$$\langle +|\psi\rangle = \langle +|\left(\alpha|+\rangle + \beta|-\rangle\right) = \alpha$$
$$\langle -|\psi\rangle = \beta.$$

Spin states are unit vectors in $\mathbb{C}^2$ i.e. $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$.
i.e. in $\mathbb{R}^4$
so $|\psi\rangle \in S^3$ = unit sphere in $\mathbb{R}^4$.

$\alpha = \alpha_1 + \alpha_2 i$, $\beta = \beta_1 + \beta_2 i$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$

$\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$

Any time we measure a spin state $|\psi\rangle \in S^3$, it collapses.
But it is possible to perform certain reversible operations $|\psi\rangle \mapsto A|\psi\rangle$ where A is a 2×2 unitary matrix $\left(AA^* = A^*A = I = \begin{pmatrix}1&0\\0&1\end{pmatrix}\right)$ over $\mathbb{C}$.

Special examples of unitary matrices are scalar matrices $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$

These perform an operation on $|\psi\rangle = \binom{\alpha}{\beta}$ whose only effect is to alter the phase of α, β

$$|\psi\rangle = \binom{\alpha}{\beta} \longmapsto A|\psi\rangle = \binom{\lambda\alpha}{\lambda\beta} = \lambda \binom{\alpha}{\beta}$$

$\lambda = e^{i\theta}$ $\quad (\theta \in [0, 2\pi))$

which has no physical significance. For this reason the so-called density matrix

$$\underbrace{\underbrace{|\psi\rangle}_{2\times 1} \underbrace{\langle\psi|}_{1\times 2}}_{2\times 2} = \binom{\alpha}{\beta} (\bar{\alpha} \quad \bar{\beta}) = \begin{pmatrix} \alpha\bar{\alpha} & \alpha\bar{\beta} \\ \beta\bar{\alpha} & \beta\bar{\beta} \end{pmatrix}$$

which holds all the physically significant information of ~~the~~ single qubit.

Hermitian 2×2 matrix

$H \in \mathbb{C}^{2\times 2}$ (2×2 complex matrix)

satisfying $H^* = H$

The map $\binom{\alpha}{\beta} \longmapsto \binom{\lambda\alpha}{\lambda\beta}$ does not change this density matrix.

Entanglement typically occurs when we include multiple electrons in our system.

Start by reviewing statistical dependence works:

Let's say we take a random individual A from a population.

Imagine the population is 40% male, 60% female; 30% short, 70% tall.

Sampling by selecting one person gives two bits: MS, MT, FS, or FT.
Combinations of attributes: 12%, 28%, 18%, 42% if gender is independent of height.